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Components of Springer fibers associated to closed orbits for the symmetric pairs $(Sp(2n), GL(n))$ and $(O(n), O(p) \times O(q))$ I

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ABSTRACT

Let (G, K) be either $(Sp(2n), GL(n))$ or $(O(n), O(p) \times O(q))$, $p+q=n$. An explicit geometric description of components of Springer fibers associated to closed K-orbits in the flag variety of G is given. This description is used to give a simple algorithm for computing associated cycles of discrete series representations for the groups $Sp(2n, \mathbf{R})$ and $SO_e(p, q)$.

1. Introduction

Suppose G is a complex semisimple algebraic group and $\mathfrak{B}$ is the flag variety of G. The Springer resolution is the moment map $\mu : T^*\mathfrak{B} \to \mathfrak{g}^* \simeq \mathfrak{g}$. The fibers of μ are equidimensional algebraic varieties in the sense that all irreducible components of any given fiber have the same dimension. The goal of this article is to describe a family of components of fibers for the classical groups $Sp(2n)$ and $O(n)$. The description we give is very explicit and is similar to that given in [1] for the group $GL(n)$. Some examples for the classical groups $Sp(2n)$ and $O(n)$ are given in [2].

The family of components of Springer fibers considered here arises naturally in the theory of Harish–Chandra modules. Suppose $G_{\mathbf{R}}$ is a real linear semisimple Lie group and $K_{\mathbf{R}}$ is a maximal compact subgroup. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the complexified Cartan decomposition and $\mathcal{N}$ for the nilpotent cone in $\mathfrak{g}$. Then, letting K be the fixed point set of the complexification of the corresponding Cartan involution of $G_{\mathbf{R}}$, K acts on $\mathfrak{B}$ with a finite number of orbits. These K-orbits partition the components of any Springer fiber $\mu^{-1}(f)$, with $f \in \mathcal{N} \cap \mathfrak{p}$, as follows. Denoting the conormal bundle in $T^*\mathfrak{B}$ to a

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K-orbit $\mathcal{Q}$ by $T^*_{\mathcal{Q}}\mathfrak{B}$, each component is contained in $\overline{T^*_{\mathcal{Q}}\mathfrak{B}}$ for exactly one $\mathcal{Q}$. Now let $\mathcal{Q}$ be a closed K-orbit in $\mathfrak{B}$ and suppose that $\mu(T^*_{\mathcal{Q}}\mathfrak{B}) = \overline{K \cdot f}$, then we call f generic. We say that a component C of $\mu^{-1}(f)$ contained in $T^*_{\mathcal{Q}}\mathfrak{B}$ is 'associated' to the closed orbit $\mathcal{Q}$; these are the components we describe in this article. The components associated to closed orbits are precisely the components that play a role in the computation of associated cycles of discrete series representations [4].

We consider pairs (G, K) of the following types:

$$\begin{aligned} &\big(Sp(2n), GL(n)\big),\\ &\big(O(n), O(p) \times O(q)\big), \quad p+q=n. \end{aligned} \tag{1}$$

These pairs arise from the real groups $Sp(2n, \mathbf{R})$ and $O(p,q)$. An algorithm is given for finding a generic f; a similar algorithm is given in [11]. Theorem 26 describes $\mu^{-1}(f) \cap T^*_{\mathcal{Q}}\mathfrak{B}$ explicitly when $\mathcal{Q}$ is a closed K-orbit in $\mathfrak{B}$. This is done in terms of a sequence of subgroups $L = L_0, L_1, \dots, L_m$ of K. These subgroups are naturally defined in terms of the closed K-orbit $\mathcal{Q}$ and root structure. The main theorem (Theorem 26) states that $\mu^{-1}(f) \cap T^*_{\mathcal{Q}}\mathfrak{B} = L_m \cdots L_1 L \cdot \mathfrak{b} \subset \mathfrak{B}$. The components are certain translates of $L_{m,e} \cdots L_{1,e} L_e \cdot \mathfrak{b}$, where the subscript e indicates identity component.

2. Preliminaries

2.1. Generalities on Springer fibers

Let (G, K) be as in (1), or more generally G may be any reductive complex algebraic group and K the fixed point group of an involution Θ. Letting θ be the differential of Θ we write the decomposition of $\mathfrak{g}$ into ± 1 eigenspaces as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Define $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{p}$, the nilpotent elements lying in $\mathfrak{p}$. The cotangent bundle of $\mathfrak{B}$ may be identified with the homogeneous bundle $G \times_B \mathfrak{n}^-$, where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ is some basepoint of $\mathfrak{B}$ and $B = N_G(\mathfrak{b})$. The moment map is given by $\mu(g, \xi) = \mathrm{Ad}(g)\xi \in \mathcal{N}$.

The action of K on the flag variety $\mathfrak{B}$ of G has a finite number of orbits. Consider one of these orbits $\mathcal{Q} = K \cdot \mathfrak{b}$. Denote by $\gamma_{\mathcal{Q}}$ the restriction of μ to the closure of the conormal bundle $T^*_{\mathcal{Q}}\mathfrak{B}$. Suppose $f \in \mathcal{N}_\theta$. As described in [5], if C is an irreducible component of $\mu^{-1}(f)$ contained in $\overline{T^*_{\mathcal{Q}}\mathfrak{B}}$, then $K \cdot C := \bigcup_{k \in K} k \cdot C$ is dense in $\overline{T^*_{\mathcal{Q}}\mathfrak{B}}$ and

$$(K \cdot C) \cap \mu^{-1}(f) = \bigcup_{z \in A_K(f)} z \cdot C,$$

where $A_K(f) = Z_K(f)/Z_K(f)_e$, the component group of the centralizer of f in K. Furthermore, all components of $\mu^{-1}(f)$ in $\overline{T^*_{\mathcal{Q}}\mathfrak{B}}$ occur in this expression. (In other words, $A_K(f)$ acts transitively on the set of components of $\mu^{-1}(f)$ contained in $\overline{T^*_{\mathcal{Q}}\mathfrak{B}}$.)

Now suppose that $\mathcal{Q} = K \cdot \mathfrak{b}$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, is a closed orbit in $\mathfrak{B}$. Then $\overline{T^*_{\mathcal{Q}}\mathfrak{B}} = T^*_{\mathcal{Q}}\mathfrak{B}$. This conormal bundle may be identified with the homogeneous bundle

$$T^*_{\mathcal{Q}}\mathfrak{B} \simeq K \underset{B \cap K}{\times} \big(\mathfrak{n}^- \cap \mathfrak{p}\big).$$

Then $\gamma_{\mathcal{Q}}$ is given by the formula $\gamma_{\mathcal{Q}}(k, \xi) = k \cdot \xi$. The image of $\gamma_{\mathcal{Q}}$ is $K \cdot (\mathfrak{n}^- \cap \mathfrak{p})$, and there is an $f \in \mathfrak{n}^- \cap \mathfrak{p}$ so that $\overline{K \cdot f} = \mathrm{im}(\gamma_{\mathcal{Q}})$. We say that such an element f is *generic in* $\mathfrak{n}^- \cap \mathfrak{p}$. The fibers may be identified with subvarieties of $\mathfrak{B}_K = K \cdot \mathfrak{b}$ (the flag variety for K) via the natural projection $\pi : K \times_{B \cap K} (\mathfrak{n}^- \cap \mathfrak{p}) \to \mathfrak{B}_K$. It follows that

$$\gamma_{\mathcal{Q}}^{-1}(f) \simeq N_K\big(f, \mathfrak{n}^- \cap \mathfrak{p}\big)^{-1} \cdot \mathfrak{b},$$

where

$$N_K\big(f, \mathfrak{n}^- \cap \mathfrak{p}\big) := \big\{k \in K\colon\ k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\big\}.$$

This is spelled out in more detail in [1, Section 2].

The above well-known facts allow us to conclude the following.

Proposition 2. *If $\mathcal{Q} = K \cdot \mathfrak{b}$ is a closed orbit and C is a component of $\mu^{-1}(f)$ ($f \in \mathcal{N}_\theta$) contained in $T^*_{\mathcal{Q}}\mathfrak{B}$, then*

$$\gamma_{\mathcal{Q}}^{-1}(f) = \bigcup_{z \in A_K(f)} z \cdot C.$$

One point still needs to be checked. Suppose $\mathfrak{b}' \in \gamma_{\mathcal{Q}}^{-1}(f)$. Then, since $\mathcal{Q}$ is closed, there is $k \in N_K(f, \mathfrak{n}^- \cap \mathfrak{p})$ so that $\mathfrak{b}' = k^{-1} \cdot \mathfrak{b}$. Since we may assume that $\mathfrak{b} \in C$, it follows that $\mathfrak{b}' \in (K \cdot C) \cap \mu^{-1}(f)$. Therefore, $\mathfrak{b}' \in \bigcup_{z \in A_K(f)} z \cdot C$.

2.2. Realizations of the pairs.

Each of the pairs (G, K) of (1) fall into one of the five types

$$\big(Sp(2n), GL(n)\big), \tag{C}$$

$$\big(O(2n+1), O(2p) \times O(2q+1)\big), \tag{B1}$$

$$\big(O(2n+1), O(2p+1) \times O(2q)\big), \tag{B2}$$

$$\big(O(2n), O(2p) \times O(2q)\big), \tag{D1}$$

$$\big(O(2n+2), O(2p+1) \times O(2q+1)\big). \tag{D2}$$

In each case $n = p + q$. We shall refer to these five pairs as being of types C, B1, B2, D1 and D2, respectively.

In this section we give realizations of each of the pairs. For our realizations of the symplectic and orthogonal groups we will use the $r \times r$ matrix

$$S_r := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \tag{3}$$

for various values of r. This matrix gives an automorphism $\eta_r(A) = -\mathrm{Ad}(S_r)A^t$ of $M_{r\times r}(\mathbf{C})$. We also use the matrix $I_{p,q} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, having p ones and q negative ones.

We consider each pair in a case by case manner and describe the realization we will use. Weight vectors in $\mathfrak{p}$ are written down explicitly.

Type C. Consider $(G, K) = (Sp(2n, \mathbf{C}), GL(n, \mathbf{C}))$. Let S_n be the $n \times n$-matrix of (3). We take G to be the complex symplectic group defined by the symplectic form ω having matrix

$$J = \begin{pmatrix} 0 & S_n \\ -S_n & 0 \end{pmatrix} \tag{4}$$

with respect to a basis $\{e_i\}$ of $\mathbf{C}^{2n}$. Therefore,

$$\mathfrak{g}=\left\{\begin{pmatrix} A & B\\ C & \eta_n(A)\end{pmatrix}:\ A,B,C\in\mathfrak{gl}(n,\mathbf{C}),\ \eta_n(B)=-B \text{ and } \eta_n(C)=-C\right\}.$$

Let Θ be conjugation of G by $I_{n,n}$ and $\theta=\mathrm{Ad}(I_{n,n})$. It follows that

$$K=G^{\Theta}=\left\{\begin{pmatrix} a & 0\\ 0 & (\mathrm{Ad}(S_n)a^t)^{-1}\end{pmatrix}\right\}\simeq GL(n,\mathbf{C}) \quad\text{and}$$
$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & B\\ C & 0\end{pmatrix}:\ \eta_n(B)=-B \text{ and } \eta_n(C)=-C\right\}.$$

Then

$$\mathfrak{h}=\left\{\mathrm{diag}(t_1,\dots,t_n,-t_n,\dots,-t_1):\ t_i\in\mathbf{C}\right\}$$

is a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$. Let $\mu_i\in\mathfrak{t}^*$ be defined by

$$\mu_i\big(\mathrm{diag}(t_1,\dots,t_n,-t_n,\dots,-t_1)\big)=t_i,\quad \text{for } 1\leqslant i\leqslant n.$$

We fix a positive system of roots in $\mathfrak{k}$ by setting

$$\Delta_c^+:=\{\mu_i-\mu_j:\ 1\leqslant i<j\leqslant n\}.$$

The set of weights of $\mathfrak{h}$ in $\mathfrak{p}$ is

$$\Delta(\mathfrak{p})=\left\{\pm(\mu_i+\mu_j):\ 1\leqslant i\leqslant j\leqslant n\right\}.$$

Let $E_{i,j}$ be the standard basis for $M_{2n\times 2n}(\mathbf{C})$, i.e., $E_{i,j}$ is the matrix having 1 in the (i,j) place and 0's elsewhere. Then a choice of $\mathfrak{h}$-weight vectors in $\mathfrak{p}$ is given by

$$\begin{aligned} X_{i+j}&:=E_{i,2n-j+1}+E_{j,2n-i+1},\quad \text{for } i\neq j,\\ X_{2\cdot i}&:=E_{i,2n-i+1},\quad \text{for } 1\leqslant i\leqslant n,\\ X_{-(i+j)}&:=(X_{i+j})^t \quad\text{and}\quad X_{-2\cdot i}:=(X_{2\cdot i})^t. \end{aligned}\tag{5}$$

The weight vectors act on the basis $\{e_k\}$ by

$$\begin{aligned} X_{i+j}(e_k)&=\delta_{2n-i+1,k}e_j+\delta_{2n-j+1,k}e_i,\\ X_{-(i+j)}(e_k)&=\delta_{i,k}e_{2n-j+1}+\delta_{j,k}e_{2n-i+1}. \end{aligned}\tag{6}$$

There is an involution τ of $\{1,2,\dots,2n\}$ defined by $e_{\tau(i)}=\pm J(e_i)$. (Therefore $\tau(i)=2n-i+1$.) The following observation will be used. Suppose that S_1 is a τ-invariant subset of $\{1,2,\dots,2n\}$. Set $W_1:=\mathrm{span}_{\mathbf{C}}\{e_i:\ i\in S_1\}$. Since S_1 is τ-invariant, ω is nondegenerate on W_1 and $\mathbf{C}^{2n}=V_1\oplus W_1$, where $V_1:=(W_1)^{\perp}=\mathrm{span}_{\mathbf{C}}\{e_i:\ i\notin S_1\}$.

