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q-difference operators for orthogonal polynomials

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ABSTRACT

In this work we apply a q-ladder operator approach to orthogonal polynomials arising from a class of indeterminate moment problems. We derive general representation of first and second order q-difference operators and we study the solution basis of the corresponding second order q-difference equations and its properties. The results are applied to the Stieltjes–Wigert and the q-Laguerre polynomials.

1. Introduction

In the nineteenth century it was realized that the orthogonal polynomials of Hermite, Jacobi, and Laguerre satisfy differential equations of the Sturm–Liouville type. It was later realized that the differential equation can be written in the form $A^*Ay = \lambda y$, where A is a linear first order differential operator and the A^* is its adjoint on a weighted L_2 space. The action of the operator A decreases the degree of a polynomial by one while A^* increases the degree by one. In the twentieth century it was realized that the orthogonal polynomials are matrix elements of irreducible representations of certain groups and that A^* and A move you up and down the irreducible representations. It is also known that only the Hermite, Laguerre and Jacobi polynomials satisfy the symmetric Sturm–Liouville type eigenvalue problem.

In the 1990s, the works [1–4] derived raising and lowering operators for polynomials orthogonal with respect to absolutely continuous measures μ under certain smoothness assumptions on μ'. Then, they showed that the polynomials satisfy $Ty = 0$ where T is a linear second order differential operator. Chen and Ismail [1] showed that T factors as $A^*(1/A_n(x))A$, for a certain function $A_n(x)$. Here A and its adjoint A^* are linear first order differential operators. It was later realized that a similar theory exists for polynomials orthogonal with respect to a measure with masses at the union of at most two geometric progressions, $\{aq^n, bq^n\}$, for some $q \in (0, 1)$, [5]. The corresponding theory for difference operators is in [6]. This theory is included in [7]. The raising and lowering operators involve two functions $A_n(x)$ and $B_n(x)$ which satisfy certain recurrence relations. In the case of differential operators, Ismail and Chen have demonstrated that the knowledge of $A_n(x)$ and $B_n(x)$ determines the polynomials uniquely in the cases of Hermite, Laguerre, and Jacobi polynomials, see [8]. This is done through recovering the properties of the polynomials including the three term recurrence relation which generates the polynomials. In [9], Chen and Ismail showed that orthogonal polynomials which arise from indeterminate moment problems have similar properties but the coefficients $A_n(x)$ and $B_n(x)$ now have integral representations instead of series representations. By composing the lowering and raising operators one can produce a second order equation satisfied by the orthogonal polynomials.

This work started from the realization that the second order equations derived using the above-mentioned technique do not reduce to the Sturm–Liouville type difference and q-difference equations for the classical polynomials. The purpose of

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this work is to produce another pair of raising and lowering operators and with the appropriate combinations to recover the classical results.

The orthogonal polynomials which arise from indeterminate moment problems have discrete and absolutely continuous orthogonality measures [10]. In many instances it is more convenient to work with absolutely continuous measures [7, Chapter 21]. In Section 2 we derive two pairs of raising and lowering operators for q-polynomials orthogonal with respect to absolutely continuous measures. In Section 3 we use compositions of these operators to construct two second order q-difference equations satisfied by the orthogonal polynomials and we find their solution bases.

We apply our results to the Stieltjes–Wigert polynomials in Section 4 and to the q-Laguerre polynomials in Section 5.

We shall assume that $\{p_n(x)\}$ are monic orthogonal polynomials, so that

$$\int_0^\infty p_m(x)p_n(x)w(x)\mathrm{d}x = \zeta_n\delta_{m,n} \tag{1.1}$$

where w is a positive weight defined on $(0,\infty)$ and such that

$$\int_0^\infty x^n w(x)\mathrm{d}x < \infty \quad \text{for all } n \geq -1.$$

A weight function w leads to a potential u defined by

$$u(x) = -\frac{D_{q^{-1}}w(x)}{w(x)}, \tag{1.2}$$

where D_q is the q-difference operator

$$D_q f(x) = \frac{f(qx)-f(x)}{qx-x}. \tag{1.3}$$

Every monic sequence of orthogonal polynomials satisfies a three term recurrence relation of the form

$$(x-\alpha_n)p_n(x) = p_{n+1}(x) + \beta_n p_{n-1}(x) \tag{1.4}$$

with $p_{-1} := 0$. A main result of [9] is that

$$D_q p_n(x) = \beta_n A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \tag{1.5}$$

holds with

$$A_n(x) := \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}p_n(y)p_n(y/q)w(y)\mathrm{d}y, \tag{1.6}$$

$$B_n(x) := \frac{1}{\zeta_{n-1}}\int_0^\infty \frac{u(qx)-u(y)}{qx-y}p_n(y)p_{n-1}(y/q)w(y)\mathrm{d}y. \tag{1.7}$$

Moreover Chen and Ismail [9] also established the supplementary relations

$$B_{n+1}(x) + B_n(x) = (x-\alpha_n)A_n(x) + x(q-1)\sum_{j=0}^n A_j(x) - u(qx), \tag{1.8}$$

$$1 + (x-\alpha_n)B_{n+1}(x) - (qx-\alpha_n)B_n(x) = \beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x). \tag{1.9}$$

In Section 2 we prove the following companion results:

Theorem 1.1. *With*

$$\frac{D_q w(x)}{w(x)} = -v(qx), \tag{1.10}$$

and

$$C_n(x) := \frac{q}{\zeta_n}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}p_n(y)p_n(qy)w(y)\mathrm{d}y, \tag{1.11}$$

$$D_n(x) := \frac{q}{\zeta_{n-1}}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}p_n(y)p_{n-1}(qy)w(y)\mathrm{d}y, \tag{1.12}$$

we have

$$D_{q^{-1}}p_n(x) = \beta_n C_n(x)p_{n-1}(x) - D_n(x)p_n(x). \tag{1.13}$$

Moreover the functions $C_n(x)$ and $D_n(x)$ satisfy the recursions

$$D_{n+1}(x) + D_n(x) = (x - \alpha_n)C_n(x) + x(1/q - 1)\sum_{j=0}^{n} C_j(x) - v(x), \tag{1.14}$$

$$(x - \alpha_n)D_{n+1}(x) - (x/q - \alpha_n)D_n(x) = -1 + \beta_{n+1}C_{n+1}(x) - \beta_n C_{n-1}(x). \tag{1.15}$$

Note that

$$v(x) = \frac{u(x)}{1 + (1 - 1/q)xu(x)}. \tag{1.16}$$

Theorem 1.1 will be proved in Section 2.

In Section 3 we provide two degree raising operators and their adjoint (degree lowering) operators with respect to the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)\,\overline{g(x)}\,w(x)\mathrm{d}x. \tag{1.17}$$

The compositions of each raising operator with the adjoint of the other raising operator give a second order q-difference equation for the polynomials p_n, the other solution of which is a certain function of the second kind.

