# Lattices associated with vector spaces over a finite field 

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#### Abstract

Let $V$ denote the $n$-dimensional row vector space over a finite field $\mathbb{F}_{q}$, and fix a subspace $W$ of dimension $n-d$. Let $\mathscr{L}(n, d)=P \cup\{V\}$, where $P$ is the set of all the subspaces of $V$ intersecting trivially with $W$. Partially ordered by ordinary or reverse inclusion, two families of finite atomic lattices are obtained. This article discusses their geometricity, and computes their characteristic polynomials.


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## 1. Introduction

Let $\mathbb{F}_{q}$ be a finite field with $q$ elements, where $q$ is a prime power. For a positive integer $n$, let $V$ be the $n$-dimensional row vector space over $\mathbb{F}_{q}$. For a fixed $(n-d)$-subspace $W$ of $V$, let $\mathscr{L}(n, d)=P \cup\{V\}$, where

$$
P=\{A \mid A \text { is a subspace of } V, A \cap W=0\} .
$$

Partially ordered by ordinary or reverse inclusion, $\mathscr{L}(n, d)$ is a finite poset, denoted by $\mathscr{L}_{O}(n, d)$ or $\mathscr{L}_{R}(n, d)$, respectively. For any two elements $A, B \in \mathscr{L}_{O}(n, d)$,

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$$
\begin{aligned}
& A \wedge B=A \cap B, \\
& A \vee B= \begin{cases}A+B, & \text { if }(A+B) \cap W=0, \\
V, & \text { otherwise }\end{cases}
\end{aligned}
$$

Similarly, for any two elements $A, B \in \mathscr{L}_{R}(n, d)$,

$$
\begin{aligned}
& A \vee B=A \cap B, \\
& A \wedge B= \begin{cases}A+B, & \text { if }(A+B) \cap W=0 \\
V, & \text { otherwise }\end{cases}
\end{aligned}
$$

Therefore, both $\mathscr{L}_{O}(n, d)$ and $\mathscr{L}_{R}(n, d)$ are finite lattices. This article shows they are atomic, discusses their geometricity, and computes their characteristic polynomials.

The results on the lattices generated by the orbits of subspaces under finite classical groups have been obtained in a series of papers by Huo and Wan [3,4,5,6,7], Gao [2], Wang and Feng [10], Wang and Guo [11,12], Wang and Li [13], and the book of Wan and Huo [9].

## 2. Preliminaries

In this section, we first recall some terminologies and definitions concerning finite posets and lattices, which can be found in [1,9], and then introduce two basic lemmas.

Let $P$ be a poset with partial order $\leqslant$. As usual, we write $a<b$ whenever $a \leqslant b$ and $a \neq b$. For any two elements $a, b \in P$, we say $a$ covers $b$, denoted by $b<\cdot a$, if $b<a$ and there exists no any element $c \in P$ such that $b<c<a$. If $P$ has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1). In this case we say that $P$ is a poset with 0 (resp. 1). Let $P$ be a finite poset with 0 . By a rank function on $P$, we mean a function $r$ from $P$ to the set of all the nonnegative integers such that
(i) $r(0)=0$.
(ii) $r(a)=r(b)+1$ whenever $b<\cdot a$.

Observe the rank function of $P$ is unique if it exists.
Let $P$ be a finite poset with 0 and 1 . The polynomial

$$
\chi(P, x)=\sum_{a \in P} \mu(0, a) x^{r(1)-r(a)}
$$

is called the characteristic polynomial of $P$, where $r$ is the rank function on $P$.
A poset $L$ is said to be a lattice if both $a \vee b:=\sup \{a, b\}$ and $a \wedge b:=\inf \{a, b\}$ exist for any two elements $a, b \in L$. Let $L$ be a finite lattice with 0 . By an atom of $L$, we mean an element of $L$ covering 0 . We say $L$ is atomic if any element of $L \backslash\{0\}$ is a union of some atoms. A finite atomic lattice $L$ is said to be geometric if $L$ admits a rank function $r$ satisfying

$$
r(a \wedge b)+r(a \vee b) \leq r(a)+r(b)
$$

for any two distinct elements $a, b \in L$.