We now turn to the orthogonal cases $(O(\hat{n}),O(\hat{p})\times O(\hat{q}))$. Consider the matrices $S_{\hat{p}}$ and $S_{\hat{q}}$, and the corresponding automorphisms $\eta_{\hat{p}}$ and $\eta_{\hat{q}}$. The orthogonal group $O(\hat{n})$ is realized as the isometry group of the nondegenerate symmetric form $(\,,)$ having matrix

$$\begin{pmatrix} S_{\hat{p}} & 0 \\ 0 & S_{\hat{q}} \end{pmatrix} \tag{7}$$

with respect to a basis $\{e_k\}$. Then the Lie algebra may be written in block form as

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -S_{\hat{q}} B^t S_{\hat{p}} & D \end{pmatrix} : \eta_{\hat{p}}(A) = A \text{ and } \eta_{\hat{q}}(D) = D \right\}.$$

Observe that $\eta_{\hat{p}}(A) = A$ means that A is 'skew symmetric with respect to the anti-diagonal'.

Let Θ be conjugation of G by $I_{\hat{p},\hat{q}}$, so $\theta = \mathrm{Ad}(I_{\hat{p},\hat{q}})$. Then

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : aS_{\hat{p}}a^t = S_{\hat{p}} \text{ and } dS_{\hat{q}}d^t = S_{\hat{q}} \right\} \simeq O(\hat{p}) \times O(\hat{q}),$$
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ -S_{\hat{q}} B^t S_{\hat{p}} & 0 \end{pmatrix} : B \in M_{\hat{p}\times\hat{q}}(\mathbf{C}) \right\}$$

and the diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$.

There is an involution τ of $\{1, 2, \dots, \hat{n}\}$, given by

$$\tau(i) = \begin{cases} \hat{p} - i + 1, & 1 \leqslant i \leqslant \hat{p}, \\ \hat{n} - (i - \hat{p}) + 1, & \hat{p} + 1 \leqslant i \leqslant \hat{n}, \end{cases}$$

having the properties that if $S_1 \subset \{1, 2, \dots, \hat{n}\}$ is τ-stable, then the symmetric form defining G is nondegenerate on $W_1 = \mathrm{span}_{\mathbf{C}}\{e_i\colon i \in S_1\}$ and $V_1 := W_1^{\perp} = \mathrm{span}_{\mathbf{C}}\{e_i\colon i \notin S_1\}$.

For each of the four orthogonal cases we fix a positive system of roots of $\mathfrak{h}$ in $\mathfrak{k}$ and we specify a basis of $\mathfrak{h}$-weight vectors in $\mathfrak{p}$.

Type B1. The pair is $(O(2n+1), O(2p) \times O(2q+1))$. The Cartan subalgebra $\mathfrak{h}$ is

$$\mathfrak{h} = \left\{ \mathrm{diag}(t_1, \dots, t_p, -t_p, \dots, -t_1, t_{p+1}, \dots, t_n, 0, -t_n, \dots, -t_{p+1})\colon t_i \in \mathbf{C} \right\}.$$

Set

$$\mu_i\big(\mathrm{diag}(t_1, \dots, t_p, -t_p, \dots, -t_1, t_{p+1}, \dots, t_n, 0, -t_n, \dots, -t_{p+1})\big) = t_i,$$

for $1 \leqslant i \leqslant n$. We let

$$\Delta_c^+ := \{\mu_i \pm \mu_j\colon 1 \leqslant i < j \leqslant p \text{ or } p+1 \leqslant i < j \leqslant n\} \cup \{\mu_j\colon p+1 \leqslant j \leqslant n\}$$

be our fixed positive system of roots in $\mathfrak{k}$. The set of $\mathfrak{h}$-weights in $\mathfrak{p}$ is

$$\Delta(\mathfrak{p}) = \left\{ \pm(\mu_i \pm \mu_j)\colon 1 \leqslant i \leqslant p < j \leqslant n \right\} \cup \{\mu_i\colon 1 \leqslant i \leqslant p\}.$$

We choose the following weight vectors in $\mathfrak{p}$:

$$\begin{aligned} X_{i-j} &:= E_{i,p+j} - E_{2n+p-j+2,2p-i+1} \\ X_{i+j} &:= E_{i,2n+p-j+2} - E_{p+j,2p-i+1}, \\ X_i &:= E_{i,n+p+1} - E_{n+p+1,2p-i+1}, \\ X_{-(i\pm j)} &:= (X_{i\pm j})^t \quad \text{and} \quad X_{-i} := (X_i)^t, \end{aligned} \tag{8}$$

for $1 \leqslant i \leqslant p < j \leqslant n$. Formulas analogous to (6) hold.

Type B2. The pair is $(O(2n+1), O(2p+1) \times O(2q))$. Now the Cartan subalgebra $\mathfrak{h}$ is

$$\mathfrak{h} = \{\operatorname{diag}(t_1, \dots, t_p, 0, -t_p, \dots, -t_1, t_{p+1}, \dots, t_n, -t_n, \dots, -t_{p+1})\}.$$

Set

$$\mu_i(\operatorname{diag}(t_1, \dots, t_p, 0, -t_p, \dots, -t_1, t_{p+1}, \dots, t_n, -t_n, \dots, -t_{p+1})) = t_i,$$

for $1 \leqslant i \leqslant n$. We fix a positive system of roots in $\mathfrak{k}$ by

$$\Delta_c^+ := \{\mu_i \pm \mu_j\colon 1 \leqslant i < j \leqslant p \text{ or } p+1 \leqslant i < j \leqslant n\} \cup \{\mu_i\colon 1 \leqslant i \leqslant p\}.$$

The set of $\mathfrak{h}$-weights in $\mathfrak{p}$ is

$$\Delta(\mathfrak{p}) = \{\pm(\mu_i \pm \mu_j)\colon 1 \leqslant i \leqslant p < j \leqslant n\} \cup \{\pm\mu_j\colon p+1 \leqslant j \leqslant n\}.$$

A basis of weight vectors in $\mathfrak{p}$ may be written as

$$\begin{aligned}
X_{i-j} &:= E_{i,p+j+1} - E_{2n+p-j+2,2p-i+2}, \\
X_{i+j} &:= E_{i,2n+p-j+2} - E_{p+j+1,2p-i+2}, \\
X_j &:= E_{p+1,2n+p-j+2} - E_{p+j+1,p+1}, \\
X_{-(i\pm j)} &:= (X_{i\pm j})^t \quad \text{and} \quad X_{-j} := (X_j)^t,
\end{aligned}$$

for $1 \leqslant i \leqslant p < j \leqslant n$.

Type D1. The pair is $(O(2n), O(2p) \times O(2q))$, $n = p+q$. The Cartan subalgebra $\mathfrak{h}$ is

$$\mathfrak{h} = \{\operatorname{diag}(t_1, \dots, t_p, -t_p, \dots, -t_1, t_{p+1}, \dots, t_n, -t_n, \dots, -t_{p+1})\}.$$

Set

$$\mu_i(\operatorname{diag}(t_1, \dots, t_p, -t_p, \dots, -t_1, t_{p+1}, \dots, t_n, -t_n, \dots, -t_{p+1})) = t_i,$$

for $1 \leqslant i \leqslant n$. We fix a positive system of roots of $\mathfrak{h}$ in $\mathfrak{k}$ by

$$\Delta_c^+ := \{\mu_i \pm \mu_j\colon 1 \leqslant i < j \leqslant p \text{ or } p+1 \leqslant i < j \leqslant n\}.$$

The set of weights of $\mathfrak{h}$ in $\mathfrak{p}$ is

$$\Delta(\mathfrak{p}) = \{\pm(\mu_i \pm \mu_j)\colon 1 \leqslant i \leqslant p < j \leqslant n\}.$$

A basis of weight vectors may be written as

$$\begin{aligned}
X_{i-j} &:= E_{i,p+j} - E_{2n+p-j+1,2p-i+1}, \\
X_{i+j} &:= E_{i,2n+p-j+1} - E_{p+j,2p-i+1}, \quad \text{and} \\
X_{-(i\pm j)} &:= (X_{i\pm j})^t,
\end{aligned} \tag{9}$$

for $1 \leqslant i \leqslant p < j \leqslant n$.

Type D2. The pair is $(O(2n+2), O(2p+1)\times O(2q+1))$, $n=p+q$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ is

$$\mathfrak{h}=\big\{\mathrm{diag}(t_1,\dots,t_p,0,-t_p,\dots,-t_1,t_{p+1},\dots,t_n,0,-t_n,\dots,-t_{p+1})\big\}.$$

Note that unlike the other cases, $\mathfrak{h}$ is not a Cartan subalgebra of $\mathfrak{g}$. Set

$$\mu_i\big(\mathrm{diag}(t_1,\dots,t_p,0,-t_p,\dots,-t_1,t_{p+1},\dots,t_n,0,-t_n,\dots,-t_{p+1})\big)=t_i,$$

for $1\leqslant i\leqslant n$. We fix a positive system of roots of $\mathfrak{h}$ in $\mathfrak{k}$ by

$$\Delta_c^+:=\{\mu_i\pm\mu_j\colon 1\leqslant i<j\leqslant p \text{ or } p+1\leqslant i<j\leqslant n\}\cup\{\mu_k\colon 1\leqslant k\leqslant n\}.$$

The set of weights of $\mathfrak{h}$ in $\mathfrak{p}$ is

$$\Delta(\mathfrak{p})=\big\{\pm(\mu_i\pm\mu_j)\colon 1\leqslant i\leqslant p<j\leqslant n\big\}\cup\{\pm\mu_k\colon 1\leqslant k\leqslant n\}\cup\{0\}.$$

A basis of weight vectors may be written as

$$\begin{aligned}
X_{i-j}&:=E_{i,p+j+1}-E_{2n+p-j+3,2p-i+2},\\
X_{i+j}&:=E_{i,2n+p-j+3}-E_{p+j+1,2p-i+2},\\
X_i&:=E_{i,n+p+2}-E_{n+p+2,2p-i+2},\\
X_j&:=E_{p+1,2n+p-j+3}-E_{p+j+1,p+1},\\
X_0&:=E_{p+1,n+p+2}-E_{n+p+2,p+1},\\
X_{-(i\pm j)}&:=(X_{i\pm j})^t \quad\text{and}\quad X_{-k}:=(X_k)^t,
\end{aligned}$$

for $1\leqslant i\leqslant p<j\leqslant n$ and $1\leqslant k\leqslant n$.

2.3. Embeddings of (G,K) into $(GL(\hat{n}), GL(\hat{p})\times GL(\hat{q}))$

Consider the pair

$$(\hat{G},\hat{K})=\big(GL(\hat{n}),GL(\hat{p})\times GL(\hat{q})\big), \tag{10}$$

where $\hat{K}$ is the fixed point group of conjugation by $I_{\hat{p},\hat{q}}$. By embedding each of our pairs (G,K) into a pair $(\hat{G},\hat{K})$, for appropriate choices of $\hat{p},\hat{q}$ and $\hat{n}$, we will be able to apply results of [1] to our study of (G,K). Our realizations of the pairs given in Section 2.2 give $G\subset\hat{G}$ and $K=\hat{K}\cap G$. Here the appropriate choices of $\hat{p},\hat{q}$ and $\hat{n}$ are $n,n,2n$ for type C, $2p,2q+1,2n+1$ for type B1, $2p+1,2q,2n+1$ for type B2, $2p,2q,2n$ for type D1, and $2p+1,2q+1,2n+2$ for type D2. We record some properties of this embedding below.

First, consider the pair $(\hat{G},\hat{K})$ and let $\hat{\mathfrak{h}}$ be the subalgebra of diagonal matrices. Then $\hat{\mathfrak{h}}$ is a Cartan subalgebra of both $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}$. Let $\epsilon_i\in\hat{\mathfrak{h}}^*$ be defined by $\epsilon_i(\mathrm{diag}(t_1,\dots,t_{\hat{n}}))=t_i$. Then the root systems for $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{k}}$ are, respectively,

$$\begin{aligned}
\hat{\Delta}&=\Delta(\hat{\mathfrak{h}},\hat{\mathfrak{g}})=\big\{\pm(\epsilon_i-\epsilon_j)\colon 1\leqslant i<j\leqslant\hat{n}\big\}\quad\text{and}\\
\hat{\Delta}_c&=\Delta(\hat{\mathfrak{h}},\hat{\mathfrak{k}})=\big\{\pm(\epsilon_i-\epsilon_j)\colon 1\leqslant i<j\leqslant\hat{p}\text{ or }\hat{p}+1\leqslant i<j\leqslant\hat{n}\big\}.
\end{aligned}$$

We fix once and for all a positive system of roots in $\mathfrak{k}$ by setting

$$\hat{\Delta}_c^+ = \{\epsilon_i - \epsilon_j\colon 1 \leqslant i < j \leqslant \hat{p} \text{ or } \hat{p}+1 \leqslant i < j \leqslant \hat{n}\}.$$

For the Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{k}$ and fixed positive systems Δ_c^+ for each type, as given in Section 2.2, the following properties are apparent.