We shall use the q-analogue of the product rule

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x). \tag{1.18}$$

We shall also use the following property: For every polynomial $s(x)$ of degree at most n,

$$\int_0^\infty \frac{s(x)p_n(t)}{x-t}w(t)\mathrm{d}t = \int_0^\infty \left(\frac{s(x)-s(t)}{x-t}\right)p_n(t)w(t)\mathrm{d}t + \int_0^\infty \frac{s(t)p_n(t)}{x-t}w(t)\mathrm{d}t = \int_0^\infty \frac{s(t)p_n(t)}{x-t}w(t)\mathrm{d}t \tag{1.19}$$

by the orthogonality relation (1.1) and the fact that $(s(x) - s(t))/(s - t)$ is a polynomial of degree less than n.

The following lemma, whose proof is a calculus exercise, will be used in the proofs of our main results.

Lemma 1.2. *If the integrals*

$$\int_0^\infty f(x)g(x)\frac{\mathrm{d}x}{x} \quad \textit{and} \quad \int_0^\infty f(x)g(qx)\frac{\mathrm{d}x}{x}$$

exist, then the following q-analogue of integration by parts holds

$$\int_0^\infty f(x)D_q g(x)\mathrm{d}x = -\frac{1}{q}\int_0^\infty g(x)D_{q^{-1}}f(x)\mathrm{d}x. \tag{1.20}$$

Immediate consequences of Lemma 1.2, (1.2), (1.1), (1.18) and (1.10) are the following relations:

$$\int_0^\infty u(y)p_n(y)p_n(y/q)w(y)\mathrm{d}y = 0, \tag{1.21}$$

$$\int_0^\infty u(y)p_{n+1}(y)p_n(y/q)w(y)\mathrm{d}y = \frac{(1-q^{n+1})q}{1-q}\zeta_n, \tag{1.22}$$

$$\int_0^\infty v(qy)p_n(y)p_n(qy)w(y)\mathrm{d}y = 0, \tag{1.23}$$

$$\int_0^\infty v(qy)p_{n+1}(y)p_n(qy)w(y)\mathrm{d}y = \frac{1-q^{-n-1}}{q-1}\zeta_n. \tag{1.24}$$

2. Proof of Theorem 1.1

We shall need the formula [7]

$$\zeta_n = \zeta_0\beta_1\beta_2\cdots\beta_n, \tag{2.1}$$

and the Christoffel–Darboux identity [11, Theorem 3.2.2], [7]

$$\sum_{k=0}^{n-1} p_k(x)p_k(y)/\zeta_k = \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{\zeta_{n-1}(x-y)}. \tag{2.2}$$

Proof of Theorem 1.1. Let $D_{q^{-1}}p_n(x) = \sum_{k=0}^{n-1} c_{n,k}p_k(x)$. Then

$$\zeta_k c_{n,k} = \int_0^\infty p_k(y)w(y)D_{q^{-1}}p_n(y)\mathrm{d}y.$$

Applying (1.18) and Lemma 1.2, (1.20), we see that

$$\begin{aligned}\zeta_k c_{n,k} &= -q\int_0^\infty p_n(y)\left[w(y)D_q p_k(y) + p_k(qy)D_q w(y)\right]\mathrm{d}y\\ &= q\int_0^\infty p_n(y)p_k(qy)\left(-\frac{D_q w(y)}{w(y)}\right)w(y)\mathrm{d}y\end{aligned}$$

where the orthogonality was used in the last step. The definition (1.10) of v yields

$$\begin{aligned}\zeta_k c_{n,k} &= q\int_0^\infty p_n(y)p_k(qy)v(qy)w(y)\mathrm{d}y\\ &= -q\int_0^\infty p_n(y)p_k(qy)\,(v(x)-v(qy))\,w(y)\mathrm{d}y,\end{aligned}$$

where we again used the orthogonality property in the last step. Therefore, using the Christoffel–Darboux identity (2.2) we obtain

$$D_{q^{-1}}p_n(x) = -\frac{q}{\zeta_{n-1}}\int_0^\infty p_n(y)\frac{v(x)-v(qy)}{x-qy}\left[p_n(x)p_{n-1}(qy) - p_n(qy)p_{n-1}(x)\right]w(y)\mathrm{d}y$$

and (1.13) now follows from (2.1). Next we prove (1.14). It is clear that

$$\begin{aligned}D_{n+1}(x) + D_n(x) &= \frac{q}{\zeta_n}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}\left[p_{n+1}(y)p_n(qy) + \beta_n p_n(y)p_{n-1}(qy)\right]w(y)\mathrm{d}y\\ &= I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \frac{q}{\zeta_n}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}\,(qy-\alpha_n)p_n(y)\,p_n(qy)w(y)\mathrm{d}y\\ I_2 &:= \frac{q}{\zeta_n}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}\left[p_{n+1}(y)p_n(qy) - p_n(y)p_{n+1}(qy)\right]w(y)\mathrm{d}y,\end{aligned}$$

after $\beta_n p_{n-1}(qy)$ is replaced by $(qy-\alpha_n)p_n(qy) - p_{n+1}(qy)$ by (1.4). It is easy to see that I_1 is given by

$$\begin{aligned}I_1 &= (x-\alpha_n)C_n(x) + \frac{q}{\zeta_n}\int_0^\infty (v(qy)-v(x))p_n(y)p_n(qy)w(y)\mathrm{d}y\\ &= (x-\alpha_n)C_n(x) - q^{n+1}v(x),\end{aligned}$$

where (1.23) and the fact that

$$p_j(qy) = q^j p_j(y) + \text{lower degree terms} \tag{2.3}$$

were used. To evaluate I_2, first note that (2.3) implies

$$\int_0^\infty p_j(y)p_j(qy)w(y)\mathrm{d}y = \zeta_j q^j.$$

Next, we apply the Christoffel–Darboux formula to

$$p_{n+1}(y)p_n(qy) - p_n(y)p_{n+1}(qy)$$

and replace $y-qy$ by $(yq-x+x)(1-q)/q$. Thus we obtain

$$\begin{aligned}I_2 &= x(1-q)/q\sum_{j=0}^n C_j(x) - (1-q)\int_0^\infty (v(x)-v(qy))\sum_{j=0}^n \frac{p_j(y)p_j(qy)}{\zeta_j}w(y)\mathrm{d}y\\ &= x(1-q)/q\sum_{j=0}^n C_j(x) - (1-q)v(x)\sum_{j=0}^n q^j + (1-q)\int_0^\infty v(qy)\sum_{j=0}^n \frac{p_j(y)p_j(qy)}{\zeta_j}w(y)\mathrm{d}y.\end{aligned}$$

The last integral vanishes by (1.23). Therefore I_2 is given by

$$I_2 = x(1/q-1)\sum_{j=0}^{n} C_j(x) + (q^{n+1}-1)v(x).$$

Simplifying $I_1 + I_2$ we establish (1.14).