$$
r(a \wedge b)+r(a \vee b) \leqslant r(a)+r(b) \quad \forall a, b \in P
$$

For any two positive integers $n \geqslant m$, let

$$
\left[\begin{array}{c}
n \\
m
\end{array}\right]=\frac{\prod_{i=n-m+1}^{n}\left(q^{i}-1\right)}{\prod_{i=1}^{m}\left(q^{i}-1\right)}
$$

For convenience, we assume that $\left[\begin{array}{l}n \\ i\end{array}\right]=0$ whenever $n<i$ and $\left[\begin{array}{l}n \\ 0\end{array}\right]=1$.

Let $V$ denote the $n$-dimensional row space over a finite field $\mathbb{F}_{q}$. Denote by $G L_{n}\left(\mathbb{F}_{q}\right)$ the set of all the $n \times n$ nonsingular matrices over $\mathbb{F}_{q}$. Then $G L_{n}\left(\mathbb{F}_{q}\right)$ forms a group under matrix multiplication, and acts on $V$ as follows:

$$
\begin{aligned}
& V \times G L_{n}\left(\mathbb{F}_{q}\right) \longrightarrow V \\
& \left(\left(x_{1}, x_{2}, \ldots, x_{n}\right), T\right) \longmapsto\left(x_{1}, x_{2}, \ldots, x_{n}\right) T .
\end{aligned}
$$

If $U$ is an $m$-subspace of $V$ with a basis $u_{1}, u_{2}, \ldots, u_{m}$, the $m \times n$ matrix

$$
\left(\begin{array}{c}
u_{1} \\
u_{2} \\
\vdots \\
u_{m}
\end{array}\right)
$$

is said to be a matrix representation of $U$. We usually denote a matrix representation of the $m$ subspace $U$ still by $U$. The above action induces an action on the set of all the subspaces. The above action is transitive on the set of all the subspaces with the same dimension by [8, Theorem 1.1].

Lemma 2.1. Let $V$ denote the n-dimensional row vector space over a finite field $\mathbb{F}_{q}$, and fix an $(n-d)$-subspace $W$ of $V$. Then the number of $i$-subspaces of $V$ intersecting trivially with $W$ is $\left[\begin{array}{c}d \\ i\end{array}\right] q^{i(n-d)}$.

Proof. By the transitivity of $G L_{n}\left(\mathbb{F}_{q}\right)$ on the set of all the subspaces with the same dimension, we may assume that $W$ has the matrix representation of the form

$$
W=\left(I^{(n-d)} \quad 0^{(n-d, d)}\right) .
$$

If $U$ is an $i$-subspaces of $V$ intersecting trivially with $W$, then $U$ has the matrix representation of the form

$$
\left(\begin{array}{ll}
Y & Z
\end{array}\right),
$$

where $Y$ is an $i \times(n-d)$ matrix and $Z$ is an $i \times d$ matrix of rank $i$. Hence the number of $i$-subspaces of $V$ intersecting trivially with $W$ is $\left[\begin{array}{l}d \\ i\end{array}\right] q^{i(n-d)}$.

