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Multiple fixed-sign solutions for a system of generalized right focal problems with deviating arguments

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Abstract

We consider the following *system* of generalized right focal boundary value problems

$$u_i'''(t) = f_i\big(t, u_1\big(\phi_1(t)\big), u_2\big(\phi_2(t)\big), \ldots, u_n\big(\phi_n(t)\big)\big), \quad t \in [a,b],$$
$$u_i(a) = u_i'(t^*) = 0, \quad \xi u_i(b) + \delta u_i''(b) = 0, \quad 1 \leqslant i \leqslant n,$$

where $\frac{1}{2}(a+b) < t^* < b$, $\xi \geqslant 0$, $\delta > 0$ and ϕ_i, $1 \leqslant i \leqslant n$ are deviating arguments. By using different fixed point theorems, we develop several criteria for the existence of three solutions of the system which are of *fixed sign* on the interval $[a,b]$, i.e., for each $1 \leqslant i \leqslant n$, $\theta_i u_i(t) \geqslant 0$ for all $t \in [a,b]$ and fixed $\theta_i \in \{1,-1\}$. Examples are also included to illustrate the results obtained.

Keywords: Fixed-sign solutions; System of generalized right focal boundary value problems; Deviating arguments

1. Introduction

In this paper we shall consider a model comprising a *system* of third-order differential equations subject to generalized right focal boundary conditions. To be precise, our model is

$$\begin{cases} u_i'''(t) = f_i(t, u_1(\phi_1(t)), u_2(\phi_2(t)), \ldots, u_n(\phi_n(t))), & t \in [a,b], \\ u_i(a) = u_i'(t^*) = 0, \quad \xi u_i(b) + \delta u_i''(b) = 0, & i = 1, 2, \ldots, n, \end{cases} \tag{F}$$

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where ϕ_i, $1 \leqslant i \leqslant n$ are deviating arguments, t^*, ξ, δ are fixed with

$$\frac{1}{2}(a+b) < t^* < b, \qquad \xi \geqslant 0, \qquad \delta > 0, \qquad \eta \equiv 2\delta + \xi(b-a)(b+a-2t^*) > 0.$$

A solution $u = (u_1, u_2, \ldots, u_n)$ of (F) will be sought in $(C[a,b])^n = C[a,b] \times C[a,b] \times \cdots \times C[a,b]$ (n times). We say that u is a solution of *fixed sign* if for each $1 \leqslant i \leqslant n$, we have $\theta_i u_i(t) \geqslant 0$ for $t \in [a,b]$, where $\theta_i \in \{1,-1\}$ is fixed. In particular, if we choose $\theta_i = 1$, $1 \leqslant i \leqslant n$, then our fixed-sign solution u becomes a *positive* solution, i.e., $u_i(t) \geqslant 0$ for $t \in [a,b]$, $1 \leqslant i \leqslant n$. We remark that in many practical problems, it is only meaningful to have *positive* solutions. Nonetheless our definition of *fixed-sign* solution is more general and gives extra *flexibility*.

Existence of positive solutions to the two-point right focal boundary value problem

$$\begin{aligned}
&(-1)^{3-k} y'''(t) = f\big(t, y(t)\big), \quad t \in [0,1],\\
&y^{(j)}(0) = 0, \quad 0 \leqslant j \leqslant k-1,\\
&y^{(j)}(1) = 0, \quad k \leqslant j \leqslant 2,
\end{aligned}$$

where $k \in \{1,2\}$, has been well discussed in the literature [1,3]. The related discrete problem can be found in [11,12,15]. Work on a three-point right focal problem, a special case of (F) when $n = 1$, $\delta = 1$, $\xi = 0$, $\phi_1(t) = t$ is available in [4,6]. Recently, Anderson [5] considered (F) when $n = 1$, $\phi_1(t) = t$ and developed the Green's function for the boundary value problem. In our present work, we generalize the problem considered in [5] to, firstly, a *system* of boundary value problems, and secondly, with very *general* nonlinear terms f_i involving *deviating arguments*—this yields a much more robust model for many nonlinear phenomena. We shall establish the existence of *three fixed-sign* solutions using *both* fixed point theorems of Leggett and Williams [10] as well as of Avery [7]. Estimates on the norms of these solutions will also be provided. Besides achieving *new* results (to date in the literature), we also discuss the generality of the results, and illustrate the importance of the results through some examples. For a special case of (F) when $\phi_i(t) = t$, $1 \leqslant i \leqslant n$, work in different aspects can be found in [13,14]. We remark that knowledge of how many solutions is probably most important from a numerical standpoint. If it is known that there are multiple solutions, then naturally one may need to develop methods that produce a specific one of the solutions for efficiency sake.

The paper is organized as follows. Section 2 contains the necessary definitions and fixed point theorems. The existence criteria are developed and discussed in Section 3. Finally, examples are presented in Section 4 to illustrate the importance of the results obtained.

2. Preliminaries

In this section we shall state some necessary definitions and the relevant fixed point theorems. Let B be a Banach space equipped with norm $\|\cdot\|$.

Definition 2.1. Let C ($\subset B$) be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

(a) if $u \in C$ and $\alpha \geqslant 0$, then $\alpha u \in C$;
(b) if $u \in C$ and $-u \in C$, then $u = 0$.

Definition 2.2. Let C $(\subset B)$ be a cone. A map ψ is called a *nonnegative continuous concave functional* on C and a map β is called a *nonnegative continuous convex functional* on C if the following conditions are satisfied:

(a) $\psi, \beta : C \to [0, \infty)$ are continuous;
(b) $\psi(ty + (1-t)z) \geqslant t\psi(y) + (1-t)\psi(z)$ and $\beta(ty + (1-t)z) \leqslant t\beta(y) + (1-t)\beta(z)$ for all $y, z \in C$ and $0 \leqslant t \leqslant 1$.

Let γ, β, Θ be nonnegative continuous convex functionals on C and α, ψ be nonnegative continuous concave functionals on C. For nonnegative numbers w_i, $1 \leqslant i \leqslant 3$, we shall introduce the following notations:

$$\begin{aligned}
&C(w_1) = \{u \in C \mid \|u\| < w_1\},\\
&C(\psi, w_1, w_2) = \{u \in C \mid \psi(u) \geqslant w_1 \text{ and } \|u\| \leqslant w_2\},\\
&P(\gamma, w_1) = \{u \in C \mid \gamma(u) < w_1\},\\
&P(\gamma, \alpha, w_1, w_2) = \{u \in C \mid \alpha(u) \geqslant w_1 \text{ and } \gamma(u) \leqslant w_2\},\\
&Q(\gamma, \beta, w_1, w_2) = \{u \in C \mid \beta(u) \leqslant w_1 \text{ and } \gamma(u) \leqslant w_2\},\\
&P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{u \in C \mid \alpha(u) \geqslant w_1,\ \Theta(u) \leqslant w_2 \text{ and } \gamma(u) \leqslant w_3\},\\
&Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{u \in C \mid \psi(u) \geqslant w_1,\ \beta(u) \leqslant w_2 \text{ and } \gamma(u) \leqslant w_3\}.
\end{aligned}$$

The following fixed point theorems are needed later. The first is usually called *Leggett–Williams' fixed point theorem*, and the second is known as the *five-functional fixed point theorem*.