(1) $\mathfrak{h} = \hat{\mathfrak{h}} \cap \mathfrak{k}$.
(2) Root vectors for $\mathfrak{g}$ are either root vectors for $\hat{\mathfrak{g}}$ or sums of two root vectors for $\hat{\mathfrak{g}}$.
(3) $\Delta_c^+ = \{\alpha|_{\mathfrak{h}}\colon \alpha \in \hat{\Delta}_c^+\}$.

Now suppose that $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then there is a regular $\lambda \in \mathfrak{h}^*$ that defines $\mathfrak{b}$ in the sense that

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-, \quad \mathfrak{n}^- = \sum_{\langle \lambda, \alpha\rangle > 0} \mathfrak{g}^{(-\alpha)}.$$

In fact, it suffices to consider only λ's that are Weyl group conjugates of $(n, n-1, \dots, 2, 1) = \sum (n-i+1)\mu_i$. For each such λ there exists a regular $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ so that $\hat{\lambda}|_{\mathfrak{h}} = \lambda$. A Borel subalgebra $\hat{\mathfrak{b}}$ is thus determined by $\hat{\lambda}$. It follows that $\mathfrak{b} = \hat{\mathfrak{b}} \cap \mathfrak{g}$. In the five cases $\hat{\lambda}$ is given by

$$\begin{aligned}
\hat{\lambda} &= \frac{1}{2}(\lambda_1, \dots, \lambda_n \mid -\lambda_n, \dots, -\lambda_1), \quad \text{in type C},\\
\hat{\lambda} &= \frac{1}{2}(\lambda_1, \dots, \lambda_p, -\lambda_p, \dots, -\lambda_1 \mid \lambda_{p+1}, \dots, \lambda_n, 0, -\lambda_n, \dots, -\lambda_{p+1}), \quad \text{in type B1},\\
\hat{\lambda} &= \frac{1}{2}(\lambda_1, \dots, \lambda_p, 0, -\lambda_p, \dots, -\lambda_1 \mid \lambda_{p+1}, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_{p+1}), \quad \text{in type B2},\\
\hat{\lambda} &= \frac{1}{2}(\lambda_1, \dots, \lambda_p, -\lambda_p, \dots, -\lambda_1 \mid \lambda_{p+1}, \dots, \lambda_n, -\lambda_n, \dots, -\lambda_{p+1}), \quad \text{in type D1},\\
\hat{\lambda} &= \frac{1}{2}(\lambda_1, \dots, \lambda_p, \epsilon, -\lambda_p, \dots, -\lambda_1 \mid \lambda_{p+1}, \dots, \lambda_n, -\epsilon, -\lambda_n, \dots, -\lambda_{p+1}), \quad \text{in type D2}.
\end{aligned}$$

In the last case $|\epsilon| < \lambda_i$, for all i. Note that there are two choices for $\hat{\lambda}$, corresponding to ϵ being positive or negative.

Recall that $\mathfrak{B}$ is the flag variety for G and let $\hat{\mathfrak{B}}$ be the flag variety for $\hat{G}$. Then we may view $\mathfrak{B}$ as a closed subvariety of $\hat{\mathfrak{B}}$. If $\lambda, \hat{\lambda}$ are as above, and $\mathfrak{b}, \hat{\mathfrak{b}}$ are the corresponding Borel subalgebras, then the K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$ satisfies $\mathcal{Q} = \hat{\mathcal{Q}} \cap \mathfrak{B}$, where $\hat{\mathcal{Q}} = \hat{K} \cdot \hat{\mathfrak{b}}$. We have described a natural way to associate to each closed K-orbit in $\mathfrak{B}$ a closed $\hat{K}$-orbit in $\hat{\mathfrak{B}}$.

2.4. Nilpotent orbits

We recall the parameterization of K-orbits in $\mathcal{N}_\theta$ in terms of signed tableaux for each of our five pairs. This may be found, for example, in [6]. Let $Y \in \mathcal{N}_\theta$ and let $\{X, H, Y\}$ be a standard triple with $X \in \mathfrak{p}$ and $H \in \mathfrak{h}$. Denote the copy of $\mathfrak{sl}(2)$ spanned by $\{X, H, Y\}$ by $\mathfrak{sl}(2)_Y$. Write the decomposition of $\mathbf{C}^{\hat{n}}$ into irreducible $\mathfrak{sl}(2)_Y$ subrepresentations as $\oplus W_i$. Order the constituents so that $\dim(W_i) \geqslant \dim(W_{i+1})$. Then the tableau associated to $K \cdot Y$ has $\dim(W_i)$ boxes in the ith row. The parameterization of K-orbits in $\mathcal{N}_\theta$ differs for the symplectic and orthogonal cases.

Type C. In this case the number of rows of a given odd length is even. The W_i may be chosen to be stable under $I_{n,n}$. Then the sign in the first box of the ith row is the sign of the eigenvalue of $I_{n,n}$ on the lowest weight vector of the $\mathfrak{sl}(2)_Y$-representation W_i. Signs are then filled in so as to alternate along each row. Of the even number of odd length rows, half will begin with a $+$ sign and half with a $-$ sign.

Types B1, B2, D1 and D2. In this case the number of rows of a given even length is even. The W_i may be chosen to be stable under $I_{2p+1,2q}$, $I_{2p,2q+1}$, $I_{2p,2q}$ or $I_{2p+1,2q+1}$ (depending on the type). The signs are placed in exactly the same manner as for type C. Of the even number of even length rows, half will begin with a $+$ sign and half with a $-$ sign.

In all cases there is a one-to-one correspondence between the set of K-orbits in $\mathcal{N}_\theta$ and the set of such signed tableaux, up to permutation of the rows of a given length.

3. Generic elements

3.1. The algorithm for finding a generic element

We continue to consider pairs (G, K) having types C, B1, B2, D1 and D2 and begin by parameterizing the closed K-orbits in $\mathfrak{B}$.

A K-orbit $\mathcal{Q}$ in the flag variety $\mathfrak{B}$ is closed if and only if each Borel subalgebra in $\mathcal{Q}$ is θ-stable (and contains a fundamental Cartan subalgebra); see for example [9, Lemma 5.9]. When $\mathrm{rank}(\mathfrak{k}) = \mathrm{rank}(\mathfrak{g})$, each closed orbit is therefore $K \cdot \mathfrak{b}$, where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, with $\mathfrak{h}$ a compact Cartan subalgebra. One easily sees in this case (from, for example, [9, Lemma 5.3]) that the closed K-orbits are parameterized by $W/W(K)$, where $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ and $W(K) = N_K(\mathfrak{h})/Z_K(\mathfrak{h})$, the Weyl groups in G and K. We consider our pairs (1) and take $\mathfrak{h}$ to be the compact Cartan subalgebra specified in Section 2.2. For type C, W (resp., $W(K)$) is the Weyl group of the root system Δ (resp., Δ_c), so the closed orbits are in one-to-one correspondence with the W-conjugates of $(n, n-1, \dots, 2, 1)$ that are Δ_c^+-dominant. In types B1, B2 and D1, both W and $W(K)$ contain *all* sign changes of the ϵ_i, due to the disconnectedness of G and K. Letting S_n denote the group of permutations of the ϵ_i (and similarly for S_p and S_q), it follows that $W/W(K) \simeq S_n/S_p \times S_q$. Therefore, the K-orbits are in one-to-one correspondence with the S_n-conjugates of $(n, n-1, \dots, 2, 1)$ that are Δ_c^+ dominant. A little more needs to be said for type D2, since $\mathrm{rank}(\mathfrak{k}) \neq \mathrm{rank}(\mathfrak{g})$. Let $\mathfrak{a} = \{a(E_{p+1,p+1} - E_{n+p+2,n+p+2})\colon\ a \in \mathbf{C}\}$. Then $\mathfrak{h} + \mathfrak{a}$ is a θ-stable fundamental Cartan subalgebra of $\mathfrak{g}$. We consider $W = N_G(\mathfrak{h} + \mathfrak{a})/Z_G(\mathfrak{h} + \mathfrak{a})$ and $W(K) = N_K(\mathfrak{h} + \mathfrak{a})/Z_K(\mathfrak{h} + \mathfrak{a})$. The closed orbits are of the form $K \cdot \mathfrak{b}$, $\mathfrak{b} = (\mathfrak{h} + \mathfrak{a}) + \mathfrak{n}^-$, with $\mathfrak{n}^-$ corresponding to a θ-stable positive system Δ^+. Suppose Δ^+ is defined by the (regular) element $(n, n-1, \dots, 2, 1) \in \mathfrak{h}^*$. Then for $w \in W$, $w\Delta^+$ is θ-stable if and only if $\theta \cdot w = w$; we denote these Weyl group elements by W^θ. But $\theta \cdot w = w$ implies that $w(\mathfrak{h}^*) = \mathfrak{h}^*$ and $w(\mathfrak{a}^*) = \mathfrak{a}^*$. It follows that $w\Delta^+$ is defined by the (regular) element $w(n, n-1, \dots, 2, 1) \in \mathfrak{h}^*$. Again, owing to the disconnectedness of G and K, all sign changes are in both W^θ and $W(K)$, and $W^\theta/W(K) \simeq S_n/S_p \times S_q$. It follows that the closed orbits are again the S_n-conjugates of $(n, n-1, \dots, 2, 1)$ that are Δ_c^+-dominant. Therefore, we let

$$\begin{aligned} &\lambda = (a_1, \dots, a_p \mid b_1, \dots, b_q) \quad \text{with} \\ &\quad a_1 > \dots > a_p > 0 > b_1 > \dots > b_q \quad (\text{for some } p, q), \text{for type C}, \\ &\quad a_1 > \dots > a_p > 0 \quad \text{and} \quad b_1 > \dots > b_q > 0, \text{otherwise}. \end{aligned} \tag{11}$$

Then λ determines a positive system of roots $\Delta^+ = \{\alpha : \langle \lambda, \alpha \rangle > 0\}$. This gives a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$, $\mathfrak{n}^- := \sum_{\Delta^+} \mathfrak{g}^{(-\alpha)}$ and a closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b} \subset \mathfrak{B}$. Note that $\alpha \in \Delta^+$ implies $\alpha|_{\mathfrak{h}} \in \Delta_c^+$.

In order to construct a generic element f in $\mathfrak{n}^- \cap \mathfrak{p}$ we associate to λ an array of n numbered dots. This array consists of two horizontal rows of dots as follows. Begin with the coordinate of λ having greatest absolute value. (The absolute value is an issue only in type C.) Note that this coordinate is either the first or the last in type C, and is either a_1 or b_1 otherwise. Begin the array by placing a dot in the upper row if this coordinate is among the a_i's and in the lower row otherwise. Next locate the coordinate with next greatest absolute value (equal to $n-1$) and place a dot to the right of the first dot, and in the upper row if this coordinate is among the a_i's and in the lower row otherwise. Continue in this manner until there are no more coordinates (and n dots have been placed in the array).

Now label each dot with the index and sign of the corresponding coordinate. This amounts to always labelling the dots in the top row with $1, 2, \dots, p$ from left to right. In type C label the dots of the lower row by $-n, -(n-1), \dots, -(p+1)$ from left to right; in the other types label the dots of the lower row by $p+1, p+2, \dots n$ from left to right.

Here are two examples. In type C take $\lambda = (7, 6, 4, 3, -1, -2, -5)$. Then $p = 4$ and $q = 3$, and the corresponding array is

$$\begin{array}{ccccccc} 1 & 2 & & 3 & 4 & & \\ \bullet & \bullet & & \bullet & \bullet & & \\ & & & & & & \\ & & \bullet & & & \bullet & \bullet \\ & & -7 & & & -6 & -5 \end{array}.$$

For the other four types, let $\lambda = (8, 5, 4, 1 \mid 7, 6, 3, 2)$. Then the array is

$$\begin{array}{cccccccc} 1 & & & 2 & 3 & & & 4 \\ \bullet & & & \bullet & \bullet & & & \bullet \\ & & & & & & & \\ & \bullet & \bullet & & & \bullet & \bullet & \\ & 5 & 6 & & & 7 & 8 & \end{array}.$$

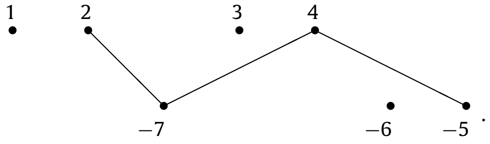
We define a *block* in the array to be a maximal set of consecutive dots lying in just one row. The blocks in the first example above are the sets of dots with labels $\{1, 2\}, \{-7\}, \{3, 4\}$ and $\{-5, -6\}$. A *string* through the array will be a sequence of dots with at most one dot in each block; they will have slightly different forms for the five types.

The algorithm for constructing our generic f in $\mathfrak{n}^- \cap \mathfrak{p}$ begins by specifying f_0. Then there is a reduction to a smaller array, where f_1 is chosen. This is continued to obtain $f_0, f_1, \dots,$ and $f = f_0 + f_1 + \cdots$. Therefore it suffices to specify f_0 and the smaller array (and the type for this smaller array). There are slight differences for the five types, so we will do this separately for each type.