It remains to prove (1.15). From the definition of $D_n(x)$ we see that

$$\begin{aligned}
&[(x-\alpha_n)D_{n+1}(x) - (x/q-\alpha_n)D_n(x)]/q = \int_0^\infty \frac{v(x)-v(qy)}{x-qy} \\
&\times \left(\left(\frac{x-\alpha_n}{\zeta_n}\right)p_{n+1}(y)p_n(qy) - \left(\frac{x/q-\alpha_n}{\zeta_{n-1}}\right)p_n(y)p_{n-1}(qy)\right)w(y)\mathrm{d}y \\
&= \int_0^\infty (v(x)-v(qy))\left(\frac{1}{\zeta_n}\left(1+\frac{qy-\alpha_n}{x-qy}\right)p_{n+1}(y)p_n(qy) - \frac{1}{\zeta_{n-1}}\left(\frac{1}{q}+\frac{y-\alpha_n}{x-qy}\right)p_n(y)p_{n-1}(qy)\right)w(y)\mathrm{d}y \\
&= -\frac{1}{\zeta_n}\int_0^\infty v(qy)p_{n+1}(y)p_n(qy)w(y)\mathrm{d}y + \frac{1}{\zeta_n}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}(qy-\alpha_n)p_n(qy)p_{n+1}(y)w(y)\mathrm{d}y \\
&\quad + \frac{1}{q\zeta_{n-1}}\int_0^\infty v(qy)p_n(y)p_{n-1}(qy)w(y)\mathrm{d}y - \frac{1}{\zeta_{n-1}}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}(y-\alpha_n)p_n(y)p_{n-1}(qy)w(y)\mathrm{d}y \\
&= -\frac{1-q^{-n-1}}{q-1} + \frac{1}{\zeta_n}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}\left[p_{n+1}(qy)+\beta_n p_{n-1}(qy)\right]p_{n+1}(y)w(y)\mathrm{d}y \\
&\quad + \frac{1-q^{-n}}{q(q-1)} - \frac{1}{\zeta_{n-1}}\int_0^\infty \frac{v(x)-v(qy)}{x-qy}\left[p_{n+1}(y)+\beta_n p_{n-1}(y)\right]p_{n-1}(qy)w(y)\mathrm{d}y \\
&= -1/q + \beta_{n+1}C_{n+1}(x)/q - \beta_n C_{n-1}(x)/q, \qquad (2.4)
\end{aligned}$$

where we have used the orthogonality, (1.24), (1.4) and (2.1). This verifies (1.15). □

3. Second order q-difference equations and fundamental solution bases

We introduce two pairs of adjoint degree raising and lowering operators.

From (1.5) and (1.13) we obtain the lowering operators

$$L_{1,n} := D_q + B_n, \qquad L_{1,n}p_n = \beta_n A_n p_{n-1} \qquad (3.1)$$

and

$$\tilde{L}_{1,n} := D_{q^{-1}} + D_n, \quad \tilde{L}_{1,n}p_n = \beta_n C_n p_{n-1}, \qquad (3.2)$$

respectively. Next, from (1.4), (1.5) and (1.13) we have

$$\begin{aligned}
D_q p_n &= A_n\left((x-\alpha_n)p_n - p_{n+1}\right) - B_n p_n, \\
D_{q^{-1}} p_n &= C_n\left((x-\alpha_n)p_n - p_{n+1}\right) - D_n p_n.
\end{aligned}$$

Thus, the corresponding raising operators can be defined as

$$L_{2,n} := -(1/A_n)D_q + (x-\alpha_n-B_n/A_n), \quad L_{2,n}p_n = p_{n+1} \qquad (3.3)$$

and

$$\tilde{L}_{2,n} := -(1/C_n)D_{q^{-1}} + (x-\alpha_n-D_n/C_n), \quad \tilde{L}_{2,n}p_n = p_{n+1}. \qquad (3.4)$$

Then, p_n is a zero of the second order operator

$$\begin{aligned}
L_{1,n+1}\tilde{L}_{2,n} - \beta_{n+1}A_{n+1} &= (D_q + B_{n+1})\left[-(1/C_n)D_{q^{-1}} + x - \alpha_n - D_n/C_n\right] - \beta_{n+1}A_{n+1} \\
&= -\frac{1}{C_n(qx)}D_q D_{q^{-1}} - \left(D_q\left(\frac{1}{C_n}\right) + \frac{B_{n+1}}{C_n}\right)D_{q^{-1}} + \left(qx - \alpha_n - \frac{D_n(qx)}{C_n(qx)}\right)D_q \\
&\quad + \left[1 - D_q(D_n/C_n) + (x-\alpha_n-D_n/C_n)B_{n+1} - \beta_{n+1}A_{n+1}\right] \qquad (3.5)
\end{aligned}$$

where we used (1.18). Similarly, p_n is a zero of the second order operator

$$\begin{aligned}
\tilde{L}_{1,n+1}L_{2,n} - \beta_{n+1}C_{n+1} &= (D_{q^{-1}} + D_{n+1})\left[-(1/A_n)D_q + x - \alpha_n - B_n/A_n\right] - \beta_{n+1}C_{n+1} \\
&= -\frac{1}{A_n(x/q)}D_{q^{-1}}D_q - \left(D_{q^{-1}}\left(\frac{1}{A_n}\right) + \frac{D_{n+1}}{A_n}\right)D_q + \left(x/q - \alpha_n - \frac{B_n(x/q)}{A_n(x/q)}\right)D_{q^{-1}} \\
&\quad + \left[1 - D_{q^{-1}}(B_n/A_n) + (x-\alpha_n-B_n/A_n)D_{n+1} - \beta_{n+1}C_{n+1}\right]. \qquad (3.6)
\end{aligned}$$

The operators (3.5) and (3.6) generate two second order q-difference equations of the form

$$a_{n,i}(x)f(qx)+b_{n,i}(x)f(x)+c_{n,i}(x)f(x/q)=0,\qquad i=1,\ 2, \tag{3.7}$$

respectively. Clearly $f=p_n$ satisfies these equations. The following formula is easily verified:

$$D_{q_1}D_{q_2}f(x)=\frac{f(q_1q_2x)-f(q_1x)-q_1f(q_2x)+q_1f(x)}{(q_1-1)(q_2-1)q_1x^2}. \tag{3.8}$$

As special cases of (3.8) we get

$$\begin{aligned} D_qD_{q^{-1}}f(x)&=\frac{f(qx)-(1+q)f(x)+qf(x/q)}{(1-q)^2x^2},\\ D_{q^{-1}}D_qf(x)&=qD_qD_{q^{-1}}f(x). \end{aligned} \tag{3.9}$$

