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PROBABILITY LAW, FLOW FUNCTION, MAXIMUM DISTRIBUTION OF WAVE-GOVERNED RANDOM MOTIONS AND THEIR CONNECTIONS WITH KIRCHOFF'S LAWS

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In this paper we derive the explicit form of the probability law and of the associated flow function of a random motion governed by the telegraph equation. Connections of this law with the transition function of Brownian motion are explored. Lower bounds for the distribution of its maximum are obtained and some particular distributions of its maximum, conditioned by the number of velocity reversals, are presented.

Finally some versions of motion admitting annihilation are proven to be connected with Kirchoff's laws of electrical circuits.

telegraph equation * Bessel functions * distribution of the maximum * Brownian motion * Kirchoff's laws

1. Introduction

The process

$$V(t) = V(0)(-1)^{N(t)} \tag{1}$$

where $N(t)$ is the number of events of an homogeneous Poisson process (with rate λ) during $(0, t)$ is usually referred to in literature as the telegraph process. The process $V(t)$ can be viewed as the velocity at time t of a point P running on the real line and whose speed performs abrupt changes of direction at Poisson times. Clearly $V(0)$ denotes the initial velocity which is either $+c$ and $-c$ with equal probability. Probably the most interesting information concerning (1) is the joint characteristic function which reads (when $c = 1$),

$$E(e^{i\alpha V(t)+i\beta V(s)}) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\, e^{-2\lambda|t-s|}. \tag{2}$$

The related process

$$X(t) = V(0)\int_0^t (-1)^{N(s)}\, ds \tag{3}$$

gives the instantaneous position of the point P.

The result that the probability law of $X(t)$, say $p(x, t; x_0, t_0)$, (or $p(x, t)$ when P starts at $x_0 = 0$, at time $t_0 = 0$), is a solution of

$$c^2 \frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} \tag{4}$$

seems due to Goldstein (1951). This is proven in many papers (for example, Cane, 1975; Orsingher, 1985) and books (Kurtz, 1986).

The proof involves the following probabilities

$$\begin{aligned} f(x, t)\,\mathrm{d}x &= \mathrm{Prob}\{P \text{ is near } x \text{ at time } t \text{ with forward velocity}\}, \\ b(x, t)\,\mathrm{d}x &= \mathrm{Prob}\{P \text{ is near } x \text{ at time } t \text{ with backward velocity}\}, \end{aligned} \tag{5}$$

and also

$$p(x, t) = f(x, t) + b(x, t), \qquad w(x, t) = f(x, t) - b(x, t).$$

In a large ensemble of particles moving according to the above prescriptions, $w(x, t)$ measures, at each time t, the excess of forward moving particles with respect to backward moving ones near point x.

In this paper we obtain the explicit form of $p(x, t)$ and $w(x, t)$ and therefore of probabilities $f(x, t)$ and $b(x, t)$ so that a complete picture of the random motion $X(t)$ is possible.

It seems relevant that all formulas are constructed by means of the function

$$G(x, t) = \begin{cases} \mathrm{e}^{-\lambda t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right), & |x| \leq ct, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

which, in the theory of vibrations, represents the instantaneous form of a string performing damped vibrations initiated at time $t = 0$ by a unit impulse at $x = 0$.

Clearly

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} (\tfrac{1}{2}x)^{2k} \tag{6'}$$

is the Bessel function with imaginary argument of order zero.

In Orsingher (1985) we obtained an expression for the probability density $p(x, t)$ (based on $G(x, t)$) defective in that the normalising factor was time-dependent. This drawback is eliminated here by combining the function (6) with its time derivative. What seems relevant is that the flow function $w(x, t)$ coincides with the space derivative of $G(x, t)$, and thus is itself related to function (6).

Since the distribution of $X(t)$ seems not directly obtainable from (3), we investigated whether moments evaluated on the basis of (3) coincide with those calculated by means of the distributions obtained analytically.

The response is affirmative and we present in detail the calculations concerning the variance.

Since equation (4), when $\lambda \to \infty$ and $c^2/\lambda \to \sigma^2$ becomes the heat equation (as is pointed out in Kac, 1974) we investigated if the probability law $p(x, t)$ of (3) tends to the usual Brownian motion transition function.

We obtained this result in Section 3 and this allows us to say that Brownian motion is a limiting case of the integrated telegraph process.

A large part of the paper is devoted to the analysis of

$$\max_{0 \leq s \leq t} X(s). \tag{7}$$

The most general results obtained, as far as the distribution of (7) is concerned, are the following simple lower bounds (based on specific properties of process (3) and valid for $0 < \beta < ct$),

$$\mathrm{Prob}\left\{\max_{0 \leq s \leq ct} X(s) < \beta \mid V(0) > 0\right\} \geq \mathrm{e}^{-\lambda t/2} - \{\mathrm{e}^{\lambda\beta/(2c)} - \mathrm{e}^{-\lambda\beta/(2c)}\}, \tag{8}$$

$$\mathrm{Prob}\left\{\max_{0 \leq s \leq ct} X(s) < \beta \mid V(0) < 0\right\} \geq \mathrm{e}^{-\lambda t/2}\, \mathrm{e}^{\lambda\beta/(2c)}. \tag{9}$$

Although the explicit law of (7) still escapes us, we are able to present the exact conditional distribution

$$\mathrm{Prob}\left\{\max_{0 \leq s \leq t} X(s) < \beta \mid N(t) = k,\ V(0) \lesseqgtr 0\right\} \tag{10}$$

when $k \leq 5$. This is clearly of interest when λ is sufficiently small.

Furthermore, the results displayed seem to indicate the existence of a rather simple analytical form for (10) which we have not been able to obtain in general because of an excessively large quantity of entangled calculations.

We observe, finally, that the basic motion dealt with in this paper has been generalised in many directions (for one-dimensional generalisations consult Orsingher, 1987; for a two-dimensional version see Orsingher, 1986).