Type C. Form a string through the array by choosing the dot in each block that is farthest to the right in the block. Let $k_1, \dots, k_l$ be the labels of the dots in the string, ordered so that each k_i is to the left of k_{i+1} in the array. Set

$$f_0 = \left(\sum_{i=1}^{l-1} X_{k_{i+1}-k_i} \right) + X_{-2k_l}. \tag{12}$$

Note that the weight vectors are in $\mathfrak{n}^- \cap \mathfrak{p}$. Also note that if there is just one block, then the string will have just one dot (so $l = 1$ and $f_0 = X_{-2k_l}$). In the example, connecting the dots in the string, we have:



(13)

Also $\{k_1, k_2, k_3, k_4\} = \{2, -7, 4, -5\}$ and

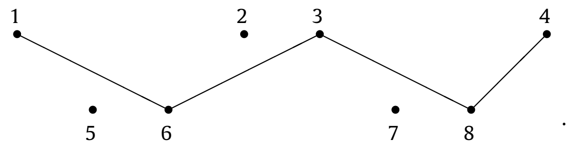
$$f_0 = X_{-(2+7)} + X_{7+4} + X_{-(4+5)} + X_{2\cdot 5}.$$

The smaller array is obtained by deleting all dots in the string; this smaller array is considered to be of type C.

Type B1. Here there are two subcases depending on whether the last dot in the array is in the upper or lower row. When the last dot is in the upper row, form a string exactly as above. List the labels of the string as above, then

$$f_0 = \left(\sum_{i=1}^{l-1} X_{k_{i+1}-k_i} \right) + X_{-k_l}. \tag{14}$$

In the earlier example (which is $(G, K) = (O(17), O(8) \times O(9))$ in type B1) the array with string is



Then $f_0 = X_{6-1} + X_{3-6} + X_{8-3} + X_{4-8} + X_{-4}$.

In the subcase for which the last dot lies in the lower row, form a string as above except that the string passes through all blocks *other than* the last (farthest right) block. Listing the labels of the string as above, f_0 is given by formula (14). (Note that in this case, if we were to allow the string to pass through the last block, then X_{-k_l} would lie in $\mathfrak{k}$, not in $\mathfrak{p}$.)

In both subcases the smaller array is obtained by deleting the dots of the string; this smaller array is treated as type D1.

Type B2. There are also two subcases here. If the last dot in the array occurs in the upper row then the string does not pass through the last block. In the other case it does. The formula for f_0 is exactly as in (14).

Again the smaller array is obtained by deleting the dots of the string; this smaller array is treated as type D1.

Type D1. Form a string passing through each block, including the last, however if there is just one block then no string is to be formed. Listing the labels as above

$$f_0 = \left(\sum_{i=1}^{l-1} X_{k_{i+1}-k_i} \right) + X_{-(k_{l-1}+k_l)}.$$

Again the smaller array is obtained by deleting the dots of the string. This smaller array is treated as type B1 (resp., B2) when the last dot in the string is in the lower (resp., upper) row.

Type D2. Form a string passing through each block and list the labels as in the other cases. Then

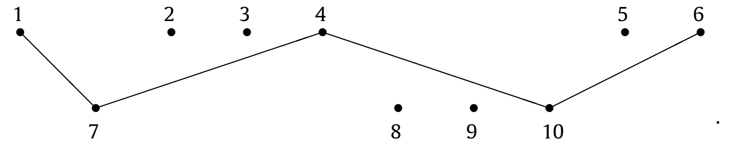
$$f_0 = \left(\sum_{i=1}^{l-1} X_{k_{i+1}-k_i} \right) + X_{-k_l}.$$

Delete the dots of the string; the smaller array is to be treated as type B1 (resp., B2) if the last dot in the string is in the lower (resp., upper) row.

For each type, f is essentially constructed by choosing f_1 in the smaller array in the same way that f_0 was chosen in the original array. However, there is an identification needed here. When (G,K) is of type D1, the smaller array is of type B1 or B2. The algorithm calls for a weight vector of the type X_{-k_l}, which does not appear to lie in the even orthogonal Lie algebra. The point is that in type D1, the reduction to type B1 or B2 (as specified above) comes from an embedding of an odd orthogonal group into the even orthogonal group; this is spelled out carefully in Section 3.2. The weight vector X_{-k_l} is in fact a sum of two root vectors in the even orthogonal Lie algebra $\mathfrak{g}$. Therefore, when constructing the $f_i, i>0$, we must do the following when f_{i-1} is constructed from an array of type D1. The weight vector X_{-k_l} of (14) must be replaced by $X_{k'-k_l} - X_{-(k'+k_l)}$, where k' is the label of the last dot in the string for the previous (type D1) array.

In this way we construct $f_0, f_1, \ldots,$ and $f = f_0 + f_1 + \cdots$.

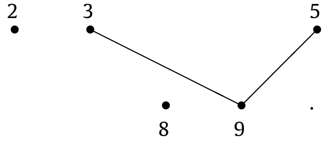
Here is an example. Consider type B1 with $\lambda = (10,8,7,6,2,1 \mid 9,5,4,3)$. The pair is therefore $(G,K) = (O(21), O(12) \times O(9))$. The array with the first string indicated is



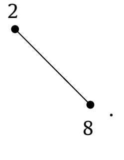
The recipe for type B1 gives

$$f_0 = X_{7-1} + X_{4-7} + X_{10-4} + X_{6-10} + X_{-6}.$$

The smaller array is the following, and is treated as type D1:



Therefore, $f_1 = X_{9-3} + X_{5-9} + X_{-(5+9)}$. Since the array for f_1 is of type D1, our smaller array is treated as type B2. The smaller array is



Eq. (14) calls for $f_2 = X_{8-2} + X_{-8}$, however we must replace X_{-8} by $X_{5-8} - X_{-(5+8)}$. Therefore, $f_2 = X_{8-2} + X_{5-8} - X_{-(5+8)}$. We have now determined $f = f_0 + f_1 + f_2$.

The proof of the following proposition is given in Section 3.3.

Proposition 15. *Given λ as in* (11) *and a corresponding closed orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$, the element f constructed above is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.*

3.2. *The doubled arrays*

To understand the algorithm for the construction of f of the previous section, as well as our description of components of the fiber (given in Section 4), it is extremely useful to consider the embeddings of (G, K) into $(\hat{G}, \hat{K})$. This leads to the formation of a 'doubled array'. This doubled array is in fact an array for the closed orbit $\hat{\mathcal{Q}}$ in $\hat{\mathfrak{B}}$ (cf. Section 2.3). Furthermore we will see that our $f \in \mathfrak{n}^- \cap \mathfrak{p}$ is (essentially) the generic element in $\hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$ constructed in [1, Section 3]. This will allow us to prove that f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.

Recall from Section 2.3 that given λ (a conjugate of $(n, n-1, \ldots, 2, 1)$) and the corresponding closed K-orbit $\mathcal{Q}$ in $\mathfrak{B}$, there is $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ and a closed $\hat{K}$-orbit $\hat{\mathcal{Q}} = \hat{K} \cdot \hat{\mathfrak{b}}$ in $\hat{\mathfrak{B}}$. We have seen how an array is determined by λ. We refer to the array for $\hat{\lambda}$ (constructed in [1] for the general linear group $\hat{G}$) as the *doubled array*.

For each of the five types we describe the following.

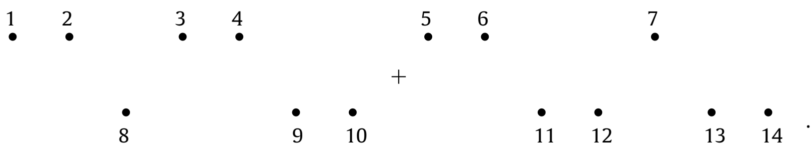
(1) The doubled array.
(2) The 'first' string through the doubled array.
(3) A subgroup $G_1 \subset G$, which forms the basis of our later induction arguments.

Type C. The doubled array is formed from the original array by reflecting about a point to the right of the array. The dots in the upper row are labelled with $1, 2, \ldots, n$ (from left to right) and the dots in the lower row are labelled with $n+1, n+2, \ldots, 2n$.

In our earlier example

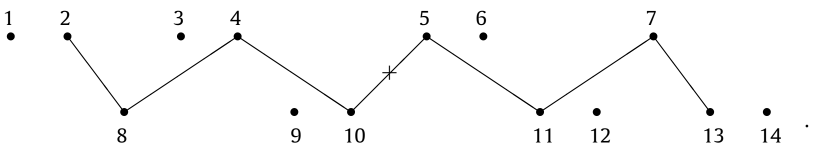
$$\lambda = (7, 6, 4, 3 \mid -1, -2, -5),$$
$$\hat{\lambda} = (7, 6, 4, 3, -1, -2, -5 \mid 5, 2, 1, -3, -4, -6, -7)$$

and the doubled array is



(Note that the original array has been reflected about the '+' sign.)

A string is formed through the doubled array by passing through the rightmost dot in each block left of center and through the leftmost dot in each block right of center. In the example this is



Therefore, the string in (13) is reflected, then connected in the middle.

The crucial observation is the following. Let the labels of the dots in the string through the doubled array be $j_1, j_2, \ldots, j_{2l}$ (with the dot labelled by j_i appearing left of the dot labelled by j_{i+1}); recall that $k_1, \ldots, k_l$ are the labels of the dots of the string in the original array. Then by Eqs. (5) we have

$$f_0 = \sum_{i=1}^{l-1} X_{k_{i+1}-k_i} + X_{-2k_l} = \sum_{i=1}^{2l-1} E_{j_{i+1}, j_i}$$

Note that

$$\begin{aligned} f_0(e_{j_i}) &= e_{j_{i+1}}, \quad \text{if } i = 1, 2, \ldots, 2l-1, \\ f_0(e_k) &= 0, \qquad \text{otherwise.} \end{aligned} \tag{16}$$

Continuing with the construction of $f_1, f_2, \ldots$ we conclude that the tableau of f coincides with that of the generic element of [1].

By the symmetry of the string, the set $S_1 = \{j_1, \ldots, j_{2l}\}$ is τ-stable. It follows that the symplectic form is nondegenerate on

$$W_1 = \operatorname{span}_{\mathbf{C}}\{e_k\colon k \in S_1\}.$$

Therefore, $\mathbf{C}^{2n} = W_1 \oplus V_1$, with $V_1 := W_1^{\perp} = \operatorname{span}_{\mathbf{C}}\{e_j\colon j \notin S_1\}$. Set

$$\begin{aligned} G_1' &= \left\{g \in G\colon g(W_1) \subset W_1 \text{ and } g|_{V_1} = Id_{V_1}\right\}, \\ G_1 &= \left\{g \in G\colon g(V_1) \subset V_1 \text{ and } g|_{W_1} = Id_{W_1}\right\}. \end{aligned}$$

Thus G_1 and G_1' are commuting complex symplectic groups of ranks $n-l$ and l.

Since f_0 lies in $\mathfrak{g}_1'$, we may choose a normal triple $\{e_0, h_0, f_0\}$ inside $\mathfrak{g}_1'$. Therefore, the $\mathfrak{sl}(2)_{f_0} := \operatorname{span}_{\mathbf{C}}\{e_0, h_0, f_0\}$ is contained in $\mathfrak{g}_1'$. Since the rank of f_0 is $2l-1$ (by (16)), f_0 is a principal nilpotent element in $\mathfrak{g}_1'$.

Now consider G_1 acting on V_1 and identify V_1 with $\mathbf{C}^{2(n-l)}$ using the ordered basis $\{e_1, e_2, \ldots, e_{2n}\} \setminus S_1$. Then the symplectic form defined by (4) restricted to V_1 has matrix of the same form as (4) (with respect to the ordered basis). Setting $K_1 = K \cap G_1$, we see that the pair (G_1, K_1) is identified with a lower rank pair of type C. Furthermore,

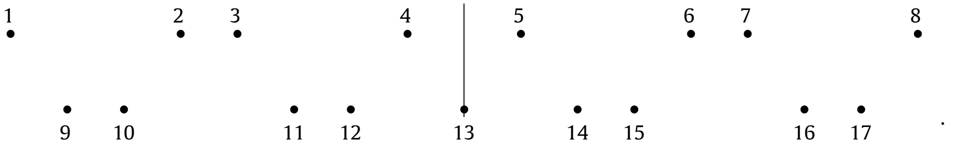
$$\hat{G}_1 = \left\{g \in \hat{G}\colon g(V_1) \subset V_1 \text{ and } g|_{W_1} = Id_{W_1}\right\}$$

may be identified with a general linear group and we have an embedding of (G_1, K_1) into $(\hat{G}_1, \hat{K}_1)$. One also sees that the restriction of λ gives a λ_1, and the array for λ_1 is obtained from that of λ by deleting dots for the first string, as described in Section 3.1. One easily checks that $\mathfrak{b}_1 := \mathfrak{b} \cap \mathfrak{g}_1$ (resp., $\mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$) is a Borel (resp., Cartan) subalgebra of $\mathfrak{g}_1$. In fact $\mathfrak{b}_1$ is defined by λ_1. Write $\mathfrak{n}_1^- = \mathfrak{n}^- \cap \mathfrak{g}_1$, then $\mathfrak{b}_1 = \mathfrak{h}_1 + \mathfrak{n}_1^-$. Then f_1 was chosen in $\mathfrak{n}_1^- \cap \mathfrak{p}$ by the same procedure that f_0 was chosen in $\mathfrak{n}^- \cap \mathfrak{p}$.