From (3.5), (3.6) and (3.9) it follows that

$$a_{n,1}=-\frac{1}{(1-q)^2x^2C_n(qx)}-\frac{1}{(1-q)x}\left(qx-\alpha_n-\frac{D_n(qx)}{C_n(qx)}\right), \tag{3.10}$$

$$a_{n,2}=-\frac{q}{(1-q)^2x^2A_n(x/q)}+\frac{1}{(1-q)x}\left(D_{q^{-1}}\left(\frac{1}{A_n}\right)+\frac{D_{n+1}}{A_n}\right), \tag{3.11}$$

$$\begin{aligned} b_{n,1}\ =\ &\frac{(1+q)}{(1-q)^2x^2C_n(qx)}+\frac{q}{(1-q)x}\left(D_q\left(\frac{1}{C_n}\right)+\frac{B_{n+1}}{C_n}\right)\\ &+\frac{1}{(1-q)x}\left(qx-\alpha_n-\frac{D_n(qx)}{C_n(qx)}\right)+1-D_q\left(\frac{D_n}{C_n}\right)+\left(x-\alpha_n-\frac{D_n}{C_n}\right)B_{n+1}-\beta_{n+1}A_{n+1}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} b_{n,2}\ =\ &\frac{q(1+q)}{(1-q)^2x^2A_n(x/q)}-\frac{1}{(1-q)x}\left(D_{q^{-1}}\left(\frac{1}{A_n}\right)+\frac{D_{n+1}}{A_n}\right)\\ &-\frac{q}{(1-q)x}\left(x/q-\alpha_n-\frac{B_n(x/q)}{A_n(x/q)}\right)+1-D_{q^{-1}}\left(\frac{B_n}{A_n}\right)+\left(x-\alpha_n-\frac{B_n}{A_n}\right)D_{n+1}-\beta_{n+1}C_{n+1}, \end{aligned} \tag{3.13}$$

$$c_{n,1}=-\frac{q}{(1-q)^2x^2C_n(qx)}-\frac{q}{(1-q)x}\left(D_q\left(\frac{1}{C_n}\right)+\frac{B_{n+1}}{C_n}\right), \tag{3.14}$$

$$c_{n,2}=-\frac{q^2}{(1-q)^2x^2A_n(x/q)}+\frac{q}{(1-q)x}\left(x/q-\alpha_n-\frac{B_n(x/q)}{A_n(x/q)}\right). \tag{3.15}$$

We will show that the second order equations (3.7) have the same solution basis.

In what follows we shall assume that there exists a domain D containing the open interval $(0,\infty)$ and such that for every $q\in(0,1)$, the weight $w(x)$ has analytic continuation in D. The function of the second kind Q_n is then defined by

$$Q_n(x):=\frac{1}{w(x)}\int_0^\infty\frac{p_n(t)}{x-t}w(t)\mathrm{d}t,\quad x\in D\setminus[0,\infty). \tag{3.16}$$

Theorem 3.1. *The function of the second kind Q_n is a zero of the operators defined in (3.5) and (3.6).*

Proof. Using (1.19) it is easy to show that $Q_n(x)$ satisfies (1.4) for all $n\geq 1$, while

$$(x-\alpha_0)Q_0(x)=Q_1(x)+\frac{\zeta_0}{w(x)}.$$

Thus, it suffices to show that $Q_n(x)$ satisfies the lowering relations (1.5) and (1.13).

Let $x\in D\setminus[0,\infty)$. First we show that

$$D_qQ_n(x)=\beta_nA_n(x)Q_{n-1}(x)-B_n(x)Q_n(x). \tag{3.17}$$

The left-hand side of (3.17) is

$$\begin{aligned} D_qQ_n(x)\ &=\ \frac{1}{(q-1)xw(x)w(qx)}\int_0^\infty\frac{[(x-t)w(x)-(qx-t)w(qx)]}{(x-t)(qx-t)}p_n(t)w(t)\mathrm{d}t\\ &=\ \frac{1}{w(x)}\int_0^\infty\frac{((x-t)u(qx)-1)}{(x-t)(qx-t)}p_n(t)w(t)\mathrm{d}t\\ &=\ \frac{1}{w(x)}\int_0^\infty\frac{u(qx)-u(t)}{qx-t}p_n(t)w(t)\mathrm{d}t+\frac{1}{w(x)}\int_0^\infty\frac{((x-t)u(t)-1)}{(x-t)(qx-t)}p_n(t)w(t)\mathrm{d}t, \end{aligned} \tag{3.18}$$

where on the second line we used that $w(x) - w(qx) = -(1-q)xu(qx)w(qx)$ which follows from (1.2). By (1.6) and (1.7) the right-hand side of (3.17) times $w(x)$ equals

$$\frac{1}{\zeta_{n-1}}\int_0^\infty \frac{u(qx)-u(t)}{qx-t}\left\{\int_0^\infty \frac{[p_n(t/q)p_{n-1}(y)-p_{n-1}(t/q)p_n(y)]}{x-y}w(y)\mathrm{d}y\right\}p_n(t)w(t)\mathrm{d}t$$
$$= \sum_{k=0}^{n-1}\frac{1}{\zeta_k}\int_0^\infty \frac{u(qx)-u(t)}{qx-t}p_k(t/q)p_n(t)\left\{\int_0^\infty \frac{(t/q-y)}{x-y}p_k(y)w(y)\mathrm{d}y\right\}w(t)\mathrm{d}t, \tag{3.19}$$

where we used (2.1) and (2.2). We apply the decomposition

$$\frac{t/q-y}{(qx-t)(x-y)} = \frac{1}{qx-t} - \frac{1/q}{x-y}$$

in (3.19) and it becomes

$$\sum_{k=0}^{n-1}\frac{1}{\zeta_k}\int_0^\infty \frac{u(qx)-u(t)}{qx-t}p_k(t/q)p_n(t)w(t)\mathrm{d}t\int_0^\infty p_k(y)w(y)\mathrm{d}y$$
$$-\sum_{k=0}^{n-1}\frac{1}{q\zeta_k}\int_0^\infty \frac{p_k(y)}{x-y}w(y)\mathrm{d}y\int_0^\infty (u(qx)-u(t))p_k(t/q)p_n(t)w(t)\mathrm{d}t$$
$$= \int_0^\infty \frac{u(qx)-u(t)}{qx-t}p_n(t)w(t)\mathrm{d}t - I_1, \tag{3.20}$$

where we used the orthogonality, and we denoted by I_1 the second line of (3.20). Replacing in I_1, $u(qx)-u(t)$ by $u(qx)-u(qy)+u(qy)-u(t)$ we obtain