Theorem 2.1. [10] *Let C $(\subset B)$ be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(u) \leqslant \|u\|$ for all $u \in \bar{C}(w_4)$, and let $S : \bar{C}(w_4) \to \bar{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1, w_2, w_3 where $0 < w_1 < w_2 < w_3 \leqslant w_4$ such that*

(a) *$\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \neq \emptyset$, and $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$;*
(b) *$\|Su\| < w_1$ for all $u \in \bar{C}(w_1)$;*
(c) *$\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.*

Then, S has (at least) three fixed points u^1, u^2 and u^3 in $\bar{C}(w_4)$. Furthermore, we have

$$\begin{aligned}
&u^1 \in C(w_1), \qquad u^2 \in \{u \in C(\psi, w_2, w_4) \mid \psi(u) > w_2\} \quad \text{and}\\
&u^3 \in \bar{C}(w_4) \backslash \big(C(\psi, w_2, w_4) \cup \bar{C}(w_1)\big).
\end{aligned} \tag{2.1}$$

Theorem 2.2. [7] *Let C $(\subset B)$ be a cone. Assume that there exist positive numbers w_5, M, nonnegative continuous convex functionals γ, β, Θ on C, and nonnegative continuous concave functionals α, ψ on C, with $\alpha(u) \leqslant \beta(u)$ and $\|u\| \leqslant M\gamma(u)$ for all $u \in \bar{P}(\gamma, w_5)$. Let $S : \bar{P}(\gamma, w_5) \to \bar{P}(\gamma, w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers w_i, $1 \leqslant i \leqslant 4$ with $0 < w_2 < w_3$ such that*

(a) *$\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \neq \emptyset$, and $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;*

(b) $\{u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(u) < w_2\} \neq \emptyset$, *and* $\beta(Su) < w_2$ *for all* $u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$;
(c) $\alpha(Su) > w_3$ *for all* $u \in P(\gamma,\alpha,w_3,w_5)$ *with* $\Theta(Su) > w_4$;
(d) $\beta(Su) < w_2$ *for all* $u \in Q(\gamma,\beta,w_2,w_5)$ *with* $\psi(Su) < w_1$.

Then, S has (*at least*) *three fixed points u^1, u^2 and u^3 in $\bar{P}(\gamma, w_5)$. Furthermore, we have*

$$\beta\big(u^1\big) < w_2, \qquad \alpha\big(u^2\big) > w_3, \quad \text{and} \quad \beta\big(u^3\big) > w_2 \quad \text{with} \quad \alpha\big(u^3\big) < w_3. \tag{2.2}$$

We also require the definition of a L^q*-Carathéodory function.*

Definition 2.3. A function $P:[a,b]\times\mathbb{R}^n \to \mathbb{R}$ is a L^q*-Carathéodory function* if the following conditions hold:

(a) the map $t \to P(t,u)$ is measurable for all $u \in \mathbb{R}^n$;
(b) the map $u \to P(t,u)$ is continuous for almost all $t \in [a,b]$;
(c) for any $r > 0$, there exists $\mu_r \in L^q[a,b]$ such that $|u| \leqslant r$ implies that $|P(t,u)| \leqslant \mu_r(t)$ for almost all $t \in [a,b]$.

3. Main results

Throughout we shall denote $u = (u_1, u_2, \ldots, u_n)$. Let the Banach space $B = (C[a,b])^n$ be equipped with the norm

$$\|u\| = \max_{1\leqslant i\leqslant n} \sup_{t\in[a,b]} \big|u_i(t)\big| = \max_{1\leqslant i\leqslant n} |u_i|_0,$$

where we denote $|u_i|_0 = \sup_{t\in[a,b]} |u_i(t)|$, $1 \leqslant i \leqslant n$.

To apply the fixed point theorems in Section 2, we need to define an operator $S: B \to B$ so that a solution u of the system (F) is a fixed point of S, i.e., $u = Su$. For this, let $g(t,s)$ be the Green's function of the boundary value problem

$$\begin{aligned}
&y'''(t) = 0, \quad t \in [a,b],\\
&y(a) = y'(t^*) = 0,\\
&\xi y(b) + \delta y''(b) = 0.
\end{aligned}$$

Hence, we shall define the operator $S: B \to B$ by

$$Su(t) = \big(S_1u(t), S_2u(t), \ldots, S_nu(t)\big), \quad t \in [a,b], \tag{3.1}$$

$$\begin{aligned}
S_iu(t) &= \int_a^b g(t,s) f_i\big(s, u_1\big(\phi_1(s)\big), u_2\big(\phi_2(s)\big), \ldots, u_n\big(\phi_n(s)\big)\big)\, ds\\
&= \int_a^b g(t,s) f_i\big(s, u\big(\phi(s)\big)\big)\, ds, \quad t \in [a,b], \quad 1 \leqslant i \leqslant n,
\end{aligned} \tag{3.2}$$

where we denote $u(\phi(s)) = (u_1(\phi_1(s)), u_2(\phi_2(s)), \ldots, u_n(\phi_n(s)))$. Clearly, a fixed point of the operator S is a solution of the system (F).

Our first lemma gives the properties of the Green's function $g(t,s)$ which will be used later.

Lemma 3.1. [5] *It is known that for $t, s \in [a, b]$,*

$$g(t,s)=\begin{cases} s\in[a,t^*]\colon & \begin{cases} \frac{t-a}{2}(2s-t-a)+\frac{\xi(t-a)}{2\eta}(s-a)^2(2t^*-a-t), & t\leqslant s,\\ \frac{(s-a)^2}{2\eta}[\eta+\xi(t-a)(2t^*-a-t)], & t\geqslant s,\end{cases}\\ s\in[t^*,b]\colon & \begin{cases} \frac{t-a}{2\eta}(2t^*-a-t)[2\delta+\xi(b-s)^2], & t\leqslant s,\\ \frac{t-a}{2\eta}(2t^*-a-t)[2\delta+\xi(b-s)^2]+\frac{(t-s)^2}{2}, & t\geqslant s.\end{cases}\end{cases} \tag{3.3}$$

Moreover,

$$g(t,s)\geqslant 0,\quad t,s\in[a,b];\qquad g(t,s)>0,\quad t,s\in(a,b], \tag{3.4}$$

$$g(t,s)\leqslant g(t^*,s),\quad t,s\in[a,b], \tag{3.5}$$

$$g(t,s)\geqslant Mg(t^*,s),\quad t\in[t^*-h,t^*+h],\ s\in[a,b], \tag{3.6}$$

where $h \in (0, b - t^*)$ is fixed and

$$M=\frac{(t^*-a+h)(t^*-a-h)}{(t^*-a)^2}\in(0,1).$$

For clarity, we shall list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1, -1\}$, $1 \leqslant i \leqslant n$ are fixed,

$$[0,\infty)_i=\begin{cases}[0,\infty), & \text{if } \theta_i=1,\\ (-\infty,0], & \text{if } \theta_i=-1,\end{cases}$$

$$\tilde K=\bigl\{u\in B \mid \text{for each } 1\leqslant i\leqslant n,\ \theta_i u_i(t)\geqslant 0 \text{ for } t\in[a,b]\bigr\},$$

$$K=\bigl\{u\in\tilde K \mid \text{for some } j\in\{1,2,\ldots,n\},\ \theta_j u_j(t)>0 \text{ for some } t\in[a,b]\bigr\}=\tilde K\backslash\{0\}.$$

(C1) For each $1 \leqslant i \leqslant n$, $f_i : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function.

