Nontrivial solutions of Kirchhoff-type problems via the Yang index

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Abstract

We obtain nontrivial solutions of a class of nonlocal quasilinear elliptic boundary value problems using the Yang index and critical groups.

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1. Introduction

In this paper we obtain a nontrivial solution of the problem

\[
\begin{cases}
- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u) & \text{in } \Omega, \\
\phantom{\Delta u} u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \), \( n = 1, 2, \) or \( 3 \), \( a, b > 0 \), and \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) such that

\[
\lim_{t \to 0} \frac{f(x, t)}{a t} = \lambda, \quad \lim_{|t| \to \infty} \frac{f(x, t)}{b t^3} = \mu \quad \text{uniformly in } x.
\]

(1.2)

This problem is related to the stationary analogue of the equation

\[
u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = g(x, t)
\]

(1.3)

proposed by Kirchhoff [11] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings. Kirchhoff’s model takes into account the changes in length of the string produced by transverse vibrations. Some early classical studies of Kirchhoff equations were Bernstein [5] and Pohozaev [14]. However, Eq. (1.3) received much attention only after Lions [12] proposed an abstract framework to the problem. Some interesting results can be found, for example, in [4,6,10]. More recently, Alves et al. [2] and Ma and Rivera [13] obtained positive solutions of such problems by variational methods. Similar nonlocal problems also model several physical and biological systems where \( u \) describes a process which depends on the average of itself, for example the population density, see [1,3,8,9,16].

Our assumptions on the asymptotic behaviors of \( f \) near zero and infinity lead us to the eigenvalue problems

\[
\begin{cases}
- \Delta u = \lambda u & \text{in } \Omega, \\
\phantom{- \Delta u} u = 0 & \text{on } \partial \Omega
\end{cases}
\]

(1.4)

and

\[
\begin{cases}
- \|u\|^2 \Delta u = \mu u^3 & \text{in } \Omega, \\
\phantom{- \|u\|^2 \Delta u} u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(1.5)

Denote by \( 0 < \lambda_1 < \lambda_2 \leq \cdots \) the eigenvalues of the linear problem (1.4). Using the Yang index we will construct an unbounded sequence of minimax eigenvalues \( 0 < \mu_1 \leq \mu_2 \leq \cdots \) of the nonlinear problem (1.5) for which the following theorem holds, where we set \( \lambda_0 = \mu_0 = -\infty \) for convenience.
Theorem 1.1. If \( \lambda \in (\lambda_l, \lambda_{l+1}) \) and \( \mu \in (\mu_m, \mu_{m+1}) \) is not an eigenvalue of (1.5), with \( l \neq m \), then problem (1.1) has a nontrivial solution.

2. Yang index

We briefly recall the definition and some basic properties of the Yang index. Yang [17] considered compact Hausdorff spaces with fixed-point-free continuous involutions and used the Čech homology theory, but for our purposes here it suffices to work with closed symmetric subsets of Banach spaces that do not contain the origin and singular homology groups.

Following [17], we first construct a special homology theory defined on the category of all pairs of closed symmetric subsets of Banach spaces that do not contain the origin and all continuous odd maps of such pairs. Let \((X, A), A \subset X\) be such a pair and \(C(X, A)\) its singular chain complex with \(\mathbb{Z}_2\) coefficients, and denote by \(T_\#\) the chain map of \(C(X, A)\) induced by the antipodal map \(T(x) = -x\). We say that a \(q\)-chain \(c\) is symmetric if \(T_\#(c) = c\), which holds if and only if \(c = c' + T_\#(c')\) for some \(q\)-chain \(c'\). The symmetric \(q\)-chains form a subgroup \(C_q(X, A; T)\) of \(C_q(X, A)\), and the boundary operator \(\partial_q\) maps \(C_q(X, A; T)\) into \(C_{q-1}(X, A; T)\), so these subgroups form a subcomplex \(C(X, A; T)\). We denote by

\[
Z_q(X, A; T) = \{ c \in C_q(X, A; T) : \partial_q c = 0 \},
\]

\[
B_q(X, A; T) = \{ \partial_{q+1} c : c \in C_{q+1}(X, A; T) \},
\]

and

\[
H_q(X, A; T) = Z_q(X, A; T) / B_q(X, A; T)
\]

the corresponding cycles, boundaries, and homology groups. A continuous odd map \(f : (X, A) \rightarrow (Y, B)\) of pairs as above induces a chain map \(f_\# : C(X, A; T) \rightarrow C(Y, B; T)\) and hence homomorphisms

\[
f_* : H_q(X, A; T) \rightarrow H_q(Y, B; T).
\]