We are able here to present some further generalisations whose probability law is connected with Kirchoff's laws of electrical circuits.

The explicit form of this law is derived from the previously described results. An example of motion with varying velocity is also produced.

2. The explicit laws

The probabilities f and b are solutions of the differential system

$$\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial x} + \lambda(b-f), \qquad \frac{\partial b}{\partial t} = c\frac{\partial b}{\partial x} + \lambda(f-b). \tag{11}$$

This is proven in Cane (1975), Orsingher (1985, 1987) and the derivation of (11) is therefore not repeated here. Furthermore the probability density p and the flow function w are solutions of

$$\frac{\partial p}{\partial t}=-c\frac{\partial w}{\partial x},\qquad \frac{\partial w}{\partial t}=-c\frac{\partial p}{\partial x}-2\lambda w, \tag{12}$$

as the reader can easily realize by adding and subtracting equations (11).

Eliminating w in (12) then swiftly yields equation (4).

In the following theorem we obtain the continuous component of $p(x, t)$ together with the flow function $w(x, t)$.

Theorem 1. *The explicit form of* $p(x, t)$ *is*

$$p(x, t)=\frac{e^{-\lambda t}}{2c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right],\quad \textit{when } |x|<ct, \tag{13}$$

while $w(x, t)$ *is given by*

$$w(x, t)=-\tfrac{1}{2}e^{-\lambda t}\frac{\partial}{\partial x}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right),\quad \textit{when } |x|<ct. \tag{14}$$

Furthermore

$$\text{Prob}\{X(t)=ct\}=\text{Prob}\{X(t)=-ct\}=\tfrac{1}{2}e^{-\lambda t}.$$

Proof. Equation (4) can be converted into

$$c^2\frac{\partial^2 v}{\partial x^2}+\lambda^2 v=\frac{\partial^2 v}{\partial t^2} \tag{15}$$

by means of

$$v(x, t)\,e^{-\lambda t}=p(x, t).$$

When

$$s=\sqrt{c^2t^2-x^2}$$

equation (15) becomes the modified Bessel differential equation

$$c^2\frac{\partial^2 v}{\partial s^2}+\frac{1}{s}\frac{\partial v}{\partial s}-\left(\frac{\lambda}{c}\right)^2 v=0 \tag{16}$$

whose general solution is

$$v(s)=AI_0\left(\frac{\lambda s}{c}\right)+BK_0\left(\frac{\lambda s}{c}\right). \tag{17}$$

We must disregard the Bessel function of the second kind K_0 which tends to infinity as s approaches 0 which clearly contradicts the features of the random motion we are analysing. Returning to the original variables we get for $|x| < ct$:

$$p(x,t) = A\,\mathrm{e}^{-\lambda t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) = AG(x,t). \tag{18}$$

We now observe that

$$\mathrm{e}^{-\lambda t}\int_{-ct}^{+ct} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x = \frac{c}{\lambda}(1-\mathrm{e}^{-2\lambda t}) \tag{19}$$

and thus to obtain

$$\int_{-ct}^{+ct} p(x,t)\,\mathrm{d}x = 1-\mathrm{e}^{-\lambda t}$$

we should select A to be a time-dependent function. In order to avoid this we combine (18) with its time derivative (which is also a solution of (4)) as follows:

$$\begin{aligned} p(x,t) &= A'G(x,t) + B'\frac{\partial}{\partial t}G(x,t) \\ &= \mathrm{e}^{-\lambda t}\left[A' I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + B'\frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]. \end{aligned} \tag{20}$$

Since

$$\begin{aligned} \int_{-ct}^{+ct} \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-c^2}\right)\mathrm{d}x &= \frac{\partial}{\partial t}\int_{-ct}^{+ct} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x - 2c \\ &= \frac{\partial}{\partial t}\left[\frac{c}{\lambda}(\mathrm{e}^{\lambda t}-\mathrm{e}^{-\lambda t})\right] - 2c \quad \text{(by (19))} \\ &= c(\mathrm{e}^{\lambda t}+\mathrm{e}^{-\lambda t}-2), \end{aligned}$$

we obtain from (20) that

$$\int_{-ct}^{+ct} p(x,t)\,\mathrm{d}x = A'\frac{c}{\lambda}(1-\mathrm{e}^{-2\lambda t}) + B'c(1-2\,\mathrm{e}^{-\lambda t}+\mathrm{e}^{-2\lambda t}) = (1-\mathrm{e}^{-\lambda t})$$

when $A' = \lambda/(2c)$ and $B' = 1/(2c)$.

In order to prove (14) we must resort to the differential system (12). On the basis of formula (13), which we have just proven, we obtain

$$\frac{\partial p}{\partial t} = \frac{\mathrm{e}^{-\lambda t}}{2c}\left[-\lambda^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{\partial^2}{\partial t^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]. \tag{21}$$

In order to establish the connection between

$$\begin{aligned} \frac{\partial^2}{\partial t^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) &= -\frac{\lambda c x^2}{\sqrt{(c^2t^2-x^2)^3}} I_0'\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \\ &\quad + \frac{(\lambda ct)^2}{c^2t^2-x^2} I_0''\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \end{aligned} \tag{22}$$

and

$$\frac{\partial}{\partial x^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) = -\frac{\lambda c t^2}{\sqrt{(c^2t^2-x^2)^3}} I_0'\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{(\lambda x)^2}{c^2(c^2t^2-x^2)} I_0''\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right), \tag{23}$$

we need to know that

$$I_0''(x) = I_0(x) - \frac{1}{x} I_0'(x), \tag{24}$$

which the reader can verify directly working on (6′). Therefore

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \\
&= \lambda c\left\{\frac{c^2t^2-x^2-c^2t^2}{\sqrt{(c^2t^2-x^2)^3}} I_0'\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right\} \\
&\quad + \lambda^2\left\{\frac{c^2t^2-x^2+x^2}{c^2t^2-x^2} I_0''\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right\} \\
&= c^2\frac{\partial^2}{\partial x^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{\lambda c}{\sqrt{c^2t^2-x^2}} I_0'\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \\
&\quad + \lambda^2 I_0''\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \quad \text{(by (23))} \\
&= c^2\frac{\partial^2}{\partial x^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{\lambda c}{\sqrt{c^2t^2-x^2}} I_0'\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \\
&\quad + \lambda^2\left\{I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) - \frac{c}{\lambda\sqrt{c^2t^2-x^2}} I_0'\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right\} \quad \text{(by (24))} \\
&= c^2\frac{\partial^2}{\partial x^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \lambda^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right).
\end{aligned} \tag{25}$$