Lemma 2.2. Let $V$ denote the $n$-dimensional row vector space over a finite field $\mathbb{F}_{q}$, and fix an $(n-d)$-subspace $W$ of $V$. For a given $l_{1}$-subspace $U_{1}$ of $V$ intersecting trivially with $W$, let $u\left(n, d ; l_{1}, l_{2}\right)$ denote the number of $l_{2}$-subspaces $U_{2}$ of $V$ satisfying $U_{2} \cap W=0$ and $U_{1} \subseteq U_{2}$. Then

$$
u\left(n, d ; l_{1}, l_{2}\right)=\frac{\left[\begin{array}{l}
d \\
l_{2}
\end{array}\right]\left[\begin{array}{l}
l_{2} \\
l_{1}
\end{array}\right] q^{\left(l_{2}-l_{1}\right)(n-d)}}{\left[\begin{array}{l}
d \\
l_{1}
\end{array}\right]}
$$

Proof. Since the subgroup $G L_{n}\left(\mathbb{F}_{q}\right)_{W}$ of $G L_{n}\left(\mathbb{F}_{q}\right)$ fixing $W$ acts transitively on the set $\{U \mid$ $\left.U \cap W=0, \operatorname{dim} U=l_{1}\right\}$, the number $u\left(n, d ; l_{1}, l_{2}\right)$ depend only on $l_{1}$ and $l_{2}$.

Let

$$
M=\left\{\left(V_{1}, V_{2}\right) \mid V_{1} \subseteq V_{2}, V_{i} \cap W=0, \operatorname{dim} V_{i}=l_{i}\right\}
$$

Counting the set $M$ in two ways, by Lemma 2.1 we obtain

$$
u\left(n, d ; l_{1}, l_{2}\right) \cdot\left[\begin{array}{l}
d \\
l_{1}
\end{array}\right] q^{l_{1}(n-d)}=\left[\begin{array}{c}
d \\
l_{2}
\end{array}\right] q^{l_{2}(n-d)} \cdot\left[\begin{array}{l}
l_{2} \\
l_{1}
\end{array}\right] .
$$

Hence the desired result follows.

## 3. The lattice $\mathscr{L}_{O}(n, d)$

The lattice $\mathscr{L}_{O}(n, d)$ has the minimum element 0 -subspace, and the maximum element $V$. Since the set of all the atoms of $\mathscr{L}_{O}(n, d)$ consists of all the 1 -subspaces intersecting trivially with $W, \mathscr{L}_{O}(n, d)$ is a finite atomic lattice.

Theorem 3.1. $\mathscr{L}_{O}(n, d)$ is a geometric lattice if and only if $d=1$.
Proof. For any $A \in \mathscr{L}_{O}(n, d)$, define

$$
r(A)= \begin{cases}d+1 & \text { if } A=V \\ \operatorname{dim} A & \text { otherwise }\end{cases}
$$

Then $r$ is the rank function of $\mathscr{L}_{O}(n, d)$.
If $d=1$, it is clear that $\mathscr{L}_{O}(n, d)$ is a geometric lattice. Now suppose that $d \geqslant 2$. By the transitivity of $G L_{n}\left(\mathbb{F}_{q}\right)$ on the set of all the subspaces with the same dimension, we may assume that $W$ has the matrix representation of the form

$$
W=\left(I^{(n-d)} \quad 0^{(n-d, d)}\right) .
$$

Let $A$ and $B$ be two 1 -subspaces of $V$ with matrix representation

$$
A=(\underbrace{0,0, \ldots, 0}_{n-d}, 1,0, \ldots, 0) \text { and } B=(1, \underbrace{0, \ldots, 0}_{n-d-1}, 1,0, \ldots, 0) \text {, }
$$

respectively. Then $A, B \in \mathscr{L}_{O}(n, d)$ and $A \vee B=V$. It follows that

$$
r(A \wedge B)+r(A \vee B)=d+1>2=r(A)+r(B)
$$

Hence $\mathscr{L}_{O}(n, d)$ is not a geometric lattice whenever $d \geqslant 2$.
Proposition 3.2 [9, Proposition 1.9]. Let $n$ be a nonnegative integer, and $q \neq 1$. Then

$$
\prod_{i=0}^{n-1}\left(1+q^{i} x\right)=\sum_{m=0}^{n} q^{\binom{m}{2}}\left[\begin{array}{c}
n \\
m
\end{array}\right] x^{m}
$$