(C2) For each $1 \leqslant i \leqslant n$,

$$\theta_i f_i(t,u)\geqslant 0,\quad u\in\tilde K,\ \text{a.e. } t\in(a,b)\quad\text{and}$$
$$\theta_i f_i(t,u)>0,\quad u\in K,\ \text{a.e. } t\in(a,b).$$

(C3) There exist continuous functions p, ν, μ_i, $1 \leqslant i \leqslant n$ with $p : \prod_{j=1}^n [0,\infty)_j \to [0,\infty)$ and $\nu, \mu_i : (a, b) \to [0,\infty)$ such that for each $1 \leqslant i \leqslant n$,

$$\mu_i(t)p(u)\leqslant\theta_i f_i(t,u)\leqslant\nu(t)p(u),\quad u\in\tilde K,\ \text{a.e. } t\in(a,b).$$

(C4) For each $1 \leqslant i \leqslant n$, there exists a number $0 < \rho_i \leqslant 1$ such that

$$\mu_i(t)\geqslant\rho_i\nu(t),\quad \text{a.e. } t\in(a,b).$$

(C5) For each $1 \leqslant i \leqslant n$, ϕ_i is continuous and ϕ_i maps $[a, b]$ into $[a, b]$.

Remark 3.1. There are many examples of a deviating function ϕ_i satisfying (C5). For instance, when $a = 0$ and $b = 1$, $\phi_i(t) = 1 - t$, $\sin \pi t$, $\sqrt{t}$.

Lemma 3.2. *Let* (C1) *hold. Then, the operator S defined in* (3.1), (3.2) *is continuous and completely continuous.*

Proof. From Lemma 3.1, we have $g(t,s) \equiv g^t(s) \in C[a,b] \subseteq L^\infty[a,b]$, $t \in [a,b]$ and the map $t \to g(t,s)$ is continuous from $[a,b]$ to $C[a,b]$. This together with $f_i : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function ensures that S is continuous and completely continuous. □

Let $h \in (0, b-t^*)$ be fixed. We define a cone C in B as

$$C = \Big\{ u \in B \mid \text{for each } 1 \leqslant i \leqslant n,\ \theta_i u_i(t) \geqslant 0 \text{ for } t \in [a,b], \text{ and} \\ \min_{t \in [t^*-h, t^*+h]} \theta_i u_i(t) \geqslant M\rho_i |u_i|_0 \Big\}, \tag{3.7}$$

where M and ρ_i are defined in (3.6) and (C4) respectively. Clearly, we have $C \subseteq \tilde{K}$.

Remark 3.2. If (C2) and (C3) hold, then it follows from (3.2) and (3.4) that for $u \in \tilde{K}$ and $t \in [a,b]$,

$$0 \leqslant \int_a^b g(t,s)\mu_i(s) p\big(u\big(\phi(s)\big)\big)\,ds \leqslant \theta_i S_i u(t) \leqslant \int_a^b g(t,s)\nu(s) p\big(u\big(\phi(s)\big)\big)\,ds, \\ 1 \leqslant i \leqslant n. \tag{3.8}$$

Lemma 3.3. *Let* (C1)–(C4) *hold. Then, the operator S maps C into C.*

Proof. Let $u \in C$. From (3.8) we have $\theta_i S_i u(t) \geqslant 0$ for $t \in [a,b]$ and $1 \leqslant i \leqslant n$.

Next, using (3.8) and (3.5) gives for $t \in [a,b]$ and $1 \leqslant i \leqslant n$,

$$\big|S_i u(t)\big| = \theta_i S_i u(t) \leqslant \int_a^b g(t,s)\nu(s) p\big(u\big(\phi(s)\big)\big)\,ds \leqslant \int_a^b g(t^*,s)\nu(s) p\big(u\big(\phi(s)\big)\big)\,ds. \tag{3.9}$$

Hence, we have

$$|S_i u|_0 \leqslant \int_a^b g(t^*,s)\nu(s) p\big(u\big(\phi(s)\big)\big)\,ds, \quad 1 \leqslant i \leqslant n \tag{3.10}$$

and therefore

$$\|Su\| = \max_{1 \leqslant i \leqslant n} |S_i u|_0 \leqslant \int_a^b g(t^*,s)\nu(s) p\big(u\big(\phi(s)\big)\big)\,ds. \tag{3.11}$$

Now, employing (3.8), (3.6), (C4) and (3.10) we find for $t \in [t^*-h, t^*+h]$ and $1 \leqslant i \leqslant n$,

$$\theta_i S_i u(t) \geqslant \int_a^b g(t,s)\mu_i(s) p\big(u\big(\phi(s)\big)\big)\,ds \\ \geqslant \int_a^b M g(t^*,s)\mu_i(s) p\big(u\big(\phi(s)\big)\big)\,ds$$

$$\geqslant \int_a^b Mg(t^*,s)\rho_i \nu(s) p\big(u\big(\phi(s)\big)\big)\, ds$$
$$\geqslant M\rho_i |S_i u|_0.$$

This leads to

$$\min_{t\in[t^*-h,t^*+h]} \theta_i S_i u(t) \geqslant M\rho_i |S_i u|_0, \quad 1 \leqslant i \leqslant n.$$

We have shown that $Su \in C$. □

Remark 3.3. From the proof of Lemma 3.3, we see that it is possible to use another cone C' ($\subset C$) given by

$$C' = \Big\{u \in B \mid \text{for each } 1 \leqslant i \leqslant n,\ \theta_i u_i(t) \geqslant 0 \text{ for } t \in [a,b], \text{ and} \\ \min_{t\in[t^*-h,t^*+h]} \theta_i u_i(t) \geqslant M\rho_i \|u\|\Big\}.$$

The arguments used will be similar.

For subsequent results, we define the following constants for $1 \leqslant i \leqslant n$, and some fixed $A, D, E \subseteq [a,b]$:

$$q = \int_a^b g(t^*,s)\nu(s)\, ds,$$
$$r_i = \min_{t\in A} \int_A g(\phi_i(t),s)\mu_i(s)\, ds,$$
$$d_{1,i} = \min_{t\in D} \int_D g\big(\phi_i(t),s\big)\mu_i(s)\, ds,$$
$$d_2 = \int_E g(t^*,s)\nu(s)\, ds,$$
$$d_3 = \int_{[a,b]\setminus E} g(t^*,s)\nu(s)\, ds = q - d_2. \tag{3.12}$$

Lemma 3.4. *Let* (C1)–(C5) *hold, and assume*

(C6) *the function* $g(t^*,s)\nu(s)$ *is nonzero on a subset of* $[a,b]$ *of positive measure.*

Suppose that there exists a number $d > 0$ *such that for* $|u_j| \in [0,d]$, $1 \leqslant j \leqslant n$,

$$p(u_1,u_2,\ldots,u_n) < \frac{d}{q}. \tag{3.13}$$

Then,

$$S\big(\bar{C}(d)\big) \subseteq C(d) \subset \bar{C}(d). \tag{3.14}$$

Proof. Let $u \in \bar{C}(d)$. Then, $|u_j| \in [0, d]$, $1 \leqslant j \leqslant n$. Applying (3.9), (C6), (C5) and (3.13), we find for $1 \leqslant i \leqslant n$ and $t \in [a, b]$,

$$\left|S_i u(t)\right| \leqslant \int_a^b g(t^*, s)\nu(s) p\bigl(u\bigl(\phi(s)\bigr)\bigr)\, ds < \int_a^b g(t^*, s)\nu(s) \frac{d}{q}\, ds = q\frac{d}{q} = d.$$

This implies $|S_i u|_0 < d$, $1 \leqslant i \leqslant n$ and so $\|Su\| < d$. Coupling with the fact that $Su \in C$ (Lemma 3.3), we have $Su \in C(d)$. The conclusion (3.14) is now immediate. □

The next lemma is similar to Lemma 3.4 and hence we shall omit the proof.

Lemma 3.5. *Let* (C1)–(C5) *hold. Suppose that there exists a number $d > 0$ such that for $|u_j| \in [0, d]$, $1 \leqslant j \leqslant n$,*

$$p(u_1, u_2, \ldots, u_n) \leqslant \frac{d}{q}.$$

Then,

$$S\bigl(\bar{C}(d)\bigr) \subseteq \bar{C}(d).$$

We are now ready to establish existence criteria for three fixed-sign solutions. Our first result employs Leggett–Williams' fixed point theorem (Theorem 2.1).