Formula (25) permits us to write $\partial p/\partial t$ as

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \frac{e^{-\lambda t}}{2c}\left[-\lambda^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + \frac{\partial^2}{\partial t^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right] \\
&= \frac{e^{-\lambda t}}{2c}\left[-\lambda^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) + c^2\frac{\partial^2}{\partial x^2} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right. \\
&\qquad \left. + \lambda^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right] \\
&= \frac{\partial}{\partial x}\left[\tfrac{1}{2}c\, e^{-\lambda t}\frac{\partial}{\partial x} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right] = -c\frac{\partial w}{\partial x}.
\end{aligned}$$

This shows that (14) is a solution of the first equation of the differential system (12). To complete the proof we show that $w(x, t)$ satisfies also the second equation of the same system. Neglecting the arguments of I_0 in (13) and (14) we obtain

$$\frac{\partial w}{\partial t}=\tfrac{1}{2}\lambda\, \mathrm{e}^{-\lambda t}\frac{\partial}{\partial x}I_0-\tfrac{1}{2}\,\mathrm{e}^{-\lambda t}\frac{\partial^2}{\partial x\,\partial t}I_0, \tag{26}$$

$$-c\frac{\partial p}{\partial x}-2\lambda w=-c\left[\frac{\mathrm{e}^{-\lambda t}}{2c}\left\{\lambda\frac{\partial}{\partial x}I_0+\frac{\partial^2}{\partial x\,\partial t}I_0\right\}\right]+\frac{2\lambda\,\mathrm{e}^{-\lambda t}}{2}\frac{\partial}{\partial x}I_0, \tag{26'}$$

and by simply comparing the second members of (26) and (26′) we obtain the desired result. □

Remark 1. An alternative form of $p(x, t)$ is

$$p(x,t)=\frac{\lambda}{c}\,\mathrm{e}^{-\lambda t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{1}{2c}\frac{\partial}{\partial t}\left\{\mathrm{e}^{-\lambda t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right\}\quad\text{for } |x|<ct. \tag{27}$$

Loosely speaking, the first term of (27) represents an overestimation of the density which is corrected by its derivative.

The results of Theorem 1 permit us to write down the explicit form of probabilities $f(x, t)$ and $b(x, t)$. In particular

$$f(x,t)=\frac{\mathrm{e}^{-\lambda t}}{4c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)+\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)-c\frac{\partial}{\partial x}I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right]$$

for $|x|<ct$, which is a fairly complex expression for a rather simple probability.

It is interesting to note that the flow function (14) shows that in $(0, ct)$ forward moving particles exceed backward moving ones (the converse happens in $(-ct, 0)$. This accords with the fact that particles diffuse out on the real line as motion develops.

3. On moments of the particle's position

We now prove that the variance of process $X(t)$ obtained by means of (3) and (13) coincide. Evaluating higher order moments involves complicated and clumsy algebra (with both approaches) and we content ourselves with presenting two independent evaluations of second order moments. We also obtain the covariance function of $X(t)$ via formula (3). Our results are contained in Lemma 1 below.

Lemma 1.

$$\operatorname{Var} X(t) = \tfrac{1}{2}c^2\left[\frac{2t}{\lambda} - \frac{(1-\mathrm{e}^{-2\lambda t})}{\lambda^2}\right], \tag{28}$$

$$\operatorname{Cov}\{X(t), X(s)\} = \tfrac{1}{4}c^2\left[\frac{4\min(t,s)}{\lambda} - \frac{(1-\mathrm{e}^{-2\lambda\min(t,s)})(1+\mathrm{e}^{-2\lambda|t-s|})}{\lambda^2}\right]. \tag{28'}$$

Proof. (i) Approach based on (13). We have

$$\begin{aligned}
EX^2(t) &= \int_{-ct}^{+ct} x^2 p(x,t)\,\mathrm{d}x + c^2t^2\,\mathrm{e}^{-\lambda t}\\
&= \frac{\mathrm{e}^{-\lambda t}}{2c}\left[\lambda\int_{-ct}^{+ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x\right.\\
&\qquad\left. + \int_{-ct}^{+ct} x^2 \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x\right] + c^2t^2\,\mathrm{e}^{-\lambda t}\\
&= \frac{\mathrm{e}^{-\lambda t}}{2c}\left[\lambda\int_{-ct}^{+ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x\right.\\
&\qquad\left. + \frac{\partial}{\partial t}\int_{-ct}^{+ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x\right].
\end{aligned}$$

Some lengthy calculations yield

$$\int_{-ct}^{+ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x = c^3\left[\frac{t}{\lambda^2}(\mathrm{e}^{\lambda t}+\mathrm{e}^{-\lambda t}) - \frac{(\mathrm{e}^{\lambda t}-\mathrm{e}^{-\lambda t})}{\lambda^3}\right],$$

$$\frac{\partial}{\partial t}\int_{-ct}^{+ct} x^2 I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\mathrm{d}x = \frac{c^3 t}{\lambda}(\mathrm{e}^{\lambda t}-\mathrm{e}^{-\lambda t}).$$

With this at hand formula (28) emerges.