Lemma 3.3. The Möbius function of $\mathscr{L}_{O}(n, d)$ is

$$
\mu(A, B)= \begin{cases}(-1)^{\operatorname{dim} B-\operatorname{dim} A} q\left(\begin{array}{ll}
(\operatorname{dim} B-\operatorname{dim} A
\end{array}\right) & \text { if } A \leqslant B \neq V \text { or } A=B=V, \\
-\prod_{i=0}^{d-1}\left(1-q^{i+n-d}\right) & \text { if } 0=A<B=V, \\
d-\operatorname{dim} A \\
\sum_{i=0}^{d}(-1)^{i+1} u(n, d ; i, \operatorname{dim} A+i) q^{\binom{i}{2}} & \text { if } 0 \neq A<B=V, \\
0 & \text { otherwise. }\end{cases}
$$

Proof. The Möbius function of $\mathscr{L}_{O}(n, d)$ is

$$
\mu(A, B)= \begin{cases}\left.(-1)^{\operatorname{dim} B-\operatorname{dim} A} q{ }^{(\operatorname{dim} B-\operatorname{dim} A}\right) & \text { if } A \leqslant B \neq V \text { or } A=B=V \\ -\sum_{A \leqslant C<B} \mu(A, C) & \text { if } A<B=V \\ 0 & \text { otherwise } .\end{cases}
$$

By Lemma 2.1 and Proposition 3.2 we have

$$
\begin{aligned}
-\sum_{0 \leqslant C<V} \mu(0, C)=\sum_{0 \leqslant C<V}(-1)^{\operatorname{dim} C} q^{\binom{\operatorname{dim} C}{2}} & =-\sum_{m=0}^{d} q^{\binom{m}{2}}\left[\begin{array}{c}
d \\
m
\end{array}\right]\left(-q^{n-d}\right)^{m} \\
& =-\prod_{i=0}^{d-1}\left(1-q^{i+n-d}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
-\sum_{0 \neq A \leqslant C<V} \mu(A, C) & =-\sum_{\substack{0 \neq A \leqslant C<V}}(-1)^{\operatorname{dim} C-\operatorname{dim} A} q\left(\frac{\operatorname{dim} C-\operatorname{dim} A}{2}\right) \\
& =\sum_{i=0}^{d-\operatorname{dim} A}(-1)^{i+1} u(n, d ; \operatorname{dim} A, \operatorname{dim} A+i) q^{\binom{i}{2}} .
\end{aligned}
$$

Hence the desired result follows.
Theorem 3.4. The characteristic polynomial of $\mathscr{L}_{O}(n, d)$ is

$$
\chi\left(\mathscr{L}_{O}(n, d), x\right)=\sum_{i=0}^{d}\left[\begin{array}{l}
d \\
i
\end{array}\right]\left(-q^{n-d}\right)^{i} q^{\binom{i}{2}} x^{d+1-i}-\prod_{i=0}^{d-1}\left(1-q^{i+n-d}\right)
$$

Proof. By Lemma 3.3 we obtain

$$
\begin{aligned}
\chi\left(\mathscr{L}_{O}(n, d), x\right) & =\sum_{0 \leqslant B \leqslant V} \mu(0, B) x^{r(V)-r(B)} \\
& =\mu(0, V)+\sum_{0 \leqslant B<V}(-1)^{\operatorname{dim} B} q^{\binom{\operatorname{dim} B}{2}} x^{d+1-\operatorname{dim} B} \\
& =\sum_{i=0}^{d}\left[\begin{array}{l}
d \\
i
\end{array}\right]\left(-q^{n-d}\right)^{i} q^{\binom{i}{2}} x^{d+1-i}-\prod_{i=0}^{d-1}\left(1-q^{i+n-d}\right),
\end{aligned}
$$

as desired.