Example 2.1 (Chung-Tao Yang [17, 1.8]). For the \(m\)-sphere,

\[
H_q(S^m; T) = \begin{cases} 
\mathbb{Z}_2 & \text{for } 0 \leq q \leq m, \\
0 & \text{for } q > m.
\end{cases}
\]

Let \(X\) be as above, and define homomorphisms \(v : Z_q(X; T) \rightarrow \mathbb{Z}_2\) inductively by

\[
v(z) = \begin{cases} 
\text{In}(c) & \text{for } q = 0, \\
v(\partial c) & \text{for } q > 0
\end{cases}
\]
if \( z = c + T_\#(c) \), where the index of a 0-chain \( c = \sum_i n_i \sigma_i \) is defined by \( \text{In}(c) = \sum_i n_i \). As in [17], \( v \) is well-defined and \( v B_q(X; T) = 0 \), so we can define the index homomorphism \( v_* : H_q(X; T) \to \mathbb{Z}_2 \) by \( v_*([z]) = v(z) \).

**Proposition 2.2** (Chung-Tao Yang [17, 2.8]). If \( F \) is a closed subset of \( X \) such that \( F \cup T(F) = X \) and \( A = F \cap T(F) \), then there is a homomorphism \( \Delta : H_q(X; T) \to H_{q-1}(A; T) \) such that \( v_*([\Delta z]) = v_*([z]) \).

Taking \( F = X \) we see that if \( v_* H_m(X; T) = \mathbb{Z}_2 \), then \( v_* H_q(X; T) = \mathbb{Z}_2 \) for \( 0 \leq q \leq m \). We define the **Yang index** of \( X \) by

\[
i(X) = \inf \{ m \geq -1 : v_* H_{m+1}(X; T) = 0 \}, \tag{2.7}\]

taking \( \inf \emptyset = \infty \). Clearly, \( v_* H_0(X; T) = \mathbb{Z}_2 \) if \( X \neq \emptyset \), so \( i(X) = -1 \) if and only if \( X = \emptyset \).

**Example 2.3** (Chung-Tao Yang [17, 3.4]). \( i(S^m) = m \).

**Proposition 2.4** (Chung-Tao Yang [17]). If \( f : X \to Y \) is as above, then \( v_* (f_*([z])) = v_*([z]) \) for \( [z] \in H_q(X; T) \), and hence \( i(X) \leq i(Y) \). In particular, this inequality holds if \( X \subset Y \).

Recall that the **Krasnoselskii Genus** of \( X \) is defined by

\[
\gamma(X) = \inf \{ m \geq 0 : \exists \text{ a continuous odd map } f : X \to S^{m-1} \}. \tag{2.8}\]

By Example 2.3 and Proposition 2.4,

**Proposition 2.5.** \( \gamma(X) \geq i(X) + 1 \).

**Proposition 2.6.** If \( i(X) = m \geq 0 \), then the reduced group \( \tilde{H}_m(X) \neq 0 \).

**Proof.** By (2.7),

\[
v_* H_q(X; T) = \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq q \leq m, \\ 0 & \text{for } q > m. \end{cases} \tag{2.9}\]

We show that if \( [z] \in H_m(X; T) \) is such that \( v_*([z]) \neq 0 \), then \( [z] \neq 0 \) in \( \tilde{H}_m(X) \). Arguing indirectly, assume that \( z \in B_m(X) \), say, \( z = \partial \sigma \). Since \( z \in B_m(X; T) \), \( T_\#(z) = z \). Let \( c' = c + T_\#(c) \). Then \( c' \in Z_{m+1}(X; T) \) since \( \partial c' = z + T_\#(z) = 2z = 0 \mod 2 \), and \( v_*([c']) = v(c') = v(\partial c) = v(z) \neq 0 \), contradicting \( v_* H_{m+1}(X; T) = 0 \).
3. Variational eigenvalues of (1.5)