(ii) Approach based on (3).

$$\begin{aligned}
EX^2(t) &= E\left[V^2(0)\int_0^t\int_0^t (-1)^{N(s)+N(z)}\,\mathrm{d}s\,\mathrm{d}z\right]\\
&= c^2\int_0^t\int_0^t E((-1)^{N(s)+N(z)})\,\mathrm{d}s\,\mathrm{d}z
\end{aligned}$$

(since $V(0)$ and Poisson process are independent).

When $z > s$, we obtain

$$\begin{aligned}
&E((-1)^{N(s)+N(z)})\\
&\quad= E((-1)^{2N(s)+N(z)-N(s)}) = E((-1)^{N(z)-N(s)})\\
&\quad= \operatorname{Prob}\{[N(z)-N(s)] = \text{even}\} - \operatorname{Prob}\{[N(z)-N(s)] = \text{odd}\}\\
&\quad= \mathrm{e}^{-2\lambda(z-s)}.
\end{aligned}$$

Therefore

$$EX^2(t) = c^2 \int_0^t \int_0^t e^{-2\lambda|z-s|}\, ds\, dz = 2c^2 \int_0^t \int_0^s e^{-2\lambda(s-z)}\, ds\, dz,$$

and performing the two-fold integral above result (28) is obtained. Slight modifications then permit us to derive also the covariance (28′). □

Remark 2. We note that as $t \to \infty$, Var $X(t) \sim c^2 t/\lambda$, i.e. it increases linearly as the variance of Brownian motion increases.

When $t \to 0$, Var $X(t) \sim c^2 t^2$ and therefore the variance increases initially more slowly than when motion has attained its limiting development.

Finally we remark that while the telegraph process $V(t)$ has the same covariance function as the Ornstein–Uhlenbeck process, $X(t)$ possesses a covariance obtained combining the covariances of Brownian motion and of the Ornstein–Uhlenbeck process. These connections are explored in Section 4.

4. Connections with Brownian motion

It has been pointed out in Kac (1974) that the wave equation (4) when $\lambda \to \infty$ and $c^2/\lambda \to \sigma^2$ tends to the heat equation.

Letting $\lambda \to \infty$ means that the velocity changes occur continuously, while $c^2/\lambda \to \sigma^2$ implies that also the speed of the moving particle must become infinite. Therefore the limiting behaviour of the integrated telegraph becomes similar to that of Brownian motion.

Our task in this section is to show that the density function (13) becomes, in the limit, the Gaussian transition function of Brownian motion.

Lemma 2. *If* $p(x, t)$ *is as in* (13) *we have*

$$\lim_{\substack{\lambda \to \infty \\ c^2/\lambda \to \sigma^2}} p(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{x^2}{2\sigma^2 t}\right\}. \tag{29}$$

Proof. Our proof is based on the integral form of Bessel function

$$I_0(x) = \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} e^{x \sin \phi}\, d\phi$$

and exploits the asymptotic estimate

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty.$$

Since

$$\frac{\lambda}{c}\sqrt{c^2t^2 - x^2} = \frac{\lambda}{c}\left[ct - \frac{x^2}{2ct} + \cdots\right] \sim \frac{\lambda}{c}\left[ct - \frac{x^2}{2ct}\right]$$

when $\lambda \to \infty$ and $c^2/\lambda \to \sigma^2$ we have

$$I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \sim \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \exp\left\{\frac{\lambda}{c}\left(ct-\frac{x^2}{2ct}\right)\sin\phi\right\} \mathrm{d}\phi = I_0\left(\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right)$$

$$\sim \exp\left\{\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right\} \Big/ \sqrt{2\pi\left(\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right)}.$$

With this at hand we have

$$\frac{\lambda\,\mathrm{e}^{-\lambda t}}{2c} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right) \sim \frac{\lambda\,\mathrm{e}^{-\lambda t}}{2c}\exp\left(\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right) \Big/ \sqrt{2\pi\frac{\lambda}{c}\left(ct-\frac{x^2}{2ct}\right)}$$

$$= \exp\left\{-\frac{\lambda x^2}{2c^2t}\right\} \Big/ \left(2\sqrt{2\pi}\sqrt{\frac{c^2t}{\lambda}-\frac{x^2}{2\lambda t}}\right). \tag{30}$$

Performing the limit in the last member of (30) yields one half of the Gaussian density. We observe now that

$$\frac{\mathrm{e}^{-\lambda t}}{2c}\frac{\partial}{\partial t}\left\{I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\right)\right\} \sim \frac{\mathrm{e}^{-\lambda t}}{2c}\frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right)$$

$$= \frac{\mathrm{e}^{-\lambda t}}{2c} I_0'\left(\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right)\left(\lambda+\frac{\lambda x^2}{2c^2t^2}\right) \tag{30'}$$

and exploiting the formula (see Bowman, 1958, p. 49),

$$\frac{I_0(x)}{I_0'(x)} = 1+\frac{1}{2x}+\cdots,$$

formula (30′) becomes

$$\frac{\mathrm{e}^{-\lambda t}}{2c} I_0\left(\frac{\lambda}{c}\left[ct-\frac{x^2}{2ct}\right]\right)\left(\lambda+\frac{\lambda x^2}{2c^2t^2}\right)$$

$$\sim \left(\exp\left\{-\frac{\lambda x^2}{2c^2t}\right\} \Big/ (2\sqrt{2\pi})\right)\left(1 \Big/ \sqrt{\frac{c^2t}{\lambda}-\frac{x^2}{2\lambda t}}\right)\left(1+\frac{x^2}{2c^2t^2}\right).$$

Carrying out the limit now yields the Gaussian density again and this concludes the proof. □

Remark 3. The reader can easily check that the variance and covariance functions (28) and (28′) converge to the variance and covariance of Brownian motion as $\lambda \to \infty$ and $c^2/\lambda \to \sigma^2$. Furthermore the initial time span where $\operatorname{Var} X(t)$ increases as t^2 increases, disappears (see Remark 2), because in the limit, particles move with infinite velocity.