## 4. The lattice $\mathscr{L}_{R}(n, d)$

The lattice $\mathscr{L}_{R}(n, d)$ has the minimum element $V$, and the maximum element 0 -subspace. The set of all the atoms of $\mathscr{L}_{R}(n, d)$ consists of all the $d$-subspaces intersecting trivially with $W$.

## Theorem 4.1

(i) $\mathscr{L}_{R}(n, d)$ is an atomic lattice.
(ii) $\mathscr{L}_{R}(n, d)$ is a geometric lattice if and only if $d=1$ or $n-d=1$.

Proof. (i) By the transitivity of $G L_{n}\left(\mathbb{F}_{q}\right)$ on the set of all the subspaces with the same dimension, we may assume that $W$ has the matrix representation of the form

$$
W=\left(I^{(n-d)} \quad 0^{(n-d, d)}\right)
$$

For any element $A \in \mathscr{L}_{R}(n, d)$ with $\operatorname{dim} A=l$, by the transitivity of $G L_{n}\left(\mathbb{F}_{q}\right)_{W}$ on the set $\{U \mid U \cap W=0$ and $\operatorname{dim} U=l\}$, we may assume that $A$ has the matrix representation of the form

$$
A=\left(0^{(l, n-d)} \quad I^{(l)} \quad 0^{(l, d-l)}\right) .
$$

Let $B$ and $C_{1}, C_{2}, \ldots, C_{d-l}$ be the subspaces of $V$ with the following matrix representations

$$
B=\left(\begin{array}{ccc}
0^{(l, n-d)} & I^{(l)} & 0^{(l, d-l)} \\
0^{(d-l, n-d)} & 0^{(d-l, l)} & I^{(d-l)}
\end{array}\right) \quad \text { and } \quad C_{i}=\left(\begin{array}{ccc}
0^{(l, n-d)} & I^{(l)} & 0^{(l, d-l)} \\
L_{i}^{(d-l, n-d)} & 0^{(d-l, l)} & I^{(d-l)}
\end{array}\right)
$$

where $1 \leqslant i \leqslant d-l$ and $L_{i}$ is the $(d-l) \times(n-d)$ matrix satisfying

$$
\left(L_{i}\right)_{s, t}= \begin{cases}1 & \text { if } s=i \text { and } t=1 \\ 0 & \text { otherwise }\end{cases}
$$

Then $B$ and $C_{1}, C_{2}, \ldots, C_{d-l}$ are atoms of $\mathscr{L}_{R}(n, d)$ satisfying $B \vee C_{1} \vee \cdots \vee C_{d-l}=A$. Therefore, $\mathscr{L}_{R}(n, d)$ is an atomic lattice.
(ii) For any element $A \in \mathscr{L}_{R}(n, d)$, define

$$
r(A)= \begin{cases}0 & \text { if } A=V \\ d+1-\operatorname{dim} A & \text { otherwise }\end{cases}
$$

Then $r$ is the rank function of $\mathscr{L}_{R}(n, d)$.
If $d=1$, it is clear that $\mathscr{L}_{R}(n, d)$ is a geometric lattice. Now suppose that $d \geqslant 2$.
Case 1. $n-d=1$. In order to prove that $\mathscr{L}_{R}(n, d)$ is a geometric lattice, it suffices to show the following inequality:

$$
\begin{equation*}
r(B \vee C)+r(B \wedge C) \leqslant r(B)+r(C) \quad \forall B, C \in \mathscr{L}_{R}(n, d) \tag{1}
\end{equation*}
$$

If $B \wedge C \neq V$, it is well known that (1) holds. If $B \wedge C=V$, then

$$
\begin{aligned}
r(B)+r(C)-r(B \vee C)-r(B \wedge C) & =d+1-\operatorname{dim}(B)-\operatorname{dim}(C)+\operatorname{dim}(B \cap C) \\
& =d+1-\operatorname{dim}(B+C) \\
& \geqslant 0 .
\end{aligned}
$$