$$I_1 = \frac{1}{q}\int_0^\infty \frac{u(qx)-u(qy)}{x-y}\sum_{k=0}^{n-1}\frac{p_k(y)}{\zeta_k}w(y)\int_0^\infty p_n(t)p_k(t/q)w(t)\mathrm{d}t\mathrm{d}y$$
$$+\frac{1}{q}\int_0^\infty \frac{w(y)}{x-y}\left\{\int_0^\infty (u(qy)-u(t))\sum_{k=0}^{n-1}\frac{p_k(y)p_k(t/q)}{\zeta_k}p_n(t)w(t)\mathrm{d}t\right\}\mathrm{d}y$$
$$= \int_0^\infty \frac{w(y)}{x-y}\left\{\int_0^\infty (u(qy)-u(t))\frac{[p_n(y)p_{n-1}(t/q)-p_n(t/q)p_{n-1}(y)]}{\zeta_{n-1}(qy-t)}p_n(t)w(t)\mathrm{d}t\right\}\mathrm{d}y$$
$$= \int_0^\infty \frac{w(y)}{x-y}\left[B_n(y)p_n(y)-\beta_nA_n(y)p_{n-1}(y)\right]\mathrm{d}y \tag{3.21}$$

where we used the orthogonality, (2.2), (2.1), (1.7) and (1.6). Now we apply (1.5) and (1.20) to obtain

$$I_1 = -\int_0^\infty \frac{w(y)}{x-y}D_qp_n(y)\mathrm{d}y = \frac{1}{q}\int_0^\infty p_n(y)D_{q^{-1}}\left(\frac{w(y)}{x-y}\right)\mathrm{d}y$$
$$= \frac{1}{q}\int_0^\infty \frac{[(x-y)w(y/q)-(x-y/q)w(y)]}{(x-y)(x-y/q)(1/q-1)y}p_n(y)\mathrm{d}y$$
$$= \frac{1}{q}\int_0^\infty \left(\frac{D_{q^{-1}}w(y)}{x-y/q}+\frac{w(y)}{(x-y)(x-y/q)}\right)p_n(y)\mathrm{d}y$$
$$= \int_0^\infty \frac{[-(x-y)u(y)+1]}{(x-y)(qx-y)}p_n(y)w(y)\mathrm{d}y \tag{3.22}$$

where (1.2) was also used. Combining (3.20) and (3.22) shows that the right-hand side of (3.17) equals its left-hand side given by (3.18).

It remains to show that $Q_n(x)$ satisfies (1.13), that is,

$$D_{q^{-1}}Q_n(x) = \beta_nC_n(x)Q_{n-1}(x) - D_n(x)Q_n(x). \tag{3.23}$$

The left-hand side of (3.23) equals

$$\begin{aligned} D_{q^{-1}}Q_n(x) &= \frac{1}{(1/q-1)xw(x)w(x/q)}\int_0^\infty \frac{[(x-t)w(x)-(x/q-t)w(x/q)]}{(x-t)(x/q-t)}p_n(t)w(t)\mathrm{d}t \\ &= \frac{1}{w(x)}\int_0^\infty \frac{(x-t)v(x)-1}{(x-t)(x/q-t)}p_n(t)w(t)\mathrm{d}t \\ &= \frac{q}{w(x)}\int_0^\infty \frac{v(x)-v(qt)}{x-qt}p_n(t)w(t)\mathrm{d}t + \frac{q}{w(x)}\int_0^\infty \frac{(x-t)v(qt)-1}{(x-t)(x-qt)}p_n(t)w(t)\mathrm{d}t, \end{aligned} \tag{3.24}$$

where we used that $w(x) - w(x/q) = (1/q-1)xv(x)w(x/q)$ which follows from (1.10). Using (1.11) and (1.12), the right-hand side of (3.23) times $w(x)$ can be written as

$$\begin{aligned} &\frac{q}{\zeta_{n-1}}\int_0^\infty \frac{v(x)-v(qt)}{x-qt}\left\{\int_0^\infty \frac{[p_n(qt)p_{n-1}(y)-p_{n-1}(qt)p_n(y)]}{x-y}w(y)\mathrm{d}y\right\}p_n(t)w(t)\mathrm{d}t \\ &= q\sum_{k=0}^{n-1}\frac{1}{\zeta_k}\int_0^\infty \frac{v(x)-v(qt)}{x-qt}p_k(qt)p_n(t)\left\{\int_0^\infty \frac{qt-y}{x-y}p_k(y)w(y)\mathrm{d}y\right\}w(t)\mathrm{d}t, \\ &= q\sum_{k=0}^{n-1}\frac{1}{\zeta_k}\int_0^\infty \frac{v(x)-v(qt)}{x-qt}p_k(qt)p_n(t)w(t)\mathrm{d}t\int_0^\infty p_k(y)w(y)\mathrm{d}y \\ &\quad -q\sum_{k=0}^{n-1}\frac{1}{\zeta_k}\int_0^\infty \frac{p_k(y)}{x-y}w(y)\mathrm{d}y\int_0^\infty (v(x)-v(qt))p_k(qt)p_n(t)w(t)\mathrm{d}t \\ &= q\int_0^\infty \frac{v(x)-v(qt)}{x-qt}p_n(t)w(t)\mathrm{d}t - I_2, \end{aligned} \tag{3.25}$$

where we used (2.1), (2.2), the decomposition

$$\frac{qt-y}{(x-qt)(x-y)} = \frac{1}{x-qt} - \frac{1}{x-y},$$

and the orthogonality relation. In the expression denoted by I_2 we replace $v(x) - v(qt)$ by $v(x) - v(y) + v(y) - v(qt)$ and we apply the orthogonality relation. We get

$$\begin{aligned} I_2 &= q\int_0^\infty \frac{w(y)}{x-y}\left\{\int_0^\infty (v(y)-v(qt))\sum_{k=0}^{n-1}\frac{p_k(y)p_k(qt)}{\zeta_k}p_n(t)w(t)\mathrm{d}t\right\}\mathrm{d}y \\ &= q\int_0^\infty \frac{w(y)}{x-y}\left\{\int_0^\infty (v(y)-v(qt))\frac{[p_n(y)p_{n-1}(qt)-p_n(qt)p_{n-1}(y)]}{\zeta_{n-1}(y-qt)}p_n(t)w(t)\mathrm{d}t\right\}\mathrm{d}y \\ &= \int_0^\infty \frac{w(y)}{x-y}[D_n(y)p_n(y)-\beta_nC_n(y)p_{n-1}(y)]\,\mathrm{d}y, \end{aligned} \tag{3.26}$$

where we also used (2.2), (2.1), (1.12) and (1.11). Thus, from (1.13), (1.20) and (1.10) we obtain

$$\begin{aligned} I_2 &= -\int_0^\infty \frac{w(y)}{x-y}D_{q^{-1}}p_n(y)\mathrm{d}y = q\int_0^\infty p_n(y)D_q\left(\frac{w(y)}{x-y}\right)\mathrm{d}y \\ &= q\int_0^\infty \frac{[(x-y)w(qy)-(x-qy)w(y)]}{(x-y)(x-qy)(q-1)y}p_n(y)\mathrm{d}y \\ &= q\int_0^\infty \left(\frac{D_qw(y)}{x-qy}+\frac{w(y)}{(x-y)(x-qy)}\right)p_n(y)\mathrm{d}y \\ &= q\int_0^\infty \frac{[-(x-y)v(qy)+1]}{(x-y)(x-qy)}p_n(y)w(y)\mathrm{d}y. \end{aligned} \tag{3.27}$$

Now, (3.23) follows from (3.24), (3.25) and (3.27). □

Since the functions p_n and Q_n satisfy both equations in (3.7), these equations have the same solution basis. Thus, we obtain the following corollary.