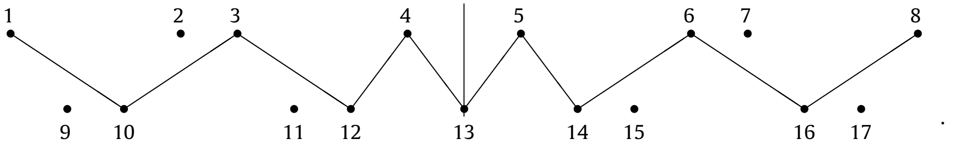
Type B1. The doubled array is formed by reflecting the original array about a vertical line to the right of the array and placing an additional dot in the lower row along the vertical line. The dots are to be numbered with $1, 2, \ldots, 2p$ along the upper row and with $2p+1, 2p+2, \ldots, 2n+1$ along the lower row. In the example of the previous section

$$\begin{aligned} \lambda &= (8, 5, 4, 1 \mid 7, 6, 3, 2), \\ \hat{\lambda} &= (8, 5, 4, 1, -1, -4, -5, -8 \mid 7, 6, 3, 2, 0, -2, -3, -6, -8) \end{aligned}$$

and the doubled array is



The first string is formed so as to pass through the blocks as for type C and to pass through the middle dot of the center block. In the example we have



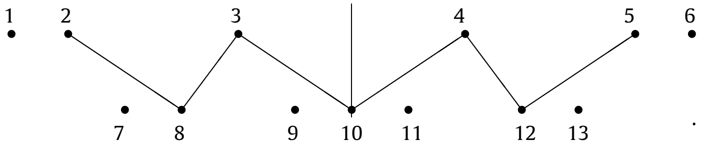
Note the symmetry of the string.

It is worth considering a second example. Take

$$\lambda = (6, 5, 2 \mid 4, 3, 1),$$

$$\hat{\lambda} = (6, 5, 2, -2, -5, -6 \mid 4, 3, 1, 0, -1, -3, -4)$$

and the doubled array is



Let $S_1 = \{j_1, j_2, \ldots, j_{2l+1}\}$ be the labels of the dots in the string (listed as they appear from left to right in the doubled array). Note that j_{l+1} labels the dot in the middle of the center block. By Eq. (8)

$$f_0 = \sum_{i=1}^{l} E_{j_{i+1}, j_i} - \sum_{i=l+1}^{2l} E_{j_{i+1}, j_i}$$

and

$$f_0(e_{j_i}) = \begin{cases} e_{j_{i+1}} & \text{if } i = 1, \ldots, l, \\ -e_{j_{i+1}} & \text{if } i = l+1, \ldots, 2l. \end{cases}$$

Therefore, as for type C, the string for f_0 passes through each block in the doubled array.

Note that the two examples illustrate the difference between the two subcases in the algorithm for type B1.

By the symmetry of the string, the set S_1 is τ-stable. Therefore the symmetric form is nondegenerate on

$$W_1 = \mathrm{span}_{\mathbf{C}}\{e_{j_1}, \ldots, e_{j_{2l+1}}\} \quad \text{and}$$

$$V_1 = (W_1)^{\perp} = \mathrm{span}_{\mathbf{C}}\{e_j \colon j \notin S_1\}.$$

Set

$$G_1' = \{g \in G\colon\ g(W_1) \subset W_1 \text{ and } g|_{V_1} = Id_{V_1}\},$$
$$G_1 = \{g \in G\colon\ g(V_1) \subset V_1 \text{ and } g|_{W_1} = Id_{W_1}\}.$$

Then G_1' and G_1 are commuting complex groups; G_1' is an odd orthogonal group of rank l and G_1 is an even orthogonal group of rank $n-l$.

As in the type C case, one may form $\mathfrak{sl}(2)_{f_0}$, a copy of $\mathfrak{sl}(2)$ which is contained in $\mathfrak{g}_1'$ and contains f_0. Since W_1 is irreducible for $\mathfrak{sl}(2)_{f_0}$, we see that f_0 is a principal nilpotent element in $\mathfrak{g}_1'$.

Now consider G_1 acting on V_1. Identify V_1 with $\mathbf{C}^{2(n-l)}$ using the ordered basis $\{e_1, e_2, \ldots, e_{2n+1}\} \setminus S_1$. Then the symmetric form defined by (7), restricted to V_1, has matrix of the same form as (7) (with respect to the ordered basis). Setting $K_1 = K \cap G_1$, we see that the pair (G_1, K_1) is identified with a pair of type D1. Furthermore,

$$\hat{G}_1 = \{g \in \hat{G}\colon\ g(V_1) \subset V_1 \text{ and } g|_{W_1} = Id_{W_1}\}$$

may be identified with a general linear group and we have an embedding of (G_1, K_1) into $(\hat{G}_1, \hat{K}_1)$. One also sees that the restriction of λ gives a λ_1, and the array for λ_1 is obtained from that of λ by deleting dots for the first string, as described in Section 3.1. A similar comment holds for $\hat{\lambda}$ and the doubled array.

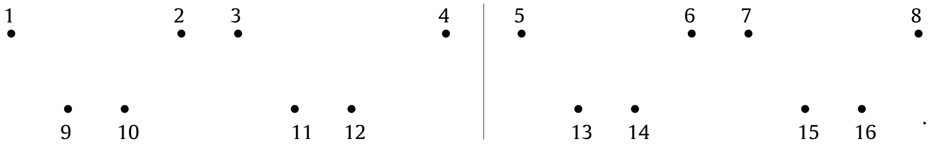
Type B2. This case is entirely analogous to the type B1 case. There is one difference. Although the doubled array is formed by reflecting the array about a vertical line right of the array, as above, the additional dot is placed in the center of the *upper* row.

Type D1. This case is slightly more complicated than the previous cases. The doubled array is formed by reflecting the array about a vertical line to the right of the array. The dots are numbered with $1, 2, \ldots, 2p$ along the upper row and by $2p+1, 2p+2, \ldots, 2n$ along the lower row.

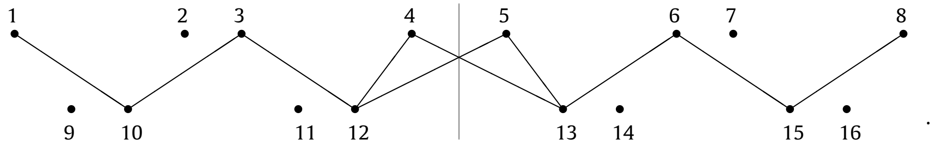
In the earlier example

$$\lambda = (8, 5, 4, 1 \mid 7, 6, 3, 2),$$
$$\hat{\lambda} = (8, 5, 4, 1, -1, -4, -5, -8 \mid 7, 6, 3, 2, -2, -3, -6, -7)$$

and the doubled array is



A string through the array is slightly different from the other cases. The string passes through the rightmost dot in each block left of the center block and through the leftmost dot in the blocks right of center, as in the other cases. However, the two dots closest to the vertical line in the center (those labelled by either $p, p+1$ or $2p+q, 2p+q+1$) are each connected to *both* dots in the string which lie in the two blocks adjoining the center block. This is illustrated in the example by



Let $S_1 = \{j_1, j_2, \dots, j_{2l}\}$ be the labels of the dots in the string (listed from left to right). Then, using Eq. (9) and the algorithm for finding f_0, we have

$$f_0 = \sum_{i=1}^{l-1} E_{j_{i+1},j_i} \pm (E_{j_{l+1},j_{l-1}} - E_{j_{l+2},j_l}) - \sum_{i=l+2}^{2l-1} E_{j_{i+1},j_i}.$$

With $+$ (resp., $-$) occurring when the dot labelled by k_l (that is the middle block of the doubled array) is in the lower (resp., upper) row. The $\pm$ signs in the following three paragraphs follow the same convention.

One easily checks that

$$\begin{aligned} f_0(e_{j_i}) &= e_{j_{i+1}}, \quad \text{for } i = 1, 2, \dots, l-2, \\ f_0(e_{j_{l-1}}) &= e_{j_l} \pm e_{j_{l+1}}, \\ f_0(e_{j_l} \pm e_{j_{l+1}}) &= \mp 2 e_{j_{l+2}}, \\ f_0(e_{j_i}) &= -e_{j_{i+1}}, \quad \text{for } i = l+2, \dots, 2l-1, \end{aligned} \tag{17}$$

and the kernel of f_0 is spanned by

$$e_{j_l} \mp e_{j_{l+1}}, e_{j_{2l}}, \quad \text{and } e_j, j \notin S_1.$$

It is a consequence that $f_0^{2l-2} \neq 0$ and $f_0^{2l-1} = 0$.

Now we describe how to pass to a lower rank orthogonal group. Note that S_1 is τ-stable. Let

$$\begin{aligned} W_1 &= \operatorname{span}_{\mathbf{C}}\{e_{j_1}, \dots, e_{j_l-1}, e_{j_l} \pm e_{j_{l+1}}, e_{j_{l+2}}, \dots, e_{j_{2l}}\}, \\ V_1 &= (W_1)^{\perp} = \operatorname{span}_{\mathbf{C}}\big(\{e_{j_l} \mp e_{j_{l+1}}\} \cup \{e_j\colon\ j \notin S_1\}\big) \end{aligned}$$

and set

$$\begin{aligned} G_1' &= \big\{g \in G\colon\ g(W_1) \subset W_1 \text{ and } g|_{V_1} = Id_{V_1}\big\}, \\ G_1 &= \big\{g \in G\colon\ g(V_1) \subset V_1 \text{ and } g|_{W_1} = Id_{W_1}\big\}. \end{aligned}$$

We have several easily verified facts. G_1 and G_1' are commuting odd orthogonal groups having ranks $n-l$ and $l-1$. $\mathfrak{h}_1' = \mathfrak{h} \cap \mathfrak{g}_1'$ is a Cartan subalgebra of $\mathfrak{g}_1'$. As in the other cases, there is a copy of $\mathfrak{sl}(2)$, which we call $\mathfrak{sl}(2)_{f_0}$, contained in $\mathfrak{g}_1'$ that contains f_0. Also, W_1 is an irreducible $\mathfrak{sl}(2)_{f_0}$ representation, by (17). Therefore, f_0 is a principal nilpotent element in $\mathfrak{g}_1'$.

Identifying V_1 with $\mathbf{C}^{2(n-l)+1}$ using the ordered basis

$$\left\{e_1, \dots, e_{k_l-1}, \frac{e_{k_l} \mp e_{k_{l+1}}}{\sqrt{\pm 2}}, e_{k_l+2}, \dots, e_{2n}\right\} \setminus \{e_k\colon\ k \in S_1\}$$

the symmetric form is as in (7). Thus, (G_1, K_1) is of type B1 (resp., B2) when the center block is in the lower (resp., upper) row. One easily sees that $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1$ is a Cartan subalgebra of $\mathfrak{g}_1$. The weights and a choice of weight vectors for $\mathfrak{p} \cap \mathfrak{g}_1$ are

$$\pm(\mu_i \pm \mu_j); X_{\pm(i\pm j)}, \quad \text{for } i \leqslant p < j, i, j \notin S_1,$$
$$\pm\mu_i; X_{i-k_l} - X_{i+k_l}; X_{-(i-k_l)} - X_{-(i+k_l)}, \quad \text{for } i \notin S_1.$$

Type D2. To form the doubled array reflect the array about a vertical line to the right of the array. Then add two dots, one in the upper row just left of the vertical line and one in the lower row just right of the vertical line. Then number the doubled array as in the other orthogonal cases.

Form a string through the doubled array by reflecting the original string about the vertical line and connecting the two pieces by passing through one of the two new dots in the middle so as to continue to alternate between upper and lower rows. Now write the labels of the string as $j_1, j_2, \ldots, j_{2l+1}$. Then

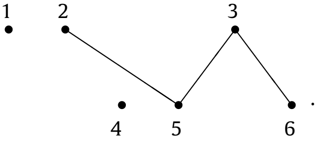
$$f_0 = \sum_{i=1}^{l} E_{j_{i+1}, j_i} - \sum_{l+1}^{2l} E_{j_{i+1}, j_i}$$

and

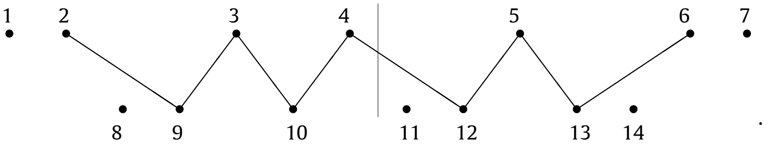
$$f_0(e_{j_i}) = \begin{cases} e_{j_{i+1}}, & i = 1, \ldots, l, \\ -e_{j_{i+1}}, & i = l+1, \ldots, 2l, \\ 0, & i = 2l+1. \end{cases}$$

Define $W_1, V_1, G_1, \hat{G}_1, \ldots$ as for types B1 and B2.

Here is an example. Let $\lambda = (6, 5, 2 \mid 4, 3, 1)$. Then the array with string is



The doubled array with string is



Note that G_1 is of type B1.

For all types we now define a sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_m$, which ends when the algorithm gives $f_m = 0$. This is done inductively. The subgroup G_1 has already been defined. The definition of G_1 uses the orthogonal subspaces W_1 and V_1 of $V = \mathbf{C}^{\hat{n}}$. To define G_i inside G_{i-1}, we first form $W_i, V_i \subset V_{i-1}$ by the same procedure that was used to define W_1, V_1. Then

$$G_i' = \{g \in G_{i-1}\colon g(W_i) \subset W_i \text{ and } g|_{V_i} = Id_{V_i}\},$$
$$G_i = \{g \in G_{i-1}\colon g(V_i) \subset V_i \text{ and } g|_{W_i} = Id_{W_i}\}.$$

Setting $K_i = K \cap G_i$, we get pairs (G_i, K_i) as in (1). It is easy to check that $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ is a Cartan subalgebra of $\mathfrak{k}_i$. Also, $\lambda_i = \lambda|_{\mathfrak{h}_i}$ is regular for $\mathfrak{g}_i$ and therefore defines a positive system $\Delta^+(\mathfrak{h}_i, \mathfrak{g}_i)$ and a Borel subalgebra $\mathfrak{b}_i$, which equals $\mathfrak{b} \cap \mathfrak{g}_i$.