Let $H = H^1_0(\Omega)$ be the usual Sobolev space, normed by

$$
\|u\| = \left( \int_{\Omega} |\nabla u|^2 \right)^{1/2},
$$

and let

$$
I(u) = \|u\|^4, \quad u \in S := \left\{ u \in H : \int_{\Omega} u^4 = 1 \right\}.
$$

By the Lagrange multiplier rule, $u \in S$ is a critical point of $I$ if and only if

$$
\|u\|^2 \int_{\Omega} \nabla u \cdot \nabla v = \mu \int_{\Omega} u^3 v \quad \forall v \in H
$$

for some $\mu \in \mathbb{R}$, i.e., $u$ is a weak solution of (1.5). Taking $v = u$ we see that the Lagrange multiplier $\mu$ equals the corresponding critical value $I(u)$. So the eigenvalues of (1.5) are precisely the critical values of the functional $I$. We use the customary notation

$$
I^a = \{ u \in S : I(u) \leq a \}
$$

for the sublevel sets of $I$.

**Lemma 3.1.** $I$ satisfies the Palais–Smale condition (PS), i.e., every sequence $(u_j)$ in $H$ such that $I(u_j)$ is bounded and $I'(u_j) \to 0$, called a (PS) sequence, has a convergent subsequence.

**Proof.** Since $\|u_j\|$ is bounded, for a subsequence, $u_j$ converges to some $u$ weakly in $H$ and strongly in $L^4(\Omega)$. Denoting by

$$
P_j v = v - \left( \int_{\Omega} u_j^3 v \right) u_j
$$

the projection of $v \in H$ onto the tangent space to $S$ at $u_j$, we have

$$
\|u_j\|^2 \int_{\Omega} \nabla u_j \cdot \nabla (u_j - u) = I(u_j) \int_{\Omega} u_j^3 (u_j - u)
$$

$$
+ \frac{1}{4} \langle I'(u_j), P_j (u_j - u) \rangle \to 0.
$$

so, passing to a subsequence, $u_j \to 0$ or $u$. $\square$
Denote by $\mathcal{A}$ the class of closed symmetric subsets of $S$, let

$$\mathcal{F}_m = \{A \in \mathcal{A} : i(A) \geq m - 1\},$$

and set

$$\mu_m := \inf_{A \in \mathcal{F}_m} \max_{u \in A} I(u).$$

**Proposition 3.2.** $\mu_m$ is an eigenvalue of (1.5) and $\mu_m \nearrow \infty$.

**Proof.** If $\mu_m$ is not a critical value of $I$, then there is an $\varepsilon > 0$ and an odd homeomorphism $\eta$ of $S$ such that $\eta(I^{\mu_m+\varepsilon}) \subset I^{\mu_m-\varepsilon}$ by the first deformation lemma. Taking $A \in \mathcal{F}_m$ with $\max I(A) \leq \mu_m + \varepsilon$, we have $A' = \eta(A) \in \mathcal{F}_m$ by Proposition 2.4, but $\max I(A') \leq \mu_m - \varepsilon$, contradicting (3.8).

Clearly, $\mu_{m+1} \geq \mu_m$. Since the sequence of Ljusternik–Schnirelmann eigenvalues $\tilde{\mu}_m$ of (1.5) defined using the genus $\gamma$ is unbounded (see, e.g., Struwe [15]) and $\mu_m \geq \tilde{\mu}_m$ by Proposition 2.5, $\mu_m \to \infty$. □

When $\mu$ is not an eigenvalue of (1.5), 0 is the only critical point of the associated variational functional

$$I_\mu(u) = \|u\|^4 - \mu \int_{\Omega} u^4, \quad u \in H$$

and hence its critical groups at 0 are defined and given by

$$C_q(I_\mu, 0) = H_q(I_\mu^0, I_\mu^0 \setminus \{0\})$$

(see, e.g., Chang [7]).