Corollary 3.2. *The functions defined with (3.10)–(3.15) satisfy*

$$\frac{b_{n,1}}{a_{n,1}} = \frac{b_{n,2}}{a_{n,2}} = -\frac{p_n(qx)Q_n(x/q) - p_n(x/q)Q_n(qx)}{p_n(x)Q_n(x/q) - p_n(x/q)Q_n(x)} \tag{3.28}$$

and

$$\frac{c_{n,1}}{a_{n,1}} = \frac{c_{n,2}}{a_{n,2}} = -\frac{p_n(x)Q_n(qx) - p_n(qx)Q_n(x)}{p_n(x)Q_n(x/q) - p_n(x/q)Q_n(x)}. \tag{3.29}$$

These identities give relations involving the functions A_n, B_n, C_n and D_n.

The next theorem provides more simple and direct relations among A_n, B_n, C_n and D_n. We define

$$\Delta_n(x) := \begin{vmatrix} p_n(x) & p_n(qx) \\ Q_n(x) & Q_n(qx) \end{vmatrix}, \quad x \in D \setminus [0, \infty). \tag{3.30}$$

Theorem 3.3. *For every $x \in D \setminus [0, \infty)$,*

$$\Delta_n(x) = -\frac{(1-q)x}{w(x)}\zeta_n A_n(x) \quad \text{and} \quad \Delta_n(x) = -\frac{(1-q)x}{w(qx)}\zeta_n C_n(qx). \tag{3.31}$$

In particular,

$$C_n(qx) = \frac{w(qx)}{w(x)}A_n(x). \tag{3.32}$$

Furthermore,

$$D_n(x) = \frac{-1 + [1 - 2(x/q - \alpha_{n-1})u(x)]w(x)/w(x/q)}{(1+1/q)x - 2\alpha_{n-1}} + [(x/q - \alpha_{n-1})A_{n-1}(x/q) - B_{n-1}(x/q)]w(x)/w(x/q). \tag{3.33}$$

Proof. For $x \in D \setminus [0, \infty)$ we have

$$\begin{aligned} \Delta_n(x) &= \frac{1}{w(qx)} \int_0^\infty \frac{p_n(x)p_n(t)}{qx - t} w(t)\mathrm{d}t - \frac{1}{w(x)} \int_0^\infty \frac{p_n(qx)p_n(t)}{x - t} w(t)\mathrm{d}t \\ &= \frac{1}{w(qx)} \int_0^\infty \frac{p_n(t/q)p_n(t)}{qx - t} w(t)\mathrm{d}t - \frac{1}{w(x)} \int_0^\infty \frac{p_n(qt)p_n(t)}{x - t} w(t)\mathrm{d}t \\ &= \frac{1}{w(qx)} \int_0^\infty \frac{p_n(t)p_n(qt)}{x - t} w(qt)\mathrm{d}t - \frac{1}{w(x)} \int_0^\infty \frac{p_n(qt)p_n(t)}{x - t} w(t)\mathrm{d}t \\ &= \frac{1}{w(x)w(qx)} \int_0^\infty \frac{[w(x)w(qt) - w(qx)w(t)]}{x - t} p_n(qt)p_n(t)\mathrm{d}t, \end{aligned} \tag{3.34}$$

where (1.19) was used. From here we can proceed in two directions. First using (1.10) we write

$$\begin{aligned} w(x)w(qt) - w(qx)w(t) &= w(x)(w(qt) - w(t)) - w(t)(w(qx) - w(x)) \\ &= w(x)(q-1)tD_q w(t) - w(t)(q-1)xD_q w(x) \\ &= (1-q)\left[w(x)tw(t)v(qt) - w(t)xw(x)v(qx)\right] \\ &= (1-q)w(x)w(t)\left[tv(qt) - xv(qx)\right]. \end{aligned}$$

Substituting in (3.34) we obtain

$$\begin{aligned} \Delta_n(x) &= -\frac{(1-q)}{w(qx)} \int_0^\infty \frac{(xv(qx) - tv(qt))}{x - t} p_n(t)p_n(qt)w(t)\mathrm{d}t \\ &= -\frac{(1-q)x}{w(qx)} \int_0^\infty \left(\frac{v(qx) - v(qt)}{x - t}\right) p_n(t)p_n(qt)w(t)\mathrm{d}t \\ &\quad - \frac{(1-q)}{w(qx)} \int_0^\infty v(qt)p_n(t)p_n(qt)w(t)\mathrm{d}t = -\frac{(1-q)x}{w(qx)}\zeta_n C_n(qx) \end{aligned} \tag{3.35}$$

by (1.23) and (1.11). Next, by (1.2) we have $w(qx) - w(x) = (1-q)xu(qx)w(qx)$ and

$$\begin{aligned} w(x)w(qt) - w(qx)w(t) &= w(qt)(w(x) - w(qx)) + w(qx)(w(qt) - w(t)) \\ &= -(1-q)xu(qx)w(qx)w(qt) + (1-q)tu(qt)w(qt)w(qx) \\ &= -(1-q)w(qx)w(qt)\left(xu(qx) - tu(qt)\right). \end{aligned}$$

Then, (3.34) leads to

$$\begin{aligned} \Delta_n(x) &= -\frac{(1-q)}{w(x)}\int_0^\infty \frac{(xu(qx) - tu(qt))}{x-t} p_n(qt)p_n(t)w(qt)\mathrm{d}t \\ &= -\frac{(1-q)x}{w(x)}\int_0^\infty \frac{(u(qx) - u(qt))}{x-t} p_n(qt)p_n(t)w(qt)\mathrm{d}t \\ &\quad - \frac{(1-q)}{w(x)}\int_0^\infty u(qt)p_n(qt)p_n(t)w(qt)\mathrm{d}t = -\frac{(1-q)x}{w(x)}\zeta_n A_n(x) \end{aligned} \tag{3.36}$$

since the last integral vanishes, which follows from the relation

$$u(qt)w(qt) = (-D_{q^{-1}}w(y))|_{y=qt} = -D_q w(t) = v(qt)w(t) \tag{3.37}$$

and (1.23).