Theorem 3.1. *Let* (C1)–(C6) *hold. Let $h \in (0, b - t^*)$ be fixed, and let A be the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [t^* - h, t^* + h]$, $1 \leqslant i \leqslant n$ for all $t \in A$. Assume*

(C7) *for each $1 \leqslant i \leqslant n$ and each $x \in [t^* - h, t^* + h]$, the function $g(x, s)\mu_i(s) \equiv g^x(s)\mu_i(s)$ is nonzero on a subset of A of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 < \frac{w_2}{M \min_{1 \leqslant i \leqslant n} \rho_i} \leqslant w_3$$

such that the following hold:

(P) $p(u_1, u_2, \ldots, u_n) < \frac{w_1}{q}$ *for* $|u_j| \in [0, w_1]$, $1 \leqslant j \leqslant n$;
(Q) *one of the following holds*:
  (Q1) $\limsup_{|u_1|, |u_2|, \ldots, |u_n| \to \infty} (p(u_1, u_2, \ldots, u_n)/|u_j|) < 1/q$ *for some* $j \in \{1, 2, \ldots, n\}$;
  (Q2) *there exists a number* d ($\geqslant w_3$) *such that* $p(u_1, u_2, \ldots, u_n) \leqslant d/q$ *for* $|u_j| \in [0, d]$, $1 \leqslant j \leqslant n$;
(R) *for each* $1 \leqslant i \leqslant n$, $p(u_1, u_2, \ldots, u_n) > w_2/r_i$ *for* $|u_j| \in [w_2, w_3]$, $1 \leqslant j \leqslant n$.

Then, the system (F) *has* (*at least*) *three fixed-sign solutions $u^1, u^2, u^3 \in C$ such that*

$$\begin{aligned} &\bigl\|u^1\bigr\| < w_1; \qquad \theta_i u_i^2(x) > w_2, \quad x \in [t^* - h, t^* + h],\ 1 \leqslant i \leqslant n; \\ &\bigl\|u^3\bigr\| > w_1 \quad \text{and} \quad \min_{1 \leqslant i \leqslant n} \min_{x \in [t^* - h, t^* + h]} \theta_i u_i^3(x) < w_2. \end{aligned} \tag{3.15}$$

Proof. We shall employ Theorem 2.1. First, we shall prove that condition (Q) implies the existence of a number w_4 where $w_4 \geqslant w_3$ such that

$$S(\bar{C}(w_4)) \subseteq \bar{C}(w_4). \tag{3.16}$$

Suppose that (Q2) holds. Then, by Lemma 3.5 we immediately have (3.16) where we pick $w_4 = d$. Suppose now that (Q1) is satisfied. Then, there exist $N > 0$ and $\epsilon < 1/q$ such that

$$\frac{p(u_1, u_2, \ldots, u_n)}{|u_j|} < \epsilon, \quad |u_1|, |u_2|, \ldots, |u_n| > N. \tag{3.17}$$

Define

$$L = \max_{|u_m| \in [0,N],\ 1 \leqslant m \leqslant n} p(u_1, u_2, \ldots, u_n).$$

In view of (3.17), it is clear that the following holds for all $(u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$,

$$p(u_1, u_2, \ldots, u_n) \leqslant L + \epsilon |u_j|, \tag{3.18}$$

where j is as in (Q1).

Now, pick the number w_4 so that

$$w_4 > \max\left\{w_3, L\left(\frac{1}{q} - \epsilon\right)^{-1}\right\}. \tag{3.19}$$

Let $u \in \bar{C}(w_4)$. For $t \in [a,b]$ and $1 \leqslant i \leqslant n$, using (3.9), (3.18), (C5) and (3.19) gives

$$\begin{aligned}
\left|S_i u(t)\right| &\leqslant \int_a^b g(t^*, s)\nu(s) p\big(u(\phi(s))\big)\, ds \\
&\leqslant \int_a^b g(t^*, s)\nu(s)\big(L + \epsilon\big|u_j\big(\phi_j(s)\big)\big|\big)\, ds \\
&\leqslant \int_a^b g(t^*, s)\nu(s)(L + \epsilon w_4)\, ds \\
&= q(L + \epsilon w_4) \\
&< q\left[w_4\left(\frac{1}{q} - \epsilon\right) + \epsilon w_4\right] \\
&= w_4.
\end{aligned}$$

This leads to $|Su_i|_0 < w_4$, $1 \leqslant i \leqslant n$. Hence, $\|Su\| < w_4$ and so $Su \in C(w_4) \subset \bar{C}(w_4)$. Thus, (3.16) follows immediately.

Let $\psi : C \to [0, \infty)$ be defined by

$$\psi(u) = \min_{1 \leqslant i \leqslant n} \min_{t \in A} \theta_i u_i\big(\phi_i(t)\big).$$

Recall that $\phi_i(t) \in [t^* - h, t^* + h]$, $1 \leqslant i \leqslant n$ for all $t \in A$. Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(u) \leqslant \|u\|$ for all $u \in C$.

We shall verify that condition (a) of Theorem 2.1 is satisfied. In fact, it is obvious that $\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \neq \emptyset$ since

$$u(t) = \left(\frac{\theta_1}{2}(w_2+w_3), \frac{\theta_2}{2}(w_2+w_3), \ldots, \frac{\theta_n}{2}(w_2+w_3)\right)$$
$$\in \{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\}.$$

Next, let $u \in C(\psi, w_2, w_3)$. Then, $w_2 \leqslant \psi(u) \leqslant \|u\| \leqslant w_3$ and hence we have, noting (C5),

$$\theta_j u_j(\phi_j(s)) = |u_j(\phi_j(s))| \in [w_2, w_3], \quad s \in A,\ 1 \leqslant j \leqslant n. \tag{3.20}$$

In view of (3.8), (3.20), (C7), (R), (3.6) and (3.12), it follows that

$$\begin{aligned}
\psi(Su) &= \min_{1\leqslant i\leqslant n} \min_{t\in A} \theta_i (S_i u)(\phi_i(t)) \\
&\geqslant \min_{1\leqslant i\leqslant n} \min_{t\in A} \int_a^b g(\phi_i(t), s)\mu_i(s) p(u(\phi(s)))\, ds \\
&\geqslant \min_{1\leqslant i\leqslant n} \min_{t\in A} \int_A g(\phi_i(t), s)\mu_i(s) p(u(\phi(s)))\, ds \\
&> \min_{1\leqslant i\leqslant n} \min_{t\in A} \int_A g(\phi_i(t), s)\mu_i(s) \frac{w_2}{r_i}\, ds \\
&= \min_{1\leqslant i\leqslant n} \frac{r_i}{r_i} w_2 \\
&= w_2.
\end{aligned}$$

Therefore, we have shown that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$.

Next, by Lemma 3.4 and condition (P), we have $S(\bar{C}(w_1)) \subseteq C(w_1)$. Hence, condition (b) of Theorem 2.1 is fulfilled.