5. On the maximum of $X(t)$

A fundamental result for the complete analysis of process $X(t)$ is the evaluation of probabilities

$$\text{Prob}\left\{\max_{0\leq s\leq t} X(s)<\beta \mid N(t)=k,\ V(0)>0\right\} \tag{31}$$

and

$$\text{Prob}\left\{\max_{0\leq s\leq t} X(s)<\beta \mid N(t)=k,\ V(0)<0\right\}. \tag{31'}$$

The dependence on the direction of the initial velocity is of fundamental importance and the analysis of the two cases must be carried out separately.

It is well known that if $N(t)=k$, the joint distribution of times $(T_1,\ldots,T_k)$ where the Poisson events occur is given by

$$\text{Prob}\{T_1\in \mathrm{d}T_1,\ldots,T_k\in \mathrm{d}T_k \mid N(t)=k\}=\frac{k!}{t^k}\,\mathrm{d}T_1\cdots \mathrm{d}T_k \tag{32}$$

where $0\leq T_1\leq T_2\leq\cdots\leq T_k\leq t$.

When $V(0)>0$ and $N(t)=k$, the jth displacement recorded has the form

$$\begin{aligned} S_j&=c\sum_{r=1}^{j}(-1)^{r-1}(T_r-T_{r-1})\\ &=c[2T_1-2T_2+\cdots+2(-1)^{j-2}T_{j-1}+(-1)^{j-1}T_j],\quad j=1,2,\ldots,k,\\ S_{k+1}&=c\sum_{r=1}^{k}(-1)^{r-1}(T_r-T_{r-1})+c(-1)^k(t-T_k). \end{aligned}$$

It is fairly obvious that

$$S_{2j}\leq S_{2j-1}$$

which implies that

$$\max(S_1,S_2,\ldots,S_{2k+1})=\max(S_1,S_3,\ldots,S_{2k-1},S_{2k+1}),$$

a fact which somewhat simplifies the problem of evaluating (31). We present in the next theorem the exact distribution of (31) when $k\leq 5$. When $k>5$, principles and techniques remain the same, but the amount of calculation increases dramatically. For the sake of simplicity we assume throughout this section that $c=1$.

Theorem 2. *If*

$$F_k^+(\beta)=\text{Prob}\left\{\max_{0\leq s\leq t} X(s)<\beta \mid V(0)>0,\ N(t)=k\right\}$$

the results of Table 1 *hold* (*when* $0<\beta<t$).

Proof. Clearly the most interesting results emerge from the density form of the maximum, which seem to indicate an underlying general law. We shall derive only the case $k=5$ since the others are simpler. Furthermore, this represents the prototype of reasoning leading (at least in principle) to the general law of (31).

Table 1

Number of velocity reversals	$F_k^+(\beta)$	$\frac{\partial}{\partial\beta} F_k^+(\beta)$
$k=1$	β/t	$1/t$
$k=2$	β/t	$1/t$
$k=3$	$\frac{\beta(3t^2-\beta^2)}{2t^3}$	$\frac{3(t^2-\beta^2)}{2t^3}$
$k=4$	$\frac{\beta(3t^2-\beta^2)}{2t^3}$	$\frac{3(t^2-\beta^2)}{2t^3}$
$k=5$	$\frac{\beta}{t^5}[\frac{5}{8}(t^2-\beta^2)(\beta^2+3t^2)+\beta^4]$	$\frac{3\cdot 5(t^2-\beta^2)^2}{8t^5}$

When $k=5$, six displacements occur (S_r, $r=1,2,\ldots,6$), but only the odd-indexed ones are relevant for the evaluation of (31). Therefore we are led to consider

$$\text{Prob}\{S_1<\beta, S_3<\beta, S_5<\beta\}$$
$$=\text{Prob}\{T_1<\beta, 2T_1-2T_2+T_3<\beta, 2T_1-2T_2+2T_3-2T_4+T_5<\beta\}. \quad (33)$$

Evaluating (33) implies taking into account three sets of time-points, namely:

(i) $T_1<\beta,\quad T_1<T_2<T_1+\frac{1}{2}(t-\beta),\quad T_2<T_3<\beta+2(T_2-T_1),$
$T_3<T_4<T_3-T_2+T_1+\frac{1}{2}(t-\beta),\quad T_4<T_5<\beta+2(T_2-T_1+T_4-T_3),$

(ii) $T_1<\beta,\quad T_1<T_2<T_1+\frac{1}{2}(t-\beta),\quad T_2<T_3<\beta+2(T_2-T_1),$
$t>T_4>T_3-T_2+T_1+\frac{1}{2}(t-\beta),\quad t>T_5>T_4,$

(iii) $T_1<\beta,\quad t>T_2>T_1+\frac{1}{2}(t-\beta),\quad t>T_3>T_2,\ t>T_4>T_3,\quad t>T_5>T_4.$

Set (i) is constructed considering that S_5 is less than β if

$$T_5<\beta+2(T_2-T_1+T_4-T_3)$$

and that the right member of this inequality must not exceed t. Therefore

$$T_4<T_3-T_2+T_1+\tfrac{1}{2}(t-\beta)$$

and since the right member cannot exceed t we obtain the constraint

$$T_3<T_2-T_1+\tfrac{1}{2}(t+\beta).$$

Since $S_3<\beta$, it follows that $T_3<\beta+2(T_2-T_1)$ and for (T_1, T_2) as in (i), $\frac{1}{2}(t+\beta)+T_2-T_1>\beta+2(T_2-T_1)$. This concludes the proof of (i).