It remains to verify (3.33). From (1.14), (3.32), (3.37) and (1.8) we have

$$\begin{aligned} D_{n+1}(qx) + D_n(qx) &= [(qx - \alpha_n)A_n(x) + (1-q)x\sum_{j=0}^{n} A_j(x) - u(qx)]w(qx)/w(x) \\ &= [((1+q)x - 2\alpha_n)A_n(x) - B_{n+1}(x) - B_n(x) - 2u(qx)]\, w(qx)/w(x). \end{aligned} \tag{3.38}$$

Furthermore, from (1.15), (3.32) and (1.9) we obtain

$$\begin{aligned} (qx - \alpha_n)D_{n+1}(qx) - (x - \alpha_n)D_n(qx) &= -1 + [\beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x)]w(qx)/w(x) \\ &= -1 + [1 + (x-\alpha_n)B_{n+1}(x) - (qx - \alpha_n)B_n(x)]w(qx)/w(x). \end{aligned} \tag{3.39}$$

Multiplying Eq. (3.38) by $x - \alpha_n$ and adding to Eq. (3.39) we obtain

$$\begin{aligned} ((1+q)x - 2\alpha_n)D_{n+1}(qx) &= -1 + [1 + (x-\alpha_n)((1+q)x - 2\alpha_n)A_n(x) \\ &\quad - ((1+q)x - 2\alpha_n)B_n(x) - 2(x-\alpha_n)u(qx)]\, w(qx)/w(x). \end{aligned}$$

From here, formula (3.33) follows by replacing n by $n-1$ and x by x/q. □

Corollary 3.4. *Assume that there exists a domain D containing $(0, \infty)$, such that for every $q \in (0, 1)$ the weight $w(x)$ is analytic in D. Then, (3.32) and (3.33) hold on $(0, \infty)$.*

Indeed, in this case $u(x)$ and $v(x)$ are analytic in D, and then by (1.6), (1.7), (1.11) and (1.12) the functions A_n, B_n, C_n and D_n are also analytic and therefore continuous in D.

4. The Stieltjes–Wigert polynomials

In this section we compute the functions A_n, B_n, C_n and D_n for the Stieltjes–Wigert polynomials. The Stieltjes–Wigert polynomials are orthogonal with respect to the Stieltjes–Wigert weight which is defined on $(0, \infty)$ by

$$w(x) = c\exp\left(\left(\log(xq^{-1/2})\right)^2/(2\log q)\right), \quad c = (2\pi\log(1/q))^{-1/2}. \tag{4.1}$$

The corresponding indeterminate moment problem was studied in [12]. The monic orthogonal polynomials are given by [13]

$$p_n(x) = q^{-n^2}(q;q)_n \sum_{k=0}^{n} \frac{(-1)^{n-k}q^{k^2}x^k}{(q;q)_k(q;q)_{n-k}}. \tag{4.2}$$

Furthermore,

$$\alpha_n = (1 + q - q^{n+1})q^{-2n-1}, \qquad \beta_n = (1-q^n)q^{-4n+1}, \tag{4.3}$$

and

$$\zeta_n = (q;q)_n q^{-2n^2-n}.$$

Simple calculations show that

$$
\begin{aligned}
u(x) &= \frac{q}{1-q}\left(\frac{1}{x}-\frac{q}{x^2}\right), \qquad v(x) = \frac{x-q}{(1-q)x},\\
A_n(x) &= \frac{R_n}{x^2}, \quad \text{where}\\
R_n &= \frac{1}{(1-q)\zeta_n}\int_0^\infty p_n(y)p_n(y/q)\frac{w(y)}{y}\mathrm{d}y,\\
B_n(x) &= \frac{r_n}{x^2}-\frac{(1-q^n)}{(1-q)x}, \quad \text{where}\\
r_n &= \frac{1}{(1-q)\zeta_{n-1}}\int_0^\infty p_n(y)p_{n-1}(y/q)\frac{w(y)}{y}\mathrm{d}y,\\
C_n(x) &= \frac{\tilde{R}_n}{x}, \quad \text{where}\\
\tilde{R}_n &= \frac{q}{(1-q)\zeta_n}\int_0^\infty p_n(y)p_n(qy)w(y)\mathrm{d}y,\\
D_n(x) &= \frac{\tilde{r}_n}{x}, \quad \text{where}\\
\tilde{r}_n &= \frac{q}{(1-q)\zeta_{n-1}}\int_0^\infty p_n(y)p_{n-1}(qy)\frac{w(y)}{y}\mathrm{d}y.
\end{aligned} \tag{4.4}
$$

Substituting in (1.9) and equating the coefficients of $1/x$ we obtain

$$r_{n+1} - qr_n = -\alpha_n q^n$$

or, equivalently, $t_{n+1} - t_n = -\alpha_n$ with $t_n := r_n/q^{n-1}$. In particular,

$$t_1 = r_1 = \frac{1}{(1-q)\zeta_0}\int_0^\infty (y-\alpha_0)\frac{w(y)}{y}\mathrm{d}y = -1/q,$$

where we used that $\alpha_0 = 1/q$, $\zeta_0 = 1$, the relation

$$w(qx) = xw(x) \tag{4.5}$$

which follows from (4.1), and the evaluation

$$\int_0^\infty \frac{w(y)}{y}\mathrm{d}y = \frac{1}{q}\int_0^\infty w(y/q)\mathrm{d}y = \int_0^\infty w(t)\mathrm{d}t = \zeta_0 = 1.$$

Thus,

$$
\begin{aligned}
r_n &= q^{n-1}t_n = q^{n-1}\left(t_1 - \sum_{j=1}^{n-1}\alpha_j\right)\\
&= -q^{n-1}\sum_{j=0}^{n-1}\left((1+q)q^{-2j-1} - q^{-j}\right) = \frac{1-q^{-n}}{1-q}
\end{aligned} \tag{4.6}
$$

and

$$B_n(x) = -\frac{(1-q^n)(x+q^{-n})}{(1-q)x^2}. \tag{4.7}$$

To compute R_n we apply (1.8) and use (4.3). Equating the coefficients of $1/x^2$ yields

$$r_{n+1} + r_n = -\alpha_n R_n + 1/(1-q).$$

Hence $R_n = (1/(1-q) - r_{n+1} - r_n)/\alpha_n = q^n/(1-q)$ and

$$A_n(x) = q^n/((1-q)x^2) \tag{4.8}$$

follows from (4.6), (4.3) and (4.4). Next, from (3.32), (4.5) and (4.8) we get

$$C_n(x) = q^{n+1}/((1-q)x). \tag{4.9}$$

It remains to determine $D_n(x)$. One option is to apply (3.33), but we use (4.4) instead. Applying (1.15) and equating the constant coefficients we obtain

$$\tilde{r}_{n+1} = \tilde{r}_n/q - 1.$$

Iteration of this relation yields

$$\tilde{r}_{n+1} = q^{-n}\tilde{r}_1 - \sum_{j=0}^{n-1} q^{-j} = -\frac{(1-q^{n+1})q^{-n}}{(1-q)},$$

where we used that

$$\tilde{r}_1 = \frac{q}{(1-q)\zeta_0}\int_0^\infty (y-\alpha_0)\frac{w(y)}{y}\mathrm{d}y = -1$$

as above. Thus,

$$D_n(x) = \frac{(1-q^{-n})q}{(1-q)x}. \tag{4.10}$$

5. The q-Laguerre polynomials

Here we derive formulas for the functions A_n, B_n, C_n and D_n for the q-Laguerre polynomials. A q-Laguerre weight is defined on $(0, \infty)$ by [7]

$$w(x) = x^\alpha/(-x; q)_\infty, \tag{5.1}$$

where $\alpha > 0$. The monic q-Laguerre orthogonal polynomials are given by [7]

$$p_n(x) = q^{-\alpha n - n^2}(q, q^{\alpha+1}; q)_n \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{\alpha k + k^2} x^k}{(q, q^{\alpha+1}; q)_k (q; q)_{n-k}}. \tag{5.2}$$