Finally, we shall show that condition (c) of Theorem 2.1 holds. Recall that

$$\phi_i(t) \in [t^* - h, t^* + h], \quad t \in A,\ 1 \leqslant i \leqslant n \quad \text{and} \quad w_3 \geqslant \frac{w_2}{M \min_{1\leqslant i\leqslant n} \rho_i}. \tag{3.21}$$

Let $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$. Using (3.8), (3.21), (3.6), (C4) and (3.11), we find

$$\begin{aligned}
\psi(Su) &\geqslant \min_{1\leqslant i\leqslant n} \min_{t\in A} \int_a^b g(\phi_i(t), s)\mu_i(s) p(u(\phi(s)))\, ds \\
&\geqslant \min_{1\leqslant i\leqslant n} \int_a^b M g(t^*, s)\rho_i \nu(s) p(u(\phi(s)))\, ds \\
&\geqslant \min_{1\leqslant i\leqslant n} M\rho_i \|Su\| \\
&> \min_{1\leqslant i\leqslant n} M\rho_i w_3 \\
&\geqslant w_2.
\end{aligned}$$

Hence, we have proved that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.

It now follows from Theorem 2.1 that the system (F) has (at least) three *fixed-sign* solutions $u^1, u^2, u^3 \in \bar{C}(w_4)$ satisfying (2.1). It is easy to see that here (2.1) reduces to (3.15). □

We shall now employ the five-functional fixed point theorem (Theorem 2.2) to give other existence criteria. In applying Theorem 2.2 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.1.

Theorem 3.2. *Let* (C1)–(C5) *hold. Let* $h \in (0, b - t^*)$ *be fixed, and let numbers* τ_j, $1 \leqslant j \leqslant 4$ *satisfying*

$$a \leqslant \tau_1 \leqslant t^* - h \leqslant \tau_2 < \tau_3 \leqslant t^* + h \leqslant \tau_4 \leqslant b$$

be such that the sets A, D, E *exist where*

A is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [t^* - h, t^* + h]$, $1 \leqslant i \leqslant n$ for all $t \in A$,

D is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [\tau_2, \tau_3]$, $1 \leqslant i \leqslant n$ for all $t \in D$,

E is the largest subset of $[a, b]$ of positive measure such that $\phi_i(t) \in [\tau_1, \tau_4]$, $1 \leqslant i \leqslant n$ for all $t \in E$.

Note that $D \subseteq A \subseteq E$. Assume

(C8) *for each $1 \leqslant i \leqslant n$ and each $x \in [\tau_2, \tau_3]$, the function $g(x, s)\mu_i(s) \equiv g^x(s)\mu_i(s)$ is nonzero on a subset of D of positive measure*;
(C9) *the function $g(t^*, s)\nu(s)$ is nonzero on a subset of E of positive measure.*

Suppose that there exist numbers w_i, $2 \leqslant i \leqslant 5$ with

$$0 < w_2 < w_3 < \frac{w_3}{M \min_{1 \leqslant i \leqslant n} \rho_i} \leqslant w_4 \leqslant w_5 \quad \text{and} \quad w_2 > \frac{w_5 d_3}{q}$$

such that the following hold:

(P) $p(u_1, u_2, \ldots, u_n) < \frac{1}{d_2}(w_2 - \frac{w_5 d_3}{q})$ *for* $|u_j| \in [0, w_2]$, $1 \leqslant j \leqslant n$;
(Q) $p(u_1, u_2, \ldots, u_n) \leqslant w_5/q$ *for* $|u_j| \in [0, w_5]$, $1 \leqslant j \leqslant n$;
(R) *for each* $1 \leqslant i \leqslant n$, $p(u_1, u_2, \ldots, u_n) > w_3/d_{1,i}$ *for* $|u_j| \in [w_3, w_4]$, $1 \leqslant j \leqslant n$.

Then, the system (F) *has* (*at least*) *three fixed-sign solutions* $u^1, u^2, u^3 \in \bar{C}(w_5)$ *such that*

$$\begin{gathered} \left|u_i^1(x)\right| < w_2, \quad x \in [\tau_1, \tau_4],\ 1 \leqslant i \leqslant n; \qquad \left|u_i^2(x)\right| > w_3, \quad x \in [\tau_2, \tau_3],\ 1 \leqslant i \leqslant n; \\ \max_{1 \leqslant i \leqslant n} \max_{x \in [\tau_1, \tau_4]} \left|u_i^3(x)\right| > w_2 \quad \text{and} \quad \min_{1 \leqslant i \leqslant n} \min_{x \in [\tau_2, \tau_3]} \left|u_i^3(x)\right| < w_3. \end{gathered} \tag{3.22}$$

Proof. To apply Theorem 2.2, we shall define the following functionals on C:

$$\begin{aligned} \gamma(u) &= \|u\|, \\ \psi(u) &= \min_{1 \leqslant i \leqslant n} \min_{t \in A} \theta_i u_i\big(\phi_i(t)\big), \\ \beta(u) &= \Theta(u) = \max_{1 \leqslant i \leqslant n} \max_{t \in E} \theta_i u_i\big(\phi_i(t)\big), \\ \alpha(u) &= \min_{1 \leqslant i \leqslant n} \min_{t \in D} \theta_i u_i\big(\phi_i(t)\big). \end{aligned} \tag{3.23}$$

First, we shall show that the operator S maps $\bar{P}(\gamma, w_5)$ into $\bar{P}(\gamma, w_5)$. Let $u \in \bar{P}(\gamma, w_5)$. Then, we have $|u_j| \in [0, w_5]$, $1 \leqslant j \leqslant n$. Using (3.9), (C5), (Q) and (3.12), for each $t \in [a, b]$ and $1 \leqslant i \leqslant n$ we find

$$\bigl|S_i u(t)\bigr| \leqslant \int_a^b g(t^*, s)\nu(s) p\bigl(u\bigl(\phi(s)\bigr)\bigr)\,ds \leqslant \int_a^b g(t^*, s)\nu(s)\frac{w_5}{q}\,ds = q\frac{w_5}{q} = w_5.$$

This implies $|S_i u|_0 \leqslant w_5$, $1 \leqslant i \leqslant n$ and so $\gamma(Su) = \|Su\| \leqslant w_5$. From Lemma 3.3, we already have $Su \in C$, thus it follows that $Su \in \bar{P}(\gamma, w_5)$. Hence, we have shown that $S : \bar{P}(\gamma, w_5) \to \bar{P}(\gamma, w_5)$.

Next, to see that condition (a) of Theorem 2.2 is fulfilled, we note that $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \neq \emptyset$ since

$$u(t) = \left(\frac{\theta_1}{2}(w_3 + w_4), \frac{\theta_2}{2}(w_3 + w_4), \ldots, \frac{\theta_n}{2}(w_3 + w_4)\right)$$
$$\in \bigl\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\bigr\}.$$

Let $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, by definition we have $\alpha(u) \geqslant w_3$ and $\Theta(u) \leqslant w_4$ which imply

$$\theta_j u_j\bigl(\phi_j(s)\bigr) = \bigl|u_j\bigl(\phi_j(s)\bigr)\bigr| \in [w_3, w_4], \quad s \in D,\ 1 \leqslant j \leqslant n. \tag{3.24}$$

Noting that $\phi_i(t) \in [\tau_2, \tau_3]$, $1 \leqslant i \leqslant n$ for all $t \in D$, we apply (3.8), (3.24), (C8), (R) and (3.12) to obtain

$$\begin{aligned}
\alpha(Su) &\geqslant \min_{1 \leqslant i \leqslant n} \min_{t \in D} \int_a^b g\bigl(\phi_i(t), s\bigr)\mu_i(s) p\bigl(u\bigl(\phi(s)\bigr)\bigr)\,ds \\
&\geqslant \min_{1 \leqslant i \leqslant n} \min_{t \in D} \int_D g\bigl(\phi_i(t), s\bigr)\mu_i(s) p\bigl(u\bigl(\phi(s)\bigr)\bigr)\,ds \\
&> \min_{1 \leqslant i \leqslant n} \min_{t \in D} \int_D g\bigl(\phi_i(t), s\bigr)\mu_i(s)\frac{w_3}{d_{1,i}}\,ds \\
&= \min_{1 \leqslant i \leqslant n} \frac{d_{1,i}}{d_{1,i}} w_3 \\
&= w_3.
\end{aligned}$$