Therefore (1) holds.
Case 2. $n-d \geqslant 2$. Let $B$ and $C$ be two subspaces of $V$ with the following matrix representations of the form:

$$
B=\left(\begin{array}{ll}
0^{(d, n-d)} & I^{(d)}
\end{array}\right) \quad \text { and } \quad C=\left(\begin{array}{cccc}
I^{(2)} & 0^{(2, n-d-2)} & I^{(2)} & 0^{(2, d-2)} \\
0^{(d-2,2)} & 0^{(d-2, n-d-2)} & 0^{(d-2,2)} & I^{(d-2)}
\end{array}\right),
$$

respectively. Then $B$ and $C$ are the elements of $\mathscr{L}_{R}(n, d)$ satisfying

$$
r(B \vee C)+r(B \wedge C)=3>2=r(B)+r(C)
$$

It follows that (1) does not hold. Therefore, $\mathscr{L}_{R}(n, d)$ is not a geometric lattice in this case.
Lemma 4.2. The Möbius function of $\mathscr{L}_{R}(n, d)$ is

$$
\mu(A, B)= \begin{cases}\left.(-1)^{\operatorname{dim} A-\operatorname{dim} B} q^{(\operatorname{dim} A-\operatorname{dim} B}\right) & \text { if } V \neq A \leqslant B \text { or } A=B=V \\ \sum_{i=0}^{d-\operatorname{dim} B}(-1)^{i+1} u(n, d ; \operatorname{dim} B, \operatorname{dim} B+i) q^{\binom{i}{2}} & \text { if } V=A<B \\ 0 & \text { otherwise. }\end{cases}
$$

Proof. The Möbius function of $\mathscr{L}_{R}(n, d)$ is

$$
\mu(A, B)= \begin{cases}(-1)^{\operatorname{dim} A-\operatorname{dim} B} q\left(\frac{\operatorname{dim} A-\operatorname{dim} B}{2}\right) & \text { if } V \neq A \leqslant B \text { or } A=B=V \\ -\sum_{A<C \leqslant B} \mu(C, B) & \text { if } V=A<B \\ 0 & \text { otherwise }\end{cases}
$$

By Lemma 2.2, we have

$$
\left.-\sum_{V<C \leqslant B}(-1)^{\operatorname{dim} C-\operatorname{dim} B} q^{(\operatorname{dim} C-\operatorname{dim} B}\right)=\sum_{i=0}^{d-\operatorname{dim} B}(-1)^{i+1} u(n, d ; \operatorname{dim} B, \operatorname{dim} B+i) q^{\binom{i}{2}}
$$

Hence the desired result follows.
Theorem 4.3. The characteristic polynomial of $\mathscr{L}_{R}(n, d)$ is

$$
\chi\left(\mathscr{L}_{R}(n, d), x\right)=x^{d+1}+\sum_{j=0}^{d}\left[\begin{array}{l}
d \\
j
\end{array}\right] q^{j(n-d)} \sum_{i=0}^{d-j}(-1)^{i+1} u(n, d ; j, i+j) q^{\binom{i}{2}} x^{j} .
$$

Proof. By Lemma 2.1 and Lemma 4.2 we obtain

$$
\begin{aligned}
\chi\left(\mathscr{L}_{R}(n, d), x\right) & =\sum_{V \leqslant B \leqslant 0} \mu(V, B) x^{r(0)-r(B)} \\
& =\mu(V, V) x^{d+1}+\sum_{V<B \leqslant 0} \mu(V, B) x^{\operatorname{dim} B} \\
& =x^{d+1}+\sum_{j=0}^{d}\left[\begin{array}{l}
d \\
j
\end{array}\right]^{j(n-d)} \sum_{i=0}^{d-j}(-1)^{i+1} u(n, d ; j, i+j) q^{\binom{i}{2}} x^{j},
\end{aligned}
$$

as desired.

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