Using (1.2), (1.10) and (5.1), it is easy to verify that

$$u(x) = \frac{(x+q-q^{1-\alpha})q}{(1-q)x(x+q)}, \qquad v(x) = \frac{((x+q)q^\alpha - q)}{(1-q)x},$$

and then

$$\begin{aligned} \frac{u(qx)-u(y)}{qx-y} &= -\frac{1}{q(x+1)}u(y) - \frac{(1-q^{-\alpha})}{(1-q)x(x+1)y}, \\ \frac{v(x)-v(qy)}{x-qy} &= -\frac{v(qy)}{x} + \frac{q^\alpha}{(1-q)x}. \end{aligned} \tag{5.3}$$

We first compute C_n and D_n, and then we obtain A_n and B_n using Theorem 3.3. For C_n, using (1.11), (5.3), (1.23) and the orthogonality we get

$$C_n(x) = -\frac{q}{\zeta_n x}\int_0^\infty v(qy)p_n(y)p_n(qy)w(y)\mathrm{d}y + \frac{q^{\alpha+1}}{(1-q)\zeta_n x}\int_0^\infty p_n(y)p_n(qy)w(y)\mathrm{d}y = \frac{q^{\alpha+n+1}}{(1-q)x}. \tag{5.4}$$

Similarly, for D_n, using (1.12), (5.3), (1.24) and the orthogonality we obtain

$$D_n(x) = -\frac{q}{\zeta_{n-1} x}\int_0^\infty v(qy)p_n(y)p_{n-1}(qy)w(y)\mathrm{d}y + \frac{q^{\alpha+1}}{(1-q)\zeta_{n-1} x}\int_0^\infty p_n(y)p_{n-1}(qy)w(y)\mathrm{d}y = \frac{(1-q^{-n})q}{(1-q)x}. \tag{5.5}$$

Then, by (3.32) we get

$$A_n(x) = C_n(qx)w(x)/w(qx) = \frac{q^n}{(1-q)x(x+1)}. \tag{5.6}$$

To compute B_n we adopt the notation $w_\alpha(x)$ for the q-Laguerre weight. From (1.7), (5.3) and (1.22) it follows that

$$\begin{aligned} B_n(x) &= -\frac{1}{q\zeta_{n-1}(\alpha)(x+1)}\int_0^\infty u(y)p_n(y)p_{n-1}(y/q)w_\alpha(y)\mathrm{d}y \\ &\quad - \frac{(1-q^{-\alpha})}{(1-q)\zeta_{n-1}(\alpha)x(x+1)}\int_0^\infty p_n(y)p_{n-1}(y/q)\frac{w_\alpha(y)}{y}\mathrm{d}y \\ &= \frac{r_n(\alpha)}{x(x+1)} - \frac{(1-q^n)}{(1-q)(x+1)}, \end{aligned} \tag{5.7}$$

where

$$r_n(\alpha) := -\frac{(1-q^{-\alpha})}{(1-q)\zeta_{n-1}(\alpha)} \int_0^\infty p_n(y)p_{n-1}(y/q)\frac{w_\alpha(y)}{y}\mathrm{d}y. \tag{5.8}$$

The coefficient $\alpha_n(\alpha)$ in (1.1) and the norm $\zeta_n(\alpha)$ can be derived from [7]:

$$\begin{aligned}\alpha_n(\alpha) &= (1+q-(1+q^\alpha)q^{n+1})q^{-2n-\alpha-1},\\ \zeta_n(\alpha) &= -\frac{\pi}{\sin(\pi\alpha)}\frac{(q^{-\alpha};q)_\infty(q^{\alpha+1};q)_n}{(q;q)_\infty(q;q)_n^3}q^{2n^2+2\alpha n-n}.\end{aligned} \tag{5.9}$$

Next, we substitute (5.7) and (5.6) into the supplementary relation (1.8) and use the formula (which follows from (5.1) and (1.2))

$$u(x) = \frac{(x+q-q^{1-\alpha})q}{(1-q)x(x+q)}$$

and (5.9) to derive

$$r_{n+1}(\alpha) + r_n(\alpha) = (-\alpha_n(\alpha)q^n - 1 + q^{-\alpha})/(1-q) = (2-(1+q)q^{-n-1})q^{-\alpha}/(1-q). \tag{5.10}$$

First we evaluate $r_1(\alpha)$ using (5.8). We have $p_0(y) = 1$, $p_1(y) = y - \alpha_0(\alpha)$, and by the orthogonality relation (1.1) and (5.1), $\int_0^\infty w_\alpha(y)\mathrm{d}y = \zeta_0(\alpha)$ and

$$\int_0^\infty w_\alpha(y)/y\mathrm{d}y = \int_0^\infty w_{\alpha-1}(y)\mathrm{d}y = \zeta_0(\alpha-1).$$

Then,

$$\begin{aligned}1/\zeta_0(\alpha)\int_0^\infty p_1(y)p_0(y/q)w_\alpha(y)/y\mathrm{d}y &= 1-\alpha_0(\alpha)\zeta_0(\alpha-1)/\zeta_0(\alpha)\\ &= 1+\left((1+q)q^{-\alpha-1}-(1+q^\alpha)q^{-\alpha}\right)/(1-q^{-\alpha})\\ &= (1-q)q^{-\alpha-1}/(1-q^{-\alpha}),\end{aligned}$$

where we used (5.9). Thus, by (5.8) we get $r_1(\alpha) = -q^{-\alpha-1}$. We can now evaluate $r_n(\alpha)$ using a telescoping sum:

$$\begin{aligned}r_n(\alpha) &= \sum_{j=2}^n (-1)^{n-j}(r_j(\alpha)+r_{j-1}(\alpha)) + (-1)^{n-1}r_1(\alpha)\\ &= \frac{(-1)^n q^{-\alpha}}{(1-q)}\left(\sum_{j=2}^n (-1)^j(2-(1+q)q^{-j}) + 1/q - 1\right)\\ &= (1-q^{-n})q^{-\alpha}/(1-q).\end{aligned} \tag{5.11}$$

Combining (5.7) and (5.11), we finally obtain

$$B_n(x) = -\frac{(1-q^n)(x+q^{-\alpha-n})}{(1-q)x(x+1)}. \tag{5.12}$$

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