For the other sets note that for $T_4>T_3-T_2+T_1+\frac{1}{2}(t-\beta)$, the displacement S_5 never exceeds the level β, no matter where the instant T_5 occurs. Analogously when $T_2>T_1+\frac{1}{2}(t-\beta)$, displacements S_3 and S_5 cannot exceed level β as a quick check shows. This is intuitively due to the fact that if the second event (which stops the first leftward step) occurs too late, the moving particle is so far to the left that it can never reach β before t.

Integrating distribution (32) (when $k=5$) on sets (i), (ii), (iii) we obtain successively:

$$\text{(i)}\quad \frac{5!}{t^5}\left\{\int_0^{\beta} \mathrm{d}T_1 \int_{T_1}^{T_1+(t-\beta)/2} \mathrm{d}T_2 \int_{T_2}^{\beta+2(T_2-T_1)} \mathrm{d}T_3 \right.$$
$$\left. \times \int_{T_3}^{T_3-T_2+T_1+(t-\beta)/2} \mathrm{d}T_4 \int_{T_4}^{\beta+2(T_2-T_1+T_4-T_3)} \mathrm{d}T_5 \right\}$$
$$=\frac{5!}{t^5}\left\{-\frac{\beta}{4!}\left(\frac{t-\beta}{2}\right)^4+\frac{1}{4!}\left[\left(\frac{t-\beta}{2}\right)\left(\frac{t+\beta}{2}\right)^4-\left(\frac{t-\beta}{2}\right)^5\right]\right.$$
$$\left.+\frac{1}{5!}\left[-\left(\frac{t+\beta}{2}\right)^5+\left(\frac{t-\beta}{2}\right)^5+\beta^5\right]\right\},$$

$$\text{(ii)}\quad \frac{5!}{t^5}\left\{\int_0^{\beta} \mathrm{d}T_1 \int_{T_1}^{T_1+(t-\beta)/2} \mathrm{d}T_2 \int_{T_2}^{\beta+2(T_2-T_1)} \mathrm{d}T_3 \int_{T_3-T_2+T_1+(t-\beta)/2}^{t} \mathrm{d}T_4 \int_{T_4}^{t} \mathrm{d}T_5\right\}$$
$$=\frac{5!}{t^5}\left\{-\frac{\beta}{4!}\left(\frac{t-\beta}{2}\right)^4-\frac{1}{4!}\left(\frac{t-\beta}{2}\right)^5+\frac{1}{4!}\left(\frac{t-\beta}{2}\right)\left(\frac{t+\beta}{2}\right)^4\right\},$$

$$\text{(iii)}\quad \frac{5!}{t^5}\left\{\int_0^{\beta} \mathrm{d}T_1 \int_{T_1+(t-\beta)/2}^{t} \mathrm{d}T_2 \int_{T_2}^{t} \mathrm{d}T_3 \int_{T_3}^{t} \mathrm{d}T_4 \int_{T_4}^{t} \mathrm{d}T_5\right\}$$
$$=\frac{5!}{t^5}\left\{\frac{1}{5!}\left[\left(\frac{t+\beta}{2}\right)^5-\left(\frac{t-\beta}{2}\right)^5\right]\right\}.$$

Summing up the above results yields the claimed distribution. Some algebra then suffices to obtain the density function.

The distribution of maximum when $k=4$ is obtained by considering that in this case the relevant displacements are

$$S_1=T_1,\qquad S_3=2T_1-2T_2+T_3,\qquad S_5=2T_1-2T_2+2T_3-2T_4+t.$$

Distribution (32) must be integrated on two sets in this case, i.e.:

(i) $T_1<\beta,\quad T_1<T_2<T_1+\frac{1}{2}(t-\beta),\quad T_2<T_3<\beta+2(T_2-T_1),$

$t>T_4>(T_1-T_2+T_3)+\frac{1}{2}(t-\beta),$

and

(ii) $T_1<\beta,\quad T_1+\frac{1}{2}(t-\beta)<T_2<t,\quad T_2<T_3<t,\quad T_3<T_4<t.$

Some calculations give respectively

$$\frac{4!}{t^4}\left\{\int_0^{\beta} \mathrm{d}T_1 \int_{T_1}^{T_1+(t-\beta)/2} \mathrm{d}T_2 \int_{T_2}^{\beta+2(T_2-T_1)} \mathrm{d}T_3 \int_{T_3-T_2+T_1+(t-\beta)/2}^{t} \mathrm{d}T_4\right\}$$
$$=\frac{t\beta(t^2-\beta^2)}{t^4}$$

and

$$\frac{4!}{t^4}\left\{\int_0^{\beta} \mathrm{d}T_1 \int_{T_1+(t-\beta)/2}^{t} \mathrm{d}T_2 \int_{T_2}^{t} \mathrm{d}T_3 \int_{T_3}^{t} \mathrm{d}T_4\right\} = \frac{t\beta(t^2+\beta^2)}{2t^4}$$

and thus the claimed distribution function quickly emerges. □

Remark 4. The results displayed lead us to conjecture a general form for (31) (when the number of Poisson events is $2k+1$) as

$$\frac{B(k)(t^2-\beta^2)^k}{t^{2k+1}} \quad \text{for } 0 \le \beta \le t,$$

$B(k)$ being the normalising constant. Unfortunately, the method employed necessitates the evaluation of $k+1$, $(2k+1)$-fold integrals. We observe that the distributions obtained are continuous and the densities decrease throughout the whole range of β (when the initial velocity is negative a discontinuity at $\beta = 0$ exists). An interesting fact emerging from Theorem 2 is that the maximum attains large values with a probability which decreases with k. Furthermore, some calculations give the results of Table 2.

Table 2

Number of velocity reversals	$E\{\max_{0<s<t} X(s) \mid V(0)>0, N(t)=k\}$
$k=1$	$2t/2^2$
$k=3$	$3t/2^3$
$k=5$	$5t/2^4$

We finally observe that the distributions of Theorem 2 can be generalised to the case $c \neq 1$ by replacing β (in the distribution function) by β/c.