Hence, $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let w_1 be such that $0 < w_1 < w_2$. Note that

$$u(t) = \left(\frac{\theta_1}{2}(w_1 + w_2), \frac{\theta_2}{2}(w_1 + w_2), \ldots, \frac{\theta_n}{2}(w_1 + w_2)\right)$$
$$\in \bigl\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\bigr\}$$

and so $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \neq \emptyset$. Let $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have $\beta(u) \leqslant w_2$ and $\gamma(u) \leqslant w_5$ which, together with (C5), imply the following for $1 \leqslant j \leqslant n$:

$$\bigl|u_j\bigl(\phi_j(s)\bigr)\bigr| \in [0, w_2], \quad s \in E; \qquad \bigl|u_j\bigl(\phi_j(s)\bigr)\bigr| \in [0, w_5], \quad s \in [a, b]. \tag{3.25}$$

In view of the fact that $\phi_i(t) \in [\tau_1, \tau_4]$, $1 \leqslant i \leqslant n$ for all $t \in E$, together with (3.9), (3.25), (C9), (P), (Q) and (3.12), we find

$$\begin{aligned}
\beta(Su) &\leqslant \int_a^b g(t^*, s)\nu(s)p\big(u\big(\phi(s)\big)\big)\,ds \\
&= \int_E g(t^*, s)\nu(s)p\big(u\big(\phi(s)\big)\big)\,ds + \int_{[a,b]\setminus E} g(t^*, s)\nu(s)p\big(u\big(\phi(s)\big)\big)\,ds \\
&< \int_E g(t^*, s)\nu(s)\frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)ds + \int_{[a,b]\setminus E} g(t^*, s)\nu(s)\frac{w_5}{q}\,ds \\
&= d_2\frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right) + d_3\frac{w_5}{q} \\
&= w_2.
\end{aligned}$$

Therefore, $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$.

Next, we shall show that condition (c) of Theorem 2.2 is met. We observe that, by (3.9), we have for $u \in C$,

$$\Theta(Su) = \max_{1\leqslant i\leqslant n}\max_{t\in E}\theta_i(S_i u)\big(\phi_i(t)\big) \leqslant \int_a^b g(t^*, s)\nu(s)p\big(u\big(\phi(s)\big)\big)\,ds. \tag{3.26}$$

Moreover, using (3.8), the fact that $D \subseteq A$, (C4) and (3.6), we get for $u \in C$,

$$\begin{aligned}
\alpha(Su) &\geqslant \min_{1\leqslant i\leqslant n}\min_{t\in D}\int_a^b g\big(\phi_i(t), s\big)\mu_i(s)p\big(u\big(\phi(s)\big)\big)\,ds \\
&\geqslant \min_{1\leqslant i\leqslant n}\min_{t\in A}\int_a^b g\big(\phi_i(t), s\big)\rho_i\nu(s)p\big(u\big(\phi(s)\big)\big)\,ds \\
&\geqslant \min_{1\leqslant i\leqslant n} M\rho_i\int_a^b g(t^*, s)\nu(s)p\big(u\big(\phi(s)\big)\big)\,ds.
\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27) yields

$$\alpha(Su) \geqslant \min_{1\leqslant i\leqslant n} M\rho_i\Theta(Su), \quad u \in C. \tag{3.28}$$

Let $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$. Then, it follows from (3.28) that

$$\alpha(Su) \geqslant \min_{1\leqslant i\leqslant n} M\rho_i\Theta(Su) > \min_{1\leqslant i\leqslant n} M\rho_i w_4 \geqslant \min_{1\leqslant i\leqslant n} M\rho_i\frac{w_3}{\min_{1\leqslant i\leqslant n} M\rho_i} = w_3. \tag{3.29}$$

Thus, $\alpha(Su) > w_3$ for all $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Let $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$. Then, we have $\beta(u) \leqslant w_2$ and $\gamma(u) \leqslant w_5$ which give (3.25). As in proving condition (b), we get $\beta(Su) < w_2$. Hence, condition (d) of Theorem 2.2 is satisfied.

It now follows from Theorem 2.2 that the system (F) has (at least) three *fixed-sign* solutions $u^1, u^2, u^3 \in \bar{P}(\gamma, w_5) = \bar{C}(w_5)$ satisfying (2.2). Furthermore, (2.2) reduces to (3.22) immediately. $\square$

Remark 3.4. Consider the special case when

$$\tau_1 = a, \qquad \tau_2 = t^* - h, \qquad \tau_3 = t^* + h \quad \text{and} \quad \tau_4 = b. \tag{3.30}$$

Then, the set $D = A$ and $E = [a, b]$ (because of (C5)), and hence we have

$$d_{1,i} = r_i, \quad 1 \leqslant i \leqslant n, \qquad d_2 = q \quad \text{and} \quad d_3 = 0. \tag{3.31}$$

In this case (C8) and (C9) are actually (C7) and (C6), respectively, and it is clear that Theorem 3.2 reduces to Theorem 3.1. Hence, Theorem 3.2 is more general than Theorem 3.1. This also shows that the five-functional fixed point theorem (Theorem 2.2), which is used to obtain Theorem 3.2, generalizes Leggett–Williams' fixed point theorem (Theorem 2.1), which is the main tool for Theorem 3.1.

Leggett–Williams' fixed point theorem is well known in the literature, possibly because of the ease to apply and also it produces easily verifiable criteria, as evidenced by the proof and result of Theorem 3.1. In fact till today many authors, e.g. [2,3,9] are still finding new applications of this theorem. As seen from the proof and result of Theorem 3.2, greater skill is needed to apply five-functional fixed point theorem and the criteria obtained are more difficult to check. Consequently, it is not as popular as Leggett–Williams' fixed point theorem. Still, a number of work, e.g. [8] has made good use of this theorem.

The next result illustrates another application of Theorem 2.2.

Theorem 3.3. *Let* (C1)–(C5) *hold. Let* $h \in (0, b - t^*)$ *be fixed, and let numbers* τ_j, $1 \leqslant j \leqslant 4$ *satisfying*

$$t^* - h \leqslant \tau_1 \leqslant \tau_2 < \tau_3 \leqslant \tau_4 \leqslant t^* + h$$

be such that the sets A, D, E (*defined in Theorem* 3.2) *exist so that* (C8) *and* (C9) *hold. Note that* $D \subseteq E \subseteq A$. *Suppose that there exist numbers* w_i, $1 \leqslant i \leqslant 5$ *with*

$$0 < w_1 \leqslant w_2 \cdot M \min_{1 \leqslant i \leqslant n} \rho_i < w_2 < w_3 < \frac{w_3}{M \min_{1 \leqslant i \leqslant n} \rho_i} \leqslant w_4 \leqslant w_5 \quad \text{and} \quad w_2 > \frac{w_5 d_3}{q}$$

such that (Q) *and* (R) *of Theorem* 3.2 *hold, and*

(P′) $p(u_1, u_2, \ldots, u_n) < \frac{1}{d_2}(w_2 - \frac{w_5 d_3}{q})$ *for* $|u_j| \in [w_1, w_2]$, $1 \leqslant j \leqslant n$.

Then, the system (F) *has* (*at least*) *three fixed-sign solutions* $u^1, u^2, u^3 \in \bar{C}(w_5)$ *satisfying* (3.22).