We may conclude the following from this discussion.

(i) For each i there is a standard triple e_i, h_i, f_i contained in $\mathfrak{g}'_{i+1}$.
(ii) Each f_i is a principal nilpotent element in $\mathfrak{g}'_{i+1}$.
(iii) Each f_i commutes with $\mathfrak{g}_j$, for $j \geqslant i+1$. In particular, the f_i are mutually commuting.
(v) $f_i + f_{i+1} + \cdots f_{m-1}$ is the generic element constructed by the algorithm applied to the pair (G_i, K_i).

Define

$$\hat{G}_i := \{ g \in \hat{G}_{i-1} \colon g(V_i) \subset V_i \text{ and } g|_{W_i} = Id_{W_i} \}$$

and $\hat{K}_i := \hat{K} \cap \hat{G}_i$. Then $\hat{G}_i$ is a general linear group and (G_i, K_i) embeds into $(\hat{G}_i, \hat{K}_i)$.

3.3. *Proof of Proposition 15*

We first check that f is generic in $\hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$. The construction of [1, Section 3] says that the generic elements in $\hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$ have tableau for which the number of boxes in the ith row is the number of blocks in the array for $\hat{\mathfrak{g}}_{i-1}$. Since the $\mathfrak{sl}(2)_{f_i}$ $(= \text{span}_{\mathbf{C}}\{e_i, h_i, f_i\})$ mutually commute we may form a standard triple $\{e, h, f\}$ by $e = \sum e_i$ and $h = \sum h_i$. Let $\mathfrak{sl}(2)_f$ denote the span of this triple. Then each W_i is an irreducible $\mathfrak{sl}(2)_f$ representation. It follows that the tableau of f has $\dim(W_i)$ boxes in the ith row. This is exactly the number of blocks in the array for $\mathfrak{g}_{i-1}$. Therefore, f is generic in $\hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$.

Now we need to show that f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$. Suppose that $f' \in \mathfrak{n}^- \cap \mathfrak{p}$. Then $f' \in \hat{\mathfrak{n}}^- \cap \hat{\mathfrak{p}}$, so $f' \in \overline{\hat{K} \cdot f} \subset \overline{\hat{G} \cdot f}$. By [6, Section 6.2], applied to $\mathfrak{gl}(2n, \mathbf{C})$, $\text{rank}((f')^k) \leqslant \text{rank}(f^k)$, for all $k \in \mathbf{Z}_{\geqslant 0}$. Applying the same closure condition to $\mathfrak{sp}(2n, \mathbf{C})$ we conclude $f' \in \overline{G \cdot f}$. In particular, $\dim(G \cdot f') \leqslant \dim(G \cdot f)$; so $\dim(K \cdot f') \leqslant \dim(K \cdot f)$. Therefore, $K \cdot f$ is the orbit of greatest dimension meeting $\mathfrak{n}^- \cap \mathfrak{p}$, so f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.

Remark 18. The shape of the tableau for $K \cdot f$ is described above. To describe the signed tableau we need to fill in the signs. This is done by beginning the ith row with a $+$ sign if the last dot in the ith string (that is, the string in the doubled array constructed in $\mathfrak{g}_i$) is in the upper row, and a $-$ sign otherwise. Observe that in the symplectic case there are always an even number of blocks in the doubled array, so all rows in the tableau have even length. Furthermore, any two rows of the same length must begin with the same sign. In the orthogonal case all rows have odd length and any two of the same length must begin with the same sign. One may see that any signed tableau described above occurs as $K \cdot f$ with f generic for some closed K-orbit $\mathcal{Q}$ in the flag variety.

4. Description of the components

Let $\mathcal{Q} = K \cdot \mathfrak{b}$ be any closed orbit and let f be the generic element in $\mathfrak{n}^- \cap \mathfrak{p}$ constructed in Section 3. As mentioned in the introduction, the components of $\mu^{-1}(f)$ 'associated' to $\mathcal{Q}$ are those that lie in the conormal bundle $T^*_{\mathcal{Q}}\mathfrak{B}$. In this section we explicitly describe these components. The statement is contained in Theorem 26.

4.1. *Parabolic subgroups*

Recall from Section 3.1 that each closed K-orbit $\mathcal{Q}$ in $\mathfrak{B}$ is associated to a positive system of roots Δ^+ with respect to a Cartan subalgebra $\mathfrak{h} + \mathfrak{a}$ (with $\mathfrak{a} \neq 0$ only in type D2) and a Borel subalgebra

$\mathfrak{b} = (\mathfrak{h} + \mathfrak{a}) + \mathfrak{n}^-$. Let Π be the set of simple roots and S the set of roots in Π that are compact (in the sense that $\mathfrak{g}^{(\alpha)} \subset \mathfrak{k}$). Note that in type D2, if a root α is compact then $\alpha|_{\mathfrak{a}} = 0$, so we may identify α with a root of $\mathfrak{h}$ in $\mathfrak{k}$. Define $\langle S \rangle$ to be the set of (compact) roots in the span of S. The subset S determines a parabolic subalgebra $\widetilde{\mathfrak{q}}$ of $\mathfrak{g}$ containing $\mathfrak{b}$:

$$\widetilde{\mathfrak{q}} = \left(\mathfrak{h} + \mathfrak{a} + \sum_{\beta \in \langle S \rangle} \mathfrak{g}^{(\beta)}\right) + \sum_{\beta \in \Delta^+ \setminus \langle S \rangle} \mathfrak{g}^{(-\beta)} = \widetilde{\mathfrak{l}} + \widetilde{\mathfrak{u}}^-,$$

and parabolic subgroup $\widetilde{Q} = N_G(\widetilde{\mathfrak{q}})$ of G.

Now define a parabolic subgroup Q of K by

$$Q := N_G(\widetilde{\mathfrak{q}}) \cap K = N_K(\widetilde{\mathfrak{q}}).$$

The Lie algebra of Q is

$$\mathfrak{q} = \left(\mathfrak{h} + \sum_{\beta \in \langle S \rangle} \mathfrak{g}^{(\beta)}\right) + \sum_{\beta \in \Delta_c^+ \setminus \langle S \rangle} \mathfrak{g}^{(-\beta)} = \mathfrak{l} + \mathfrak{u}^-. \tag{19}$$

One easily sees that $Q \subset N_K(\mathfrak{q})$ and that equality fails in general.

Similarly define parabolic subgroups $\widetilde{Q_i}$ of G_i and Q_i of K_i. Write $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i^-$ as in (19). Using the convention that $G_0 = G$, $K_0 = K$ and $Q_0 = Q$, and similarly for the Lie algebras, we now state several very simple properties of these parabolics.

Lemma 20. *The following hold.*

(1) *L_i normalizes $\mathfrak{n}_i^- \cap \mathfrak{p}$, $i = 0, \dots, m$.*
(2) *$R_i := L_i \cap Q_{i-1}$ is a parabolic subgroup of L_i, $i = 1, 2, \dots, m-1$.*
(3) *$\mathfrak{u}_i^- \subset \mathfrak{u}_{i-1}^-$, $i = 1, 2, \dots, m-1$.*

Proof. Note that $L_i \subset \widetilde{Q_i}$, so L_i normalizes $\widetilde{\mathfrak{u}_i}^-$. Since $L_i \subset K_i$, L_i also normalizes $\widetilde{\mathfrak{u}_i}^- \cap \mathfrak{p}$. But $\widetilde{\mathfrak{u}_i}^- \cap \mathfrak{p} = \mathfrak{n}_i^- \cap \mathfrak{p}$. Statement (1) follows.

Since $\mathfrak{q}_{i-1}$ contains the Borel subalgebra $\mathfrak{b}_{i-1} \cap \mathfrak{k}_{i-1} = \mathfrak{b} \cap \mathfrak{k}_{i-1}$, $\mathfrak{r}_i$ contains the Borel subalgebra $\mathfrak{b}_i \cap \mathfrak{l}_i$. This proves that $\mathfrak{l}_i \cap \mathfrak{q}_{i-1}$ is a parabolic subalgebra of $\mathfrak{l}_i$.

Part (3) follows from the fact that $\mathfrak{l}_{i-1} \cap \mathfrak{k}_i \subset \mathfrak{l}_i$, which is verified for each case by considering the compact roots simple for $\mathfrak{g}_{i-1}$ and $\mathfrak{g}_i$. For example, in type C, if a root $-(\epsilon_j - \epsilon_{j+1})$ is simple for $\mathfrak{g}_{i-1}$ then $j, j+1$ or $-j, -(j+1)$ are in the same block in the array for $\mathfrak{g}_{i-1}$. But if the root space for $-(\epsilon_j - \epsilon_{j+1})$ is in $\mathfrak{k}_i$, then both $j, j+1$ or $-j, -(j+1)$ are also in the same block in the array for $\mathfrak{g}_i$, so $-(\epsilon_j - \epsilon_{j+1})$ is simple and compact for $\mathfrak{g}_i$. $\square$

Remark 21. It is not always the case that $L_i \subset L_{i-1}$. A simple example occurs when $(G, K) = (Sp(10), GL(5))$ and $\lambda = (5, 4, 2, 1, -3)$. Then $L_1 \not\subseteq L_0 = L$, since $\Delta(\mathfrak{l}) = \{\pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_3 - \epsilon_4)\}$, but $\Delta(\mathfrak{l}_1) = \{\pm(\epsilon_1 - \epsilon_3)\}$. This is roughly because in passing from the original array to the array obtained by removing the first string, the two blocks in the upper row 'collapse' down to one block.

Remark 22. The sequence $G = G_0 \supset G_1 \supset \cdots$ ends when $G_m = K_m$.

Remark 23. For each of the five types, the simple roots that are compact (i.e., the roots in S) may be read off the array. For type C, the set S consists of $\epsilon_i - \epsilon_{i+1}$ when i and $i+1$ are labels of dots in the same block. For the orthogonal cases, these same roots are in S along with one more in the following cases:

ϵ_n in type B1 when the last dot is in the lower row,

ϵ_p in type B2 when the last dot is in the upper row,

$\epsilon_{n-1} + \epsilon_n$ in type D1 when the last two dots are in the lower row,

$\epsilon_{p-1} + \epsilon_p$ in type D1 when the last two dots are in the upper row.

The Levi factors L have the following form. Let ℓ be the number of blocks in the array (so $\ell = l$ for types C, D1 and D2, and $\ell = l$ or $l + 1$ for the two subcases of types B1 and B2). We let $b_1, \dots, b_\ell$ be the sizes of the blocks in the array:

$$\begin{aligned}
\text{Type C}, \quad & L \simeq GL(b_1) \times \cdots \times GL(b_\ell). \\
\text{Types B1/B2}, \quad & L \simeq GL(b_1) \times \cdots \times GL(b_\ell) \times O(1) \quad \text{or} \\
& L \simeq GL(b_1) \times \cdots \times GL(b_{\ell-1}) \times O(2b_\ell + 1). \\
\text{Type D1}, \quad & L \simeq GL(b_1) \times \cdots \times GL(b_{\ell-1}) \times O(2b_\ell). \\
\text{Type D2}, \quad & L \simeq GL(b_1) \times \cdots \times GL(b_\ell) \times O(1) \times O(1).
\end{aligned}$$

The two subcases for types B1/B2 correspond to the two subcases in the construction of the strings in Section 3.1. By considering each case, one sees that $\mathfrak{l} \cap \mathfrak{l}_1$ is as follows:

$$\begin{aligned}
\text{Type C}, \quad & \mathfrak{gl}(b_1 - 1) \oplus \cdots \oplus \mathfrak{gl}(b_\ell - 1). \\
\text{Types B1/B2}, \quad & \mathfrak{gl}(b_1 - 1) \oplus \cdots \oplus \mathfrak{gl}(b_\ell - 1) \quad \text{or} \\
& \mathfrak{gl}(b_1 - 1) \oplus \cdots \oplus \mathfrak{gl}(b_{\ell-1} - 1) \oplus \mathfrak{so}(2b_\ell). \\
\text{Type D1}, \quad & \mathfrak{gl}(b_1 - 1) \oplus \cdots \oplus \mathfrak{gl}(b_{\ell-1} - 1) \oplus \mathfrak{so}(2b_\ell - 1). \\
\text{Type D2}, \quad & \mathfrak{gl}(b_1 - 1) \oplus \cdots \oplus \mathfrak{gl}(b_\ell - 1).
\end{aligned}$$

A useful description of the parabolic subalgebras $\mathfrak{q}_i$ may be given in terms of stabilizers of certain flags in $\mathbf{C}^{\hat{n}}$. To describe this it suffices to see how $\mathfrak{q} = \mathfrak{q}_0$ in $\mathfrak{k}$ is the stabilizer of a flag; the same description applies to $\mathfrak{q}_i$ in $\mathfrak{k}_i$.