**Proposition 3.3.** If $\mu \in (\mu_m, \mu_{m+1})$ is not an eigenvalue of (1.5), then

$$C_m(I_\mu, 0) \neq 0.$$  

**Proof.** Since $I_\mu$ is positive homogeneous, $I_\mu^0$ is radially contractible to 0 and $I_\mu^0 \setminus \{0\}$ is homotopic to $I_\mu^0 \cap S$ via the radial projection onto $S$, so it follows from the long exact sequence of reduced homology groups of the pair $\left( I_\mu^0, I_\mu^0 \setminus \{0\} \right)$ that

$$C_m(I_\mu, 0) = H_m \left( I_\mu^0, I_\mu^0 \setminus \{0\} \right) \cong \tilde{H}_{m-1} \left( I_\mu^0 \cap S \right) = \tilde{H}_{m-1}(I^\mu),$$

where the last equality follows from $I_\mu|_S = I - \mu$. Since $\mu > \mu_m$, there is an $A \in \mathcal{F}_m$ such that $A \subset I^\mu$, so $i(I^\mu) \geq i(A) \geq m - 1$ by Proposition 2.4. On the other hand,
4. Proof of Theorem 1.1

Recall that a function $u \in H$ is called a weak solution of (1.1) if

$$
(a + b \|u\|^2) \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f(x, u) v \quad \forall v \in H.
$$

Weak solutions are the critical points of the $C^1$ functional

$$
\Phi(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \int_{\Omega} F(x, u), \quad u \in H.
$$

They are also classical solutions if $f$ is locally Lipschitz on $\overline{\Omega} \times \mathbb{R}$.

**Lemma 4.1.** If $\mu$ is not an eigenvalue of (1.5), $\Phi$ satisfies (PS).

**Proof.** As usual, it suffices to show that every (PS) sequence $(u_j)$ of $\Phi$ is bounded (see, e.g., Alves et al. [2, Lemma 1]). Suppose that $\rho_j = \|u_j\| \to \infty$ for a subsequence. Setting $v_j = u_j / \rho_j$ and passing to a further subsequence, $v_j$ converges to some $v$ weakly in $H$, strongly in $L^4(\Omega)$, and a.e. in $\Omega$. Passing to the limit in

$$
\int_{\Omega} \nabla v_j \cdot \nabla w - \int_{\Omega} \frac{f(x, u_j)}{\rho_j^2} v_j^3 \int_{\Omega} w = \frac{(\Phi'(u_j), w)}{(a + b \rho_j^2) \rho_j}
$$

gives

$$
\int_{\Omega} \nabla v \cdot \nabla w - \mu v^3 w = 0
$$

for each $w \in H$, and passing to the limit with $w = v_j - v$ shows that $\|v\| = 1$, so $\mu$ is an eigenvalue of (1.5), contrary to assumption. □

Let

$$
\Phi_0(u) = \frac{a}{2} \left( \|u\|^2 - \int_{\Omega} u^2 \right), \quad \Phi_\infty(u) = \frac{b}{4} \left( \|u\|^4 - \mu \int_{\Omega} u^4 \right).
$$

(4.5)
Proposition 4.2. If \( \lambda \) and \( \mu \) are not eigenvalues of (1.4) and (1.5), respectively, then for all sufficiently small \( \rho > 0 \) and sufficiently large \( R > 4\rho \) there is a functional \( \widetilde{\Phi} \in C^1(H, \mathbb{R}) \) such that 

\[
\begin{align*}
\Phi(0) = & \begin{cases} \Phi_0(u), & \|u\| \leq \rho, \\
\Phi(u), & 2\rho \leq \|u\| \leq R/2, \\
\Phi_\infty(u), & \|u\| \geq R,
\end{cases} \\
\Phi' = & \begin{cases} \widetilde{\Phi}_0(u), & \|u\| \leq 2\rho, \\
\widetilde{\Phi}(u), & R/2 \leq \|u\| \leq R/2, \\
\widetilde{\Phi}_\infty(u), & \|u\| \geq R,
\end{cases}
\end{align*}
\]

in particular,

\[
C_q(\widetilde{\Phi}, 0) = C_q(\Phi_0, 0), \quad C_q(\widetilde{\Phi}, \infty) = C_q(\Phi_\infty, 0),
\]

(ii) \( u = 0 \) is the only critical point of \( \Phi \) and \( \widetilde{\Phi} \) with \( \|u\| \leq 2\rho \) or \( \|u\| \geq R/2 \), in particular, critical points of \( \widetilde{\Phi} \) are the solutions of (1.1), (iii) \( \Phi \) satisfies (PS). 