Proof. To apply Theorem 2.2, we shall define the following functionals on C:

$$\gamma(u) = \|u\|,$$
$$\psi(u) = \min_{1 \leqslant i \leqslant n} \min_{t \in E} \theta_i u_i\big(\phi_i(t)\big),$$
$$\beta(u) = \max_{1 \leqslant i \leqslant n} \max_{t \in E} \theta_i u_i\big(\phi_i(t)\big),$$

$$\alpha(u) = \min_{1\leqslant i\leqslant n} \min_{t\in D} \theta_i u_i\big(\phi_i(t)\big),$$
$$\Theta(u) = \max_{1\leqslant i\leqslant n} \max_{t\in D} \theta_i u_i\big(\phi_i(t)\big). \tag{3.32}$$

Using a similar argument as in the proof of Theorem 3.2, we can show that $S:\bar{P}(\gamma, w_5) \to \bar{P}(\gamma, w_5)$, and condition (a) of Theorem 2.2 is fulfilled.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. As in the proof of Theorem 3.2, we see that $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \neq \emptyset$. Let $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have $\psi(u) \geqslant w_1$, $\beta(u) \leqslant w_2$ and $\gamma(u) \leqslant w_5$ which, in view of (C5), give the following for $1 \leqslant j \leqslant n$:

$$\big|u_j\big(\phi_j(s)\big)\big| \in [w_1, w_2], \quad s \in E,$$
$$\big|u_j\big(\phi_j(s)\big)\big| \in [0, w_5], \quad s \in [a, b]. \tag{3.33}$$

In view of (3.9), (3.33), (C9), (P′), (Q) and (3.12), we find, as in the proof of Theorem 3.2, $\beta(Su) < w_2$. Therefore, condition (b) of Theorem 2.2 is fulfilled.

Next, we shall show that condition (c) of Theorem 2.2 is met. In view of (3.9), we have for $u \in C$,

$$\Theta(Su) = \max_{1\leqslant i\leqslant n} \max_{t\in D} \theta_i (S_i u)\big(\phi_i(t)\big) \leqslant \int_a^b g(t^*, s)\nu(s) p\big(u\big(\phi(s)\big)\big)\, ds. \tag{3.34}$$

Moreover, using (3.8), (C4) and (3.6), we get (3.27) for $u \in C$. Combining (3.27) and (3.34) yields (3.28). The rest then follows as in the proof of Theorem 3.2.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Using (3.9), we see that for $u \in C$,

$$\beta(Su) = \max_{1\leqslant i\leqslant n} \max_{t\in E} \theta_i (S_i u)\big(\phi_i(t)\big) \leqslant \int_a^b g(t^*, s)\nu(s) p\big(u\big(\phi(s)\big)\big)\, ds. \tag{3.35}$$

On the other hand, similar to (3.27) it follows from (3.8), the fact $D \subseteq A$, (C4) and (3.6) that for $u \in C$,

$$\psi(Su) = \min_{1\leqslant i\leqslant n} \min_{t\in E} \theta_i (S_i u)\big(\phi_i(t)\big) \geqslant \min_{1\leqslant i\leqslant n} M\rho_i \int_a^b g(t^*, s)\nu(s) f\big(u(s)\big)\, ds. \tag{3.36}$$

A combination of (3.35) and (3.36) gives

$$\psi(Su) \geqslant \min_{1\leqslant i\leqslant n} M\rho_i \beta(Su), \quad u \in C. \tag{3.37}$$

Let $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$. Then, (3.37) leads to

$$\begin{aligned}
\beta(Su) &\leqslant \frac{1}{\min_{1\leqslant i\leqslant n} M\rho_i} \psi(Su) \\
&< \frac{1}{\min_{1\leqslant i\leqslant n} M\rho_i} w_1 \\
&\leqslant \frac{1}{\min_{1\leqslant i\leqslant n} M\rho_i} w_2 \cdot \min_{1\leqslant i\leqslant n} M\rho_i \\
&= w_2.
\end{aligned}$$

Thus, $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$.

The conclusion is now immediate from Theorem 2.2. □

4. Examples

In this section we shall provide examples to illustrate the usefulness of the results obtained in Section 3.

Example 4.1. Consider the boundary value problem (F) when

$$\begin{aligned} &n=2, \quad a=0, \quad b=1, \quad t^*=0.55, \quad \xi=1, \quad \delta=0.5,\\ &\phi_1(t)=\phi_2(t)=t^2, \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_1(t,u_1,u_2) &= f_2(t,u_1,u_2) = p(u_1,u_2)\\ &= \begin{cases} \dfrac{w_1}{2q}, & (u_1,u_2)\in[0,w_1]\times[0,w_1]\equiv E_1,\\ \dfrac{1}{2}\left(\dfrac{d}{q}+\dfrac{w_2}{\min\{r_1,r_2\}}\right), & (u_1,u_2)\in[w_2,\infty)\times[w_2,\infty)\equiv E_2,\\ \ell(u_1,u_2), & (u_1,u_2)\in\mathbb{R}^2\backslash(E_1\cup E_2), \end{cases} \end{aligned} \tag{4.2}$$

where $\ell(u_1,u_2)$ is continuous in each argument and satisfies

$$\begin{cases} \ell(0,u_2)=\ell(w_1,u_2)=\ell(u_1,0)=\ell(u_1,w_1)=\dfrac{w_1}{2q}, & u_1,u_2\in[0,w_1];\\ \ell(w_2,u_2)=\ell(u_1,w_2)=\dfrac{1}{2}\left(\dfrac{d}{q}+\dfrac{w_2}{\min\{r_1,r_2\}}\right), & u_1,u_2\in[w_2,\infty);\\ 0\leqslant \ell(u_1,u_2)\leqslant \dfrac{1}{2}\left(\dfrac{d}{q}+\dfrac{w_2}{\min\{r_1,r_2\}}\right), & (u_1,u_2)\in\mathbb{R}^2\backslash(E_1\cup E_2); \end{cases} \tag{4.3}$$

and w_i's and d are as in the context of Theorem 3.1 and fulfill

$$0<w_1<w_2<\frac{w_2}{M\min\{\rho_1,\rho_2\}}\leqslant w_3\leqslant d, \quad d>\frac{qw_2}{\min\{r_1,r_2\}}. \tag{4.4}$$

Fix $h=0.1$, $\theta_1=\theta_2=1$ and the functions $\mu_1=\mu_2=\nu\equiv 1$ (this implies $\rho_1=\rho_2=1$). Then, $[t^*-h,t^*+h]=[0.45,0.65]$ and $A=[\sqrt{0.45},\sqrt{0.65}]$, and it is clear that (C1)–(C7) are fulfilled. Moreover, by direct computation we have $M=\frac{117}{121}$, $q=0.1178$, $r_1=r_2=0.03507$. Hence, (4.4) reduces to

$$0<w_1<w_2<\frac{121}{117}w_2\leqslant w_3\leqslant d \quad\text{and}\quad d>3.3590w_2. \tag{4.5}$$

We shall check the conditions of Theorem 3.1. First, condition (P) is obviously satisfied. Next, from (4.4) we have $\frac{w_2}{\min\{r_1,r_2\}}<\frac{d}{q}$, therefore it follows for $(u_1,u_2)\in[0,d]\times[0,d]$,

$$p(u_1,u_2)\leqslant\frac{1}{2}\left(\frac{d}{q}+\frac{w_2}{\min\{r_1,r_2\}}\right)<\frac{1}{2}\left(\frac{d}{q}+\frac{d}{q}\right)=\frac{d}{q}.$$