Let (G, K) be any of the five types. Let l be the length of the first string through the array (as in Section 3.1). Now consider the doubled array. Define an isotropic flag

$$\begin{aligned}
&\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq \mathbf{C}^{\hat{n}}, \quad \text{types C, B1, B2 and D2}, \\
&\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{l-1} \subsetneq \mathbf{C}^{\hat{n}}, \quad \text{type D1}
\end{aligned} \tag{24}$$

by

$$F_j = \operatorname{span}_{\mathbf{C}}\{e_i \colon i \text{ is the label of a dot in one of the } j \text{ rightmost blocks}\}.$$

It is easy to verify that $\mathfrak{q}$ (resp., $\tilde{\mathfrak{q}}$) is the stabilizer in $\mathfrak{k}$ (resp., $\mathfrak{g}$) of this flag.

The following is also easily checked.

Lemma 25. *The stabilizer of the flag* (24) *in K is contained in Q.*

This lemma will be used in the proof of Lemma 31 below.

4.2. The main theorem

The structure of the components of Springer fibers associated to closed K-orbits in $\mathfrak{B}$ is given by the following theorem. We continue with a closed K-orbit in $\mathfrak{B}$ and the generic element f as constructed in Section 3. Let $L_{i,e}$ (resp. $Q_{i,e}$) denote the identity component of L_i (resp., Q_i). Define $C_f := L_{m,e} \cdots L_{1,e} L_e \cdot \mathfrak{b}$.

Theorem 26. *C_f is a component of the Springer fiber $\mu^{-1}(f)$ that is contained in $\gamma_{\mathcal{Q}}^{-1}(f)$. Furthermore,*

$$\gamma_{\mathcal{Q}}^{-1}(f) = \bigcup_{z \in A_K(f)} z C_f = L_m \cdots L_1 L \cdot \mathfrak{b}.$$

The proof of this theorem follows from several lemmas. The argument is to see that (a) C_f is a closed irreducible subset of $\gamma_{\mathcal{Q}}^{-1}(f)$ and (b) $\dim(C_f) = \dim(\mu^{-1}(f))$. Then we show $A_K(f) \cdot C_f = L_m \cdots L_1 L \cdot \mathfrak{b}$.

Since each $L_{i,e}$ is irreducible, so is C_f. In order to show that C_f is closed in $\mathfrak{B}$ we apply [7, Section 0.15]. In particular, the statement we use is that if a connected reductive algebraic group H acts on a variety B and A is a closed subset of B that is stable under a parabolic subgroup of H, then $H \cdot A$ is closed in B.

Lemma 27. *The following hold for $i = 0, 1, 2, \ldots, m$.*

(1) $L_{i,e} \cdots L_{1,e} L \cdot \mathfrak{b} = Q_{i,e} \cdots Q_{1,e} Q_e \cdot \mathfrak{b}$.
(2) $R_{i,e} = L_{i,e} \cap Q_{i-1,e}$ *stabilizes* $L_{i-1,e} \cdots L_{1,e} L \cdot \mathfrak{b}$, $i > 0$.
(3) $L_{i,e} \cdots L_{1,e} L_e \cdot \mathfrak{b}$ *is closed in* $\mathfrak{B}$.

Proof. The proof of (1) is by induction on i. The $i = 0$ case is $L_e \cdot \mathfrak{b} = Q_e \cdot \mathfrak{b}$, which is immediate since $U^- \subset B$. Assume $L_{i-1,e} \cdots L_{1,e} L_e \cdot \mathfrak{b} = Q_{i-1,e} \cdots Q_{1,e} Q_e \cdot \mathfrak{b}$. Then

$$\begin{aligned} Q_{i,e} \cdots Q_{1,e} Q_e \cdot \mathfrak{b} &= L_{i,e} U_i^- Q_{i-1,e} \cdots Q_{1,e} Q_e \cdot \mathfrak{b} \\ &= L_{i,e} Q_{i-1,e} \cdots Q_{1,e} Q_e \cdot \mathfrak{b}, \quad \text{by Lemma 20(3)}, \\ &= L_{i,e} L_{i-1,e} \cdots L_{1,e} L_e \cdot \mathfrak{b}, \quad \text{by induction.} \end{aligned}$$

Now (2) is immediate.

Part (3) is also proved by induction on i, using the comments preceding the statement of the lemma. The $i = 0$ case is clear since $L_e \cdot \mathfrak{b}$ is a flag variety. The inductive hypothesis is that $L_{i-1,e} \cdots L_{1,e} L_e \cdot \mathfrak{b}$ is closed. By (2) this set is also $R_{i,e}$-stable. Since $R_{i,e}$ is a parabolic subgroup of $L_{i,e}$ (by Lemma 20(2)), it follows that $L_{i,e} \cdots L_{1,e} L_e \cdot \mathfrak{b}$ is closed. □

Lemma 28. $L_m \cdots L_1 L \cdot \mathfrak{b} \subset \gamma_{\mathcal{Q}}^{-1}(f)$.

Proof. It is enough to show that $L L_1 \cdots L_m \subset N(f, \mathfrak{n}^- \cap \mathfrak{p})$ (cf. Section 2.1). We apply induction on $\dim(G)$. Let $\ell \in L$, $\ell_i \in L_i$, $i = 1, 2, \ldots, m-1$. Since G_i, $i > 0$, stabilizes f_0

$$\ell \ell_1 \cdots \ell_m \cdot f = \ell\bigl(f_0 + \ell_1 \cdots \ell_m \cdot (f_1 + \cdots + f_m)\bigr)$$

This lies in $\ell \cdot (f_0 + \mathfrak{n}_1^- \cap \mathfrak{p})$ by induction. But $f_0 \in \mathfrak{n}^- \cap \mathfrak{p}$, which is normalized by L (by Lemma 20(3)), and $\mathfrak{n}_1^- \cap \mathfrak{p} \subset \mathfrak{n}^- \cap \mathfrak{p}$. Therefore, $\ell \ell_1 \cdots \ell_m \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}$. □

Lemma 29. *The dimension of* $\mu^{-1}(f)$ *equals*

$$\dim(L/L \cap B) + \sum_{j=1}^{m} \dim(L_j/R_j).$$

Proof. For this proof we set d_i equal to the number of boxes in the ith row of the tableau corresponding to f, so $d_i = \dim(W_i)$. Let r be the number of rows. As in Remark 23, ℓ is the number of blocks in the array. Note that $d_1 = 2l$ in type C, $d_1 = 2l + 1$ in types B1, B2 and D2, and $d_1 = 2l - 1$ in type D1. We also use the temporary notation f^1 for $f - f_0 = f_1 + f_2 + \cdots$. Note that f^1 is the generic element in $\mathfrak{n}_1^- \cap \mathfrak{p}$ constructed by our algorithm and the tableau of f^1 is the tableau of f with the first row removed.

Claim 1.

$$\dim\bigl(Z_G(f)\bigr) - \dim\bigl(Z_{G_1}\bigl(f^1\bigr)\bigr) = \begin{cases} \dim(\mathfrak{l}) - \dim(\mathfrak{l} \cap \mathfrak{l}_1), & \text{types C, B1, B2, D1}, \\ \dim(\mathfrak{l}) - \dim(\mathfrak{l} \cap \mathfrak{l}_1) + 1, & \text{type D2}. \end{cases}$$

The proof is case by case. In type C the formula

$$\dim\bigl(Z_G(f)\bigr) = \frac{1}{2}\dim(V) + \sum_{i=1}^{r}(i-1)d_i,$$

with $V = \mathbf{C}^{\hat{n}} = \mathbf{C}^{2n}$, is well known and may be found in [8, Section 3]. Applying this formula also to $f^1 \in \mathfrak{g}_1$ gives

$$\begin{aligned} \dim\bigl(Z_G(f)\bigr) - \dim\bigl(Z_{G_1}\bigl(f^1\bigr)\bigr) &= \frac{1}{2}\dim(V) + \sum_{i=1}^{r}(i-1)d_i - \frac{1}{2}\bigl(\dim(V) - d_1\bigr) - \sum_{i=2}^{r}(i-2)d_i \\ &= \frac{d_1}{2} + \sum_{i=2}^{r} d_i \\ &= \dim(V) - \frac{d_1}{2} \\ &= 2n - \ell. \end{aligned}$$

By Remark 23, with b_i equal to the number of dots in the ith block, we have

$$\dim(\mathfrak{l}) - \dim(\mathfrak{l} \cap \mathfrak{l}_1) = \sum_{i=1}^{\ell} b_i^2 - \sum_{i=1}^{\ell}(b_i - 1)^2 = 2\sum_{i=1}^{\ell} b_i - \ell = 2n - \ell.$$

This proves the claim for type C.

In the orthogonal cases the formula for the dimension of the centralizer is also given in [8, Section 3]. Since the tableau of f has only rows of odd length, the formula says

$$\dim\bigl(Z_G(f)\bigr) = \frac{1}{2}\dim(V) + \sum_{i=1}^{r}(i-1)d_i - \frac{r}{2}.$$

Applying this formula to f^1 we get

$$\dim\bigl(Z_G(f)\bigr) - \dim\bigl(Z_{G_1}\bigl(f^1\bigr)\bigr) = \dim(V) - \frac{d_1+1}{2}.$$

In types B1 and B2 (again using Remark 23) we get

$$\dim(\mathfrak{l}) - \dim(\mathfrak{l}\cap\mathfrak{l}_1) = \begin{cases} \dim(V) - (\ell+1), \\ \dim(V) - \ell \end{cases}$$

for the two subcases. As noted above, $\frac{1}{2}(d_1+1) = \ell+1, \ell$ in the two subcases. For types D1 and D2

$$\dim(\mathfrak{l}) - \dim(\mathfrak{l}\cap\mathfrak{l}_1) = \begin{cases} \dim(V) - \ell, & \text{type D1}, \\ \dim(V) - \ell - 2, & \text{type D2}. \end{cases}$$

In these cases $\frac{1}{2}(d_1+1) = \ell, \ell+1$. The claim now follows for types B1, B2, D1 and D2.

Claim 2.

$$\dim\bigl(Z_G(f)\bigr) = \begin{cases} \dim(\mathfrak{l}) + 2\sum_{i=1}^m \dim(\mathfrak{l}_i\cap\mathfrak{u}_{i-1}), & \text{types C, B1, B2, D1}, \\ \dim(\mathfrak{l}) + 2\sum_{i=1}^m \dim(\mathfrak{l}_i\cap\mathfrak{u}_{i-1}) + 1, & \text{type D2}. \end{cases}$$

We prove this by induction on m. Consider the cases of types C, B1, B2 and D1. In these cases all pairs (G_i, K_i) have type C, B1, B2 or D1.

$$\begin{aligned} \dim\bigl(Z_G(f)\bigr) &= \dim\bigl(Z_{G_1}\bigl(f^1\bigr)\bigr) + \dim(\mathfrak{l}) - \dim(\mathfrak{l}\cap\mathfrak{l}_1), \quad \text{by Claim 1}, \\ &= \left(\dim(\mathfrak{l}_1) + 2\sum_{i=2}^m \dim(\mathfrak{l}_i\cap\mathfrak{u}_{i-1})\right) + \dim(\mathfrak{l}) - \dim(\mathfrak{l}\cap\mathfrak{l}_1), \quad \text{by induction}, \\ &= \dim(\mathfrak{l}) + 2\sum_{i=2}^m \dim(\mathfrak{l}_i\cap\mathfrak{u}_{i-1}) + 2\dim(\mathfrak{l}_1\cap\mathfrak{u}_0), \quad \text{since } \mathfrak{l}_0\cap\mathfrak{l}_1 + \mathfrak{l}_1\cap\mathfrak{u}_0 \text{ is} \\ &\qquad \text{a Levi decomposition of a parabolic subalgebra } \mathfrak{r}_1 \text{ of } \mathfrak{l}_1. \end{aligned}$$

The case of type D2 is proved in the same way.