Theorem 3. *For the continuous component of probability*

$$F_k^-(\beta) = \text{Prob}\left\{\max_{0 \le s \le t} X(s) < \beta \mid N(t)=k,\ V(0)<0\right\}$$

the results of Table 3 *hold.*

Proof. Since the reasoning involved is similar to that of Theorem 2, details are omitted. □

Remark 5. The reader will notice that no explicit law for the continuous part of $F_k^-(\beta)$ is suggested by the results reported. This is probably due to the perturbating influence exerted by the discontinuity at $\beta = 0$ on the continuous part.

Table 3

Number of velocity reversals	$F_k^-(\beta)$	$\frac{\partial}{\partial\beta} F_k^-(\beta)$
$k=1$	$\frac{1}{t}\left(\frac{t+\beta}{2}\right)$	$\frac{1}{2t}$
$k=2$	$\frac{1}{2t^2}(t^2-\beta^2+2t\beta)$	$\frac{t-\beta}{t^2}$
$k=3$	$\frac{1}{(2t)^3}[3(t^2-\beta^2)(t+\beta)$ $+2\beta(\beta^2+3t^2)]$	$\frac{3}{(2t)^3}[t^2-\beta^2$ $+2t(t-\beta)]$
$k=4$	$\frac{1}{2^3t^4}[3(t^2-\beta^2)^2+4t\beta(3t^2-\beta^2)]$	$\frac{4\cdot 3}{2^3t^4}(t-\beta)(t^2-\beta^2)$
$k=5$	$\frac{1}{2^4t^5}[5t(t^2-\beta^2)^2+\beta(5t^2-\beta^2)^2]$	$\frac{5(t^2-\beta^2)}{2^4t^5}(t^2-\beta^2$ $+4t(t-\beta))$

We now present the proof of lower bounds (8) and (9).

Theorem 4. *When* $0<\beta<ct$,

$$\text{Prob}\left\{\max_{0\leq s\leq t} X(s)<\beta \mid V(0)>0\right\} \geq e^{-\lambda t/2}\{e^{\lambda\beta/(2c)}-e^{-\lambda\beta/(2c)}\}, \tag{34}$$

$$\text{Prob}\left\{\max_{0\leq s\leq t} X(s)<\beta \mid V(0)<0\right\} \geq e^{-\lambda t/2}\, e^{\lambda\beta/(2c)}. \tag{35}$$

Proof. When $V(0)>0$, if $cT_1<\beta$ and $T_2\geq T_1+\frac{1}{2}(t-\beta/c)$, all displacements do not exceed level β, no matter what the values of $T_3\leq T_4\leq\cdots\leq T_n\leq t$.

In fact, the general form of displacements $(2\leq j\leq n)$ is

(i) $2c(T_1-T_2+\cdots-T_{2j-2})+cT_{2j-1}$,

(ii) $2c(T_1-T_2+\cdots+T_{2j-1})-cT_{2j}$,

and thus for $T_2\geq T_1+\frac{1}{2}(t-\beta/c)$ we obtain

$$\begin{aligned} &2c(T_1-T_2+\cdots-T_{2j-2})+cT_{2j-1}\\ &\quad\leq 2cT_1-2cT_1-ct+\beta+2cT_3-2cT_4+\cdots-2cT_{2j-2}+cT_{2j-1}\\ &\quad=\beta-2c(T_4-T_3)-\cdots-2(T_{2j-2}-T_{2j-3})-c(t-T_{2j-1})\leq\beta. \end{aligned}$$

The same inequality is true, a fortiori, for displacements of type (ii). Therefore when $N(t)=n$ and $V(0)>0$,

$$\begin{aligned} &\left\{T_1<\frac{\beta}{c},\ t\geq T_2\geq T_1+\tfrac{1}{2}\left(t-\frac{\beta}{c}\right),\ t\geq T_3\geq T_2,\ldots,\ t\geq T_n\geq T_{n-1}\right\}\\ &\quad\subset\left\{\max_{0\leq s\leq t} X(s)<\beta\right\}, \end{aligned}$$

Thus,

$$\begin{aligned}
&\operatorname{Prob}\left\{\max_{0\le s\le t} X(s)<\beta \mid N(t)=n,\ V(0)>0\right\}\\
&\ge \frac{n!}{t^n}\left[\int_0^{\beta/c} \mathrm{d}T_1 \int_{T_1+(t-\beta/c)/2}^{t} \mathrm{d}T_2 \int_{T_2}^{t} \mathrm{d}T_3 \cdots \int_{T_{n-1}}^{t} \mathrm{d}T_n\right]\\
&=\frac{1}{t^n}\left[\left\{\tfrac{1}{2}\left(t+\frac{\beta}{c}\right)\right\}^n-\left\{\tfrac{1}{2}\left(t-\frac{\beta}{c}\right)\right\}^n\right].
\end{aligned}$$

Conditioning out with respect to n, inequality (34) energes. As far as the other result is concerned, it suffices to note that when $V(0)<0$ and $T_1 \ge \frac{1}{2}(t-\beta/c)$ all displacements do not exceed level β, regardless of the instants where the velocity switches occur. Imitating the scheme of the above proof then yields (35). □

Clearly we have also

$$\operatorname{Prob}\left\{\max_{0\le s\le t} X(s)=0 \mid V(0)<0\right\} \ge \mathrm{e}^{-\lambda t/2}$$

because the moving particle P can remain, with positive probability, on the negative axis (when $V(0)<0$) up to time t.

6. Connection with Kirchoff's laws of circuits

The differential system governing variations of voltage $V(x, t)$ and current $I(x, t)$ in a long wire is

$$L\frac{\partial I}{\partial t}+RI+\frac{\partial V}{\partial x}=0, \qquad \frac{\partial I}{\partial x}+K\frac{\partial V}{\partial t}+GV=0. \tag{36}$$

The resemblance between (36) and (12) is striking. The fundamental difference between the two systems is the term RI in the first equation of (36) which has no counterpart in (12).