**Proof.** Since \( \lambda \) and \( \mu \) are not eigenvalues of (1.4) and (1.5), respectively, \( \Phi_0 \) and \( \Phi_\infty \) satisfy (PS) and have no critical points with \( \|u\| = 1 \), so

\[
\delta_0 := \inf_{\|u\| = 1} \|\Phi'(u)\| > 0, \quad \delta_\infty := \inf_{\|u\| = 1} \|\Phi'_\infty(u)\| > 0
\]

and

\[
\inf_{\|u\| = \rho} \|\Phi'_0(u)\| = \rho \delta_0, \quad \inf_{\|u\| = R} \|\Phi'_\infty(u)\| = R^3 \delta_\infty
\]

by homogeneity. Let

\[
\begin{align*}
\Psi_0(u) = & \frac{b}{4} \|u\|^4 + \int_\Omega \frac{a\lambda}{2} u^2 - F(x, u), \\
\Psi_\infty(u) = & \frac{a}{2} \|u\|^2 + \int_\Omega \frac{b\mu}{4} u^4 - F(x, u).
\end{align*}
\]

By (1.2),

\[
\sup_{\|u\| = \rho} |\Psi_0(u)| = o(\rho^3), \quad \sup_{\|u\| = R} |\Psi_\infty(u)| = o(R^4)
\]

and

\[
\sup_{\|u\| = \rho} \|\Psi'_0(u)\| = o(\rho), \quad \sup_{\|u\| = R} \|\Psi'_\infty(u)\| = o(R^3)
\]
as \( \rho \to 0 \) and \( R \to \infty \). Since \( \Phi = \Phi_0 + \Psi_0 = \Phi_\infty + \Psi_\infty \), it follows from (4.8) and (4.11) that

\[
\inf_{\|u\| = \rho} \left\| \Phi'(u) \right\| = \rho (\delta_0 + o(1)), \quad \inf_{\|u\| = R} \left\| \Phi'(u) \right\| = R^3 (\delta_\infty + o(1)). \tag{4.12}
\]

Take smooth functions \( \varphi_0, \varphi_\infty : [0, \infty) \to [0, 1] \) such that

\[
\varphi_0(t) = \begin{cases} 1, & t \leq 1, \\ 0, & t \geq 2, \end{cases} \quad \varphi_\infty(t) = \begin{cases} 0, & t \leq 1/2, \\ 1, & t \geq 1, \end{cases} \tag{4.13}
\]

and set

\[
\widetilde{\Phi}(u) = \Phi(u) - \varphi_0(\|u\|/\rho) \Psi_0(u) - \varphi_\infty(\|u\|/R) \Psi_\infty(u). \tag{4.14}
\]

Since

\[
\left\| d(\varphi_0(\|u\|/\rho)) \right\| = O(\rho^{-1}), \quad \left\| d(\varphi_\infty(\|u\|/R)) \right\| = O(R^{-1}), \tag{4.15}
\]

(4.12) holds with \( \Phi \) replaced by \( \widetilde{\Phi} \) also, and (i) and (ii) follow.

As for (iii) \( \left\| \Phi' \right\| \) is bounded away from 0 for \( \rho \leq \|u\| \leq 2\rho \) and for \( \|u\| \geq R/2 \) by construction, so every (PS) sequence for \( \Phi \) has a subsequence in \( \|u\| < \rho \) or in \( 2\rho < \|u\| < R/2 \), which is then a (PS) sequence for \( \Phi_0 \) or for \( \Phi \), respectively. \( \square \)

We are now ready to prove Theorem 1.1. Since \( \lambda \in (\lambda_l, \lambda_{l+1}) \) and \( l \neq m \)

\[
C_m(\widetilde{\Phi}, 0) = C_m(\Phi_0, 0) = 0, \tag{4.16}
\]

but

\[
C_m(\widetilde{\Phi}, \infty) = C_m(\Phi_\infty, 0) = C_m(I_\mu, 0) \neq 0 \tag{4.17}
\]

by Proposition 3.3, so \( \widetilde{\Phi} \) must have a nontrivial critical point.

References