Now we are ready to prove the lemma. The components of $\mu^{-1}(f)$ all have the same dimension [10]. This dimension is given by $\dim(\mathfrak{n}) - \frac{1}{2}\dim(G\cdot f)$, a formula of Springer and Steinberg (see [7, Chapter 6] for a discussion of this formula). Note that

$$\begin{aligned} \dim(\mathfrak{n}) - \frac{1}{2}\bigl(\dim(\mathfrak{g}) - \dim(\mathfrak{l})\bigr) &= \frac{1}{2}\bigl(\dim(\mathfrak{l}) - \operatorname{rank}(\mathfrak{g})\bigr) \\ &= \begin{cases} \dim(L/L\cap B), & \text{types C, B1, B2, D1}, \\ \dim(L/L\cap B) - \frac{1}{2}, & \text{type D2}. \end{cases} \end{aligned}$$

Therefore, in all cases,

$$\begin{aligned}\dim\bigl(\gamma_Q^{-1}(f)\bigr) &= \dim(\mathfrak{n}) - \frac{1}{2}\bigl(\dim(G) - \dim\bigl(Z_G(f)\bigr)\bigr) \\ &= \dim(\mathfrak{n}) - \frac{1}{2}\bigl(\dim(\mathfrak{g}) - \dim(\mathfrak{l})\bigr) + \frac{1}{2}\bigl(\dim\bigl(Z_G(f)\bigr) - \dim(\mathfrak{l})\bigr) \\ &= \dim(L/L \cap B) + \sum_{i=1}^{m} \dim(\mathfrak{l}_i \cap \mathfrak{u}_{i-1}).\end{aligned}$$

The lemma now follows by noting that

$$\dim(L_i/R_i) = \dim(L_i/L_i \cap Q_{i-1}) = \dim(\mathfrak{l}_i \cap \mathfrak{u}_{i-1}). \qquad \square$$

Lemma 30.

$$\dim(C_f) = \dim(L/L \cap B) + \sum_{j=1}^{m} \dim(L_j/R_j).$$

Proof. By Lemma 28, $C_f \subset \gamma_Q^{-1}(f)$, so it suffices to prove that the left-hand side is greater than or equal to the right-hand side. Consider $\mathfrak{l}_i \cap \mathfrak{u}_{i-1}$ (which has the same dimension as L_i/R_i), where $\mathfrak{u}_{i-1}$ is opposite to $\mathfrak{u}_{i-1}^-$. Let $\Delta(i)$ be the compact roots β so that $\mathfrak{g}^{(\beta)} \subset \mathfrak{l}_i \cap \mathfrak{u}_{i-1}$ and let $\Delta(0)$ be the roots β so that $\mathfrak{g}^{(\beta)} \subset \mathfrak{l} \cap \mathfrak{n}$. Then $\Delta(0), \Delta(1), \ldots, \Delta(m)$ are pairwise disjoint and $\sum_{i=0}^{m} |\Delta(i)|$ equals the right-hand side of the expression in the statement of the lemma. Write N for the connected group with Lie algebra $\sum_{\beta \in \Delta^+} \mathfrak{g}^{(\beta)}$. Then the map $\varphi : N \to N \cdot \mathfrak{b}, \varphi(n) = n \cdot \mathfrak{b}$ is an isomorphism of varieties. Enumerating the roots in $\bigcup_{i=0}^{m} \Delta(i)$ as $\beta_1, \beta_2, \ldots, \beta_d$, listing roots in $\Delta(m)$ first, then those in $\Delta(m-1)$, etc., gives a map

$$\begin{gathered} U_{\beta_1} \times \cdots \times U_{\beta_d} \to N, \\ (u_1, \ldots, u_d) \to u_1 \cdots u_d, \end{gathered}$$

where $U_\beta = \exp(\mathbf{C} X_\beta)$. By [3, Section 14.4] the image of this map is the d-dimensional subvariety $U_{\beta_1} U_{\beta_2} \cdots U_{\beta_d}$. Therefore, $\varphi(U_{\beta_1} U_{\beta_2} \cdots U_{\beta_d})$ is a d-dimensional subvariety of $\mathfrak{B}$ contained in C_f. $\square$

At this point we conclude that C_f is an irreducible component of $\mu^{-1}(f)$. This proves the first part of the theorem. By Proposition 2, the second part of the theorem will follow if we show that $A_K(f) \cdot C_f \subset L_m \cdots L_1 L \cdot \mathfrak{b}$. Since each element of $A_K(f)$ is represented by an element of the reductive part of $Z_K(f)$, it suffices to prove the following lemma.

Lemma 31. *If $k \in Z_K(f)_{\mathrm{red}}$, then $k \cdot C_f \subset L_m \cdots L_1 L \cdot \mathfrak{b}$.*

Proof. The reductive part of $Z_K(f)$ is the set of $k \in K$ that commute with $\mathfrak{sl}(2)_f$, that is, the set of intertwining operators for the representation of $\mathfrak{sl}(2)_f$ on V $(= \mathbf{C}^{\hat{n}})$ coming from K.

Recall that $V = W_1 \oplus \cdots \oplus W_r$ as $\mathfrak{sl}(2)_f$-representation, with W_i as at the end of Section 3.2. An intertwining operator must preserve isotypic subspaces and is a product of intertwining operators, each acting by the identity on all isotypic subspaces except one. Therefore we may assume that k is of this form. Therefore, assume that k preserves an isotypic subspace $W_i \oplus W_{i+1} \oplus \cdots \oplus W_j$, for some $i < j$, and acts by the identity on other isotypic subspaces. Then k commutes with $G_j, G_{j+1}, \ldots, G_m$. We claim that $L_{i-1} \supset \cdots \supset L_{j-2} \supset L_{j-1}$ and $k \in L_{i-1}$. If this is the case, then

$$\begin{aligned} k \cdot C_f &= kL_m \dots L_1 L \cdot \mathfrak{b} \\ &= L_m \dots L_j (kL_{j-1} \dots L_{i-1}) L_{i-2} \dots L \cdot \mathfrak{b} \\ &= L_m \dots L_j (kL_{i-1}) L_{i-2} \dots L \cdot \mathfrak{b} \\ &= L_m \dots L_j (L_{i-1}) L_{i-2} \dots L \cdot \mathfrak{b} \\ &= C_f, \end{aligned}$$

and the lemma will be proved.

By working in the pair (G_{i-1}, K_{i-1}) we may assume that k acts on the isotypic component for the $\mathfrak{sl}(2)_f$-subrepresentation of greatest dimension. Therefore, the isotypic component is $W_1 \oplus \cdots \oplus W_j$, for some j, and the claim is that

$$L = L_0 \supset L_1 \supset \cdots \supset L_{j-1} \quad \text{and } k \in L.$$

We prove this claim.

If $j = 1$ then k acts as a scalar and lies in the diagonal Cartan subgroup H, which is in L, and the claim is proved. Suppose that $j > 1$. As noted earlier, it is not always the case that $L \supset L_1$. However, in the present situation the subspaces $W_1, \dots, W_j$ all have the same dimension. This means that in passing from an array to the smaller array, no blocks collapse, i.e., the arrays for $G, G_1, \dots, G_{j-1}$ all have the same number of blocks. In particular, by considering the flags preserved by the Q_i, we see that $Q = Q_0 \supset Q_1 \supset \cdots \supset Q_{j-1}$. The first part of the claim follows. For the second part, note that k preserves weight spaces of the $\mathfrak{sl}(2)_f$ action. Since the first j strings pass through the *same* blocks, any weight space in $W_1 \oplus \cdots \oplus W_j$ is contained in the span of all e_i with i in a given block. By considering the description of Q in terms of flags, we see that $k \in L$. $\square$

The theorem is now proved, by the remarks following its statement. $\square$

There are several consequences of the theorem worth mentioning. Recall that $\gamma_Q^{-1}(f) = N(f, \mathfrak{n}^- \cap \mathfrak{p})^{-1} \cdot \mathfrak{b}$, with $N(f, \mathfrak{n}^- \cap \mathfrak{p}) = \{k \in K\colon\ k \cdot f \in \mathfrak{n}^- \cap \mathfrak{p}\}$.

Corollary 32. *The following hold.*

(1) $N(f, \mathfrak{n}^- \cap \mathfrak{p}) = (B \cap K) L L_1 \cdots L_m$.
(2) *Suppose Y is generic in $\mathfrak{n}^- \cap \mathfrak{p}$, then there exist $b \in B \cap K$, $\ell \in L$ and $\ell_i \in L_i$ so that $Y = b\ell\ell_1 \cdots \ell_m \cdot f$.*

Proof. Suppose $k \in N(f, \mathfrak{n}^- \cap \mathfrak{p})$. Then $k^{-1} \cdot \mathfrak{b} \in \gamma_Q^{-1}(f) = L_m \cdots L_1 L \cdot \mathfrak{b}$, so $k\ell_m \cdots \ell_1 \ell \in B \cap K$. This proves (1).

For (2) note that the set of generic elements in $\mathfrak{n}^- \cap \mathfrak{p}$ is $(K \cdot f) \cap (\mathfrak{n}^- \cap \mathfrak{p})$, so $k \cdot f$ is generic if and only if $k \in N(f, \mathfrak{n}^- \cap \mathfrak{p})$. $\square$

The statement of the theorem can be improved slightly for the symplectic group.

Corollary 33. *For the pair $(Sp(2n), GL(n))$, there is just one component associated to each closed K-orbit in $\mathfrak{B}$, i.e., $C_f = L_m \cdots L_1 L \cdot \mathfrak{b}$.*

Proof. This follows from the theorem because $A_K(f) = \{e\}$. The stated form of C_f follows since K and all K_i are connected, so L and L_i are also connected. $\square$

5. Associated cycles of discrete series representations

The description of $\gamma_{\mathcal{Q}}^{-1}(f)$ given in Theorem 26, along with a theorem of J.-T. Chang, allows us to compute associated cycles of discrete series representations of the groups $Sp(2n,\mathbf{R})$ and $SO_e(p,q)$ (with pq even). In the case of $SU(p,q)$, this is discussed is some detail in [1, Section 6]. We do not repeat this discussion here, but we give the algorithm for the computation.

Let X_λ be the Harish–Chandra module of a discrete series representation with Harish–Chandra parameter λ (which we may assume is Δ_c^+-dominant). Therefore X_λ has infinitesimal character λ and the lowest K-type has highest weight $\tau=\lambda+\rho-2\rho_c$. Since λ is regular, we may associate to λ a positive system Δ^+ and a closed K-orbit $\mathcal{Q}=K\cdot\mathfrak{b}$, $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}^-$. This gives a generic element f in $\mathfrak{n}^-\cap\mathfrak{p}$ as in Section 3. The associated cycle is $AC(X_\lambda)=m_{\mathcal{Q}}(\lambda)\overline{K\cdot f}$. Chang's theorem says that the multiplicity $m_{\mathcal{Q}}(\lambda)$ is given as follows. The weight τ determines a K-equivariant line bundle $\mathcal{L}_\tau\to\mathcal{Q}$. Then

$$m_{\mathcal{Q}}(\lambda)=\dim\bigl(H^0\bigl(\gamma_{\mathcal{Q}}^{-1}(f),\mathcal{O}(\mathcal{L}_\tau|_{\gamma_{\mathcal{Q}}^{-1}(f)})\bigr)\bigr).$$

Our description of $\gamma_{\mathcal{Q}}^{-1}(f)$ is used to compute the dimension of this space of sections.

Assume for the moment that $(G,K)=(Sp(2n),GL(n))$. In this case each L_i is a general linear group and is therefore connected. The Borel–Weil Theorem states that $H^0(\mathcal{Q},\mathcal{O}(\mathcal{L}_\tau))\simeq(W_{-\tau})^*$, where $W_{-\tau}$ is the irreducible finite dimensional representation of K having lowest weight $-\tau$. Letting $w_{-\tau}$ be a lowest weight vector in $W_{-\tau}$,

$$\dim\bigl(H^0\bigl(\gamma_{\mathcal{Q}}^{-1}(f),\mathcal{O}(\mathcal{L}_\tau|_{\gamma_{\mathcal{Q}}^{-1}(f)})\bigr)\bigr)=\dim\bigl(\operatorname{span}_{\mathbf{C}}\bigl\{k^{-1}w_{-\tau}\colon k\in N\bigl(f,\mathfrak{n}^-\cap\mathfrak{p}\bigr)\bigr\}\bigr).$$

Using our description of $N(f,\mathfrak{n}^-\cap\mathfrak{p})$ given in Corollary 32 we have

$$\dim\bigl(H^0\bigl(\gamma_{\mathcal{Q}}^{-1}(f),\mathcal{O}(\mathcal{L}_\tau|_{\gamma_{\mathcal{Q}}^{-1}(f)})\bigr)\bigr)=\dim\bigl(\operatorname{span}_{\mathbf{C}}\{L_m\ldots L_1L\cdot w_{-\tau}\}\bigr).$$

This may be computed inductively as follows. Write

$$\operatorname{span}_{\mathbf{C}}\{L_i\ldots L_1L\cdot w_{-\tau}\}=\sum_j F_{-\tau_j},$$

with $F_{-\tau_j}$ the irreducible representation of L_i of lowest weight $-\tau_j$. Then decompose each $F_{-\tau_j}$ under the action of $L_i\cap L_{i+1}$. Write this as

$$F_{-\tau_j}|_{L_i\cap L_{i+1}}=\sum_k U_{-\tau_{jk}},$$

with $U_{-\tau_{jk}}$ the irreducible $L_i\cap L_{i+1}$-representation of lowest weight $-\tau_{jk}$. The following lemma shows that the span of $L_{i+1}\cdot U_{-\tau_{jk}}$ is the irreducible L_{i+1}-representation $E_{-\tau_{jk}}$ of lowest weight $-\tau_{jk}$. It then follows that

$$\operatorname{span}_{\mathbf{C}}\{L_{i+1}\ldots L_1L\cdot w_{-\tau}\}=\sum_{j,k}E_{-\tau_{jk}}.$$

Lemma 34. *For each i, $\operatorname{span}_{\mathbf{C}}\{L_{i+1}\ldots L_1L\cdot w_{-\tau}\}$ is annihilated by $\mathfrak{l}_{i+1}\cap\mathfrak{u}_i^-$.*

Proof. It suffices to show that $(L_{i+1} \cap U_i^-)L_i \ldots L_1 L \cdot w_{-\tau} \subset L_i \ldots L_1 L \cdot w_{-\tau}$. We show this by applying induction on i to show that

$$Q_i L_{i-1} \ldots L_1 L \subset L_i \ldots L_1 L(B \cap K).$$

We have

$$\begin{aligned} Q_i L_{i-1} \ldots L_1 L &= L_i U_i^- L_{i-1} \ldots L_1 L \\ &\subset L_i U_{i-1}^- L_{i-1} \ldots L_1 L, \quad \text{by Lemma 20(3)}, \\ &= L_i Q_{i-1} L_{i-2} \ldots L_1 L, \\ &\subset L_i L_{i-1} \ldots L_1 L(B \cap K), \quad \text{by induction.} \qquad \square \end{aligned}$$

Note that the form of the subgroups L_i (Remark (23)) shows that the branching rules needed are those for $GL(r-1) \subset GL(r)$.

In the orthogonal group cases the same holds, except there are minor complications due to the fact that the L_i that appear are not necessarily connected. In this case branching rules for and $SO(r-1) \subset SO(r)$ are also used.

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