Hence, condition (Q2) is met. Finally, (R) is satisfied since for $(u_1,u_2)\in[w_2,w_3]\times[w_2,w_3]$, we have

$$p(u_1,u_2)=\frac{1}{2}\left(\frac{d}{q}+\frac{w_2}{\min\{r_1,r_2\}}\right)>\frac{1}{2}\left(\frac{w_2}{\min\{r_1,r_2\}}+\frac{w_2}{\min\{r_1,r_2\}}\right)=\frac{w_2}{\min\{r_1,r_2\}}.$$

By Theorem 3.1, the boundary value problem (F) with (4.1)–(4.3), (4.5) has (at least) three positive solutions $u^1, u^2, u^3 \in C$ such that

$$\begin{aligned} &\|u^1\| < w_1, \qquad u_1^2(t), u_2^2(t) > w_2, \quad t \in [0.45, 0.65], \\ &\|u^3\| > w_1 \quad \text{and} \quad \min\Big\{\min_{t\in[0.45,0.65]} u_1^3(t), \min_{t\in[0.45,0.65]} u_2^3(t)\Big\} < w_2. \end{aligned} \tag{4.6}$$

Example 4.2. Consider the boundary value problem (F) with (4.1) and the nonlinear term

$$\begin{aligned} &f_1(t, u_1, u_2) = f_2(t, u_1, u_2) = p(u_1, u_2) \\ &= \begin{cases} \dfrac{1}{2d_2}\left(w_2 - \dfrac{w_5 d_3}{q}\right), & (u_1, u_2) \in [0, w_2] \times [0, w_2] \equiv E_3, \\ \dfrac{1}{2}\left(\dfrac{w_3}{\min\{r_1, r_2\}} + \dfrac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right), & (u_1, u_2) \in [w_3, \infty) \times [w_3, \infty) \equiv E_4, \\ \ell(u_1, u_2), & (u_1, u_2) \in \mathbb{R}^2 \backslash (E_3 \cup E_4), \end{cases} \end{aligned} \tag{4.7}$$

where $\ell(u_1, u_2)$ is continuous in each argument and satisfies

$$\begin{cases} \ell(0, u_2) = \ell(w_2, u_2) = \ell(u_1, 0) = \ell(u_1, w_2) = \dfrac{1}{2d_2}\left(w_2 - \dfrac{w_5 d_3}{q}\right), & u_1, u_2 \in [0, w_2]; \\ \ell(w_3, u_2) = \ell(u_1, w_3) = \dfrac{1}{2}\left(\dfrac{w_3}{\min\{r_1, r_2\}} + \dfrac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right), & u_1, u_2 \in [w_3, \infty); \\ 0 \leqslant \ell(u_1, u_2) \leqslant \dfrac{1}{2}\left(\dfrac{w_3}{\min\{r_1, r_2\}} + \dfrac{w_3}{\min\{d_{1,1}, d_{1,2}\}}\right), & (u_1, u_2) \in \mathbb{R}^2 \backslash (E_3 \cup E_4); \end{cases} \tag{4.8}$$

and w_i's and d are as in the context of Theorem 3.2 and satisfy

$$0 < w_2 < w_3 < \frac{w_3}{M \min\{\rho_1, \rho_2\}} \leqslant w_4 \leqslant w_5 < \frac{q w_2}{d_3}, \quad w_5 > \frac{q w_3}{\min\{r_1, r_2\}}. \tag{4.9}$$

Fix $h = 0.4$, $\theta_1 = \theta_2 = 1$, the functions $\mu_1 = \mu_2 = \nu \equiv 1$ (this implies $\rho_1 = \rho_2 = 1$),

$$\tau_1 = t^* - h = 0.15, \qquad \tau_2 = 0.2, \qquad \tau_3 = 0.9 \quad \text{and} \quad \tau_4 = t^* + h = 0.95.$$

Then, it is clear that $A = E = [\sqrt{0.15}, \sqrt{0.95}]$ and $D = [\sqrt{0.2}, \sqrt{0.9}]$, and (C1)–(C5), (C8) and (C9) are fulfilled. Moreover, by direct computation we have $M = \frac{57}{121}$, $q = 0.1178$, $r_1 = r_2 = 0.05117$, $d_{1,1} = d_{1,2} = 0.05539$, $d_2 = 0.1086$, $d_3 = 0.009161$. Hence, (4.9) reduces to

$$0 < w_2 < w_3 < \frac{121}{57} w_3 \leqslant w_4 \leqslant w_5 < 12.8589 w_2 \quad \text{and} \quad w_5 > 2.3021 w_3. \tag{4.10}$$

We shall check the conditions of Theorem 3.2. First, condition (P′) is obviously satisfied. Next, since

$$\min\{r_1, r_2\} < \min\{d_{1,1}, d_{1,2}\} < d_2 \quad \text{and} \quad \frac{w_3}{\min\{r_1, r_2\}} < \frac{w_5}{q} \quad (\text{i.e., } w_5 > 2.3021 w_3), \tag{4.11}$$

we find for $(u_1, u_2) \in [0, w_5] \times [0, w_5]$,

$$\begin{aligned}
p(u_1,u_2) &\leqslant \frac{1}{2}\left(\frac{w_3}{\min\{r_1,r_2\}}+\frac{w_3}{\min\{d_{1,1},d_{1,2}\}}\right)\\
&< \frac{1}{2}\left(\frac{w_3}{\min\{r_1,r_2\}}+\frac{w_3}{\min\{r_1,r_2\}}\right)\\
&= \frac{w_3}{\min\{r_1,r_2\}}\\
&< \frac{w_5}{q}.
\end{aligned}$$

Hence, condition (Q) is met. Finally, (R) is satisfied since for $(u_1,u_2)\in[w_3,w_4]\times[w_3,w_4]$, using (4.11) we get

$$\begin{aligned}
p(u_1,u_2) &= \frac{1}{2}\left(\frac{w_3}{\min\{r_1,r_2\}}+\frac{w_3}{\min\{d_{1,1},d_{1,2}\}}\right)\\
&> \frac{1}{2}\left(\frac{w_3}{\min\{d_{1,1},d_{1,2}\}}+\frac{w_3}{\min\{d_{1,1},d_{1,2}\}}\right)\\
&= \frac{w_3}{\min\{d_{1,1},d_{1,2}\}}.
\end{aligned}$$

It follows from Theorem 3.2 that the boundary value problem (F) with (4.1), (4.7), (4.8) and (4.10) has (at least) three positive solutions $u^1,u^2,u^3\in\bar{C}(w_5)$ such that

$$\begin{aligned}
&u_1^1(t),u_2^1(t)<w_2,\quad t\in[0.15,0.95];\qquad u_1^2(t),u_2^2(t)>w_3,\quad t\in[0.2,0.9];\\
&\max_{i=1,2}\max_{t\in[0.15,0.95]}u_i^3(t)>w_2\quad\text{and}\quad \min_{i=1,2}\min_{t\in[0.2,0.9]}u_i^3(t)<w_3.
\end{aligned}\tag{4.12}$$

Remark 4.1. In Example 4.2, we see that for $(u_1,u_2)\in[w_3,w_4]\times[w_3,w_4]$,

$$\begin{aligned}
p(u_1,u_2) &= \frac{1}{2}\left(\frac{w_3}{\min\{r_1,r_2\}}+\frac{w_3}{\min\{d_{1,1},d_{1,2}\}}\right)\\
&< \frac{1}{2}\left(\frac{w_3}{\min\{r_1,r_2\}}+\frac{w_3}{\min\{r_1,r_2\}}\right)\\
&= \frac{w_3}{\min\{r_1,r_2\}}.
\end{aligned}$$

Thus, condition (R) of Theorem 3.1 is *not* satisfied. Example 4.2 illustrates the case when Theorem 3.2 is applicable but not Theorem 3.1. Hence, this example shows that Theorem 3.2 is indeed more general than Theorem 3.1.

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