We present here a changed version of the random motion treated above possessing the peculiarity that its probability law and the related flow function are solutions of a differential system coinciding with (36).

This system can therefore be viewed as a probabilistic counterpart of Kirchoff's laws.

Assume that the particle P (moving forward and backward with Poisson-paced velocity reversals) can be annihilated while moving forward (decay is assumed exponentially distributed with rate μ). We need also the notation

$$\hat{f}(x, t)\,\mathrm{d}x = \operatorname{Prob}\{P \text{ is located at } x \text{ at time } t \text{ with forward velocity}\},$$

$$\hat{b}(x, t)\,\mathrm{d}x = \operatorname{Prob}\{P \text{ is located at } x \text{ at time } t \text{ with backward velocity}\},$$

and

$$\hat{p}=\hat{f}+\hat{b}, \qquad \hat{w}=\hat{f}-\hat{b}.$$

Keeping in mind that P moves with velocity c and can disappear with probability $\mu\Delta t$ while moving forward, we can establish the system

$$\begin{aligned}\hat{f}(x,t+\Delta t)&=(1-\mu\Delta t)(1-\lambda\Delta t)\hat{f}(x-c\Delta t,t)+\lambda\Delta t\hat{b}(x+c\Delta t,t)+o(\Delta t),\\ \hat{b}(x,t+\Delta t)&=(1-\lambda\Delta t)\hat{b}(x+c\Delta t,t)+(1-\mu\Delta t)\lambda\Delta t\hat{f}(x-c\Delta t,t)+o(\Delta t).\end{aligned} \tag{37}$$

Expanding equations (37) and letting Δt and $\Delta x\to 0$ we obtain

$$\frac{\partial\hat{f}}{\partial t}=-c\frac{\partial\hat{f}}{\partial x}+\lambda(\hat{b}-\hat{f})-\mu\hat{f}, \qquad \frac{\partial\hat{b}}{\partial t}=c\frac{\partial\hat{b}}{\partial x}+\lambda(\hat{f}-\hat{b}). \tag{38}$$

System (38) can be recast as

$$\frac{\partial\hat{p}}{\partial t}=-c\frac{\partial\hat{w}}{\partial x}-\tfrac{1}{2}\mu\hat{p}-\tfrac{1}{2}\mu\hat{w}, \qquad \frac{\partial\hat{w}}{\partial t}=-c\frac{\partial\hat{p}}{\partial x}-\tfrac{1}{2}\mu\hat{p}-(2\lambda+\tfrac{1}{2}\mu)\hat{w}. \tag{39}$$

Eliminating the flow function $\hat{w}$ in (39) we obtain the complete telegraph equation

$$\frac{\partial^2\hat{p}}{\partial t^2}=c^2\frac{\partial^2\hat{p}}{\partial x^2}+c\mu\frac{\partial\hat{p}}{\partial x}-(2\lambda+\mu)\frac{\partial p}{\partial t}-\lambda\mu\hat{p}, \tag{40}$$

which clearly reduces to (4) when $\mu=0$.

The reader can easily check that the explicit law of $\hat{p}$ is given by

$$\hat{p}(x,t)=\mathrm{e}^{-\mu t/2-\mu x/(2c)}p(x,t) \tag{41}$$

where p coincides with (13). Result (41) shows that the annihilation assumed implies an asymmetry of the probability law (p and $\hat{p}$ differ more and more as x passes from $-ct$ to $+ct$) and also

$$\int_{-ct}^{+ct}\hat{p}(x,t)\,\mathrm{d}x<1, \quad \text{when } t>0.$$

When annihilation occurs while P moves in both directions its probability law, say $\tilde{p}$, is a solution of

$$\frac{\partial^2\tilde{p}}{\partial t^2}+2(\lambda+\mu)\frac{\partial\tilde{p}}{\partial t}+\mu(2\lambda+\mu)\tilde{p}=c^2\frac{\partial^2\tilde{p}}{\partial x^2} \tag{42}$$

and its explicit form turns out to be

$$\tilde{p}(x,t)=\mathrm{e}^{-\mu t}p(x,t)$$

as simple probabilistic (and also analytic) arguments show. It is of interest to note that the role of resistivity in Kirchoff's equations is here played by the chance of annihilation of the moving particle P. We conclude this section devoted to generalisations by observing that if P moves forward with velocity c_1 and backward with velocity c_2 the differential system (11) governing probabilities f and p must be replaced by

$$\frac{\partial f}{\partial t}=-c_1\frac{\partial f}{\partial x}+\lambda(b-f), \qquad \frac{\partial b}{\partial t}=c_2\frac{\partial b}{\partial x}+\lambda(f-b),$$

and the equation governing probability $p=f+b$ becomes

$$\frac{\partial^2 p}{\partial t^2}+2\lambda\frac{\partial p}{\partial t}=c_1c_2\frac{\partial^2 p}{\partial x^2}+(c_2-c_1)\frac{\partial^2 p}{\partial x\,\partial t}+\lambda(c_2-c_1)\frac{\partial p}{\partial x}. \tag{43}$$

It is of interest to remark that equation (43) can be reduced to the form (4) by the Galilean transformation

$$x'=x-\tfrac{1}{2}(c_1-c_2)t, \qquad t'=t.$$

Some calculations show that in the (x', t') frame the probability law $p(x', t')$ is a solution of

$$(\tfrac{1}{2}(c_1+c_2))^2\frac{\partial^2 p}{\partial x'^2}=\frac{\partial^2 p}{\partial t'^2}+2\lambda\frac{\partial p}{\partial t'}.$$

The random motion with different forward and backward velocities has the same law as the basic motion treated above with velocity $\frac{1}{2}(c_1+c_2)$ plus a drift of intensity $\frac{1}{2}(c_1-c_2)$. Its explicit probability law now emerges easily from (13).

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