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Symmetric Composition Algebras

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A construction of all the Okubo algebras over fields of characteristic 3 is provided and the complete classification of the composition algebras with associative norm over fields is finished. © 1997 Academic Press

1. INTRODUCTION

An algebra A over a field F is said to be a *composition algebra* if there exists a strictly nondegenerate quadratic form (called the *norm*) $n: A \to F$ such that

$$n(xy) = n(x)n(y) \tag{1}$$

for any $x, y \in A$. The form being strictly nondegenerate means that its polarization, defined by

$$n(x, y) = n(x + y) - n(x) - n(y),$$

is a nondegenerate bilinear form.

Linearization of (1) gives

$$\begin{aligned} n(xy, xz) = n(yx, zx) = n(x)n(y, z), \\ n(xy, zt) + n(zy, xt) = n(x, z)n(y, t), \end{aligned} \tag{2}$$

for any $x, y, z, t \in A$.

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Unital composition algebras, also called *Hurwitz algebras*, constitute a well-known class of algebras of dimension 1, 2, 4, or 8, and are classified by the "Generalized Hurwitz Theorem" [ZSSS82, Chap. II; KMRT, (35.16)].

Through the work of Petersson [P69], Okubo [O78, O95], Okubo and Myung [O-M80], Okubo and Osborn [O-O181, O-O281], Faulkner [F88], Elduque and Myung [E-M91, E-M93], and Elduque and Pérez [E-P96], those composition algebras in which the norm is associative, that is, it satisfies

$$n(xy, z) = n(x, yz) \tag{3}$$

for any x, y, z, have been considered.

Because of their nice and symmetric properties, these composition algebras of dimension ≥ 2 with associative norm have been called *symmetric composition algebras* in [KMRT] and this notation will be used here.

It turns out [M86, p. 86] that for a strictly nondegenerate quadratic form n defined on an algebra, conditions (1) and (3) are equivalent to the validity of

$$(xy)x = x(yx) = n(x)y \tag{4}$$

for any x, y. In particular, by taking an element $0 \neq x \in A$ with $n(x) \neq 0$, (4) shows that the left and right multiplications by x are bijective, whence it follows (see [K53]) that any symmetric composition algebra is finite dimensional, its dimension being restricted to 2, 4, or 8.

The linearization of (4);

$$(xy)z + (zy)x = x(yz) + z(yx) = n(x, z)y, \tag{5}$$

will be used throughout the paper.

Two classes of symmetric composition algebras have played a key role: the para-Hurwitz and Okubo algebras. Given a Hurwitz algebra C, with multiplication xy, the *para-Hurwitz* algebra associated to C is the algebra defined on the same vector space C, but with new multiplication $x \cdot y = \bar{x}\bar{y}$ (see [O-M80]), where $x \mapsto \bar{x}$ is the standard conjugation of C.

By definition, the Okubo algebras are the forms of the pseudo-octonion algebra—which will be defined shortly—that is, those algebras which become isomorphic to the pseudo-octonion algebra after extending scalars. The original definition in [O78] of the pseudo-octonion algebra required some restrictions on the ground field. In order to avoid this, a new definition, which extended the one given by Okubo, valid over any field was proposed in [E-P96], inspired in a previous construction of Petersson [P69]. Given a Hurwitz algebra C and an automorphism τ of C such that $\tau^3 = 1$, the *Petersson algebra* C_τ is the algebra defined on C with the new multiplication

$$x * y = \bar{x}^{\tau}\bar{y}^{\tau^{-1}}$$

(see [E-P96, (1.2)]). The name Petersson algebra appeared for the first time in [KMRT, Sect. 36B]. Now, in case $C = C(F)$ is Zorn's vector matrix algebra

$$C(F) = \left\{ \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} : \alpha, \beta \in F, u, v \in F \times F \times F \right\}$$

with multiplication

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}\begin{pmatrix} \alpha' & u' \\ v' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + u \cdot v' & \alpha u' + \beta' u - v \times v' \\ \alpha' v + \beta v' + u \times u' & \beta\beta' + u \cdot u' \end{pmatrix},$$

where $u \cdot v$ and $u \times v$ denote the usual dot and vector product in $V = F \times F \times F$, let us take any $\varphi \in \mathrm{End}_F(V)$ with minimum polynomial $X^3 - 1$ and define

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}^{\tau} = \begin{pmatrix} \alpha & u^{\varphi} \\ v^{(\varphi^*)-1} & \beta \end{pmatrix},$$

where φ^* is the adjoint relative to the dot product. Then τ is an automorphism of $C(F)$ of order 3 and the Petersson algebra $C(F)_{\tau}$ (unique up to isomorphism) is called the *pseudo-octonion algebra* and denoted by $P_8(F)$.

The main results in [E-P96], extending those in [O-O181, O-O281], assert that the symmetric composition algebras with nonzero idempotents are exactly the Petersson algebras, and that these are either para-Hurwitz or some Okubo algebras constructed in a specific way.

Since it is known [O-O181; O-O281; KMRT, (36.11)] that given a symmetric composition algebra, either it already contains a nonzero idempotent or one finds these idempotents after a scalar extension of degree three (see also the paragraph before Corollary 3.4 here), it follows that the symmetric composition algebras are the forms of the para-Hurwitz algebras and the Okubo algebras. On the other hand, any form of a para-Hurwitz algebra of dimension $\neq 2$ is again a para-Hurwitz algebra [E-P96, Lemma 3.3], so only the forms of the two-dimensional para-Hurwitz algebras and the Okubo algebras need to be determined.

Along the same lines of the definition by Okubo of the pseudo-octonion algebra, but independently, Faulkner [F88] obtained composition algebras from separable alternative algebras of generic degree 3 over fields of characteristic $\neq 3$. Following this idea, the symmetric composition algebras over fields of characteristic $\neq 3$ are classified in [E-M93] as follows.

Let F be a field of characteristic $\neq 3$ and assume first that F contains a primitive cube root ω of 1. Let A be a separable alternative algebra of degree 3 over F with generic minimum polynomial

$$p_x(\lambda) = \lambda^3 - T(x)\lambda^2 + S(x)\lambda - N(x)1,$$

for a linear form T, a quadratic form S, and a cubic one N. Consider the subspace $A_0 = \{x \in A: T(x) = 0\}$ with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{2\omega + 1}{3} T(xy)1.$$

Then $(A_0, *)$ is a composition algebra relative to S.

If, on the contrary, F does not contain the primitive cube roots of 1, consider the quadratic field extension $K = F[\omega]$ obtained by adjoining one of them, ω, to F. Let A be a separable alternative algebra of degree 3 over K with generic minimum polynomial $p_x(\lambda)$ as above and equipped with an involution J of the second kind $(\omega^J = \omega^2)$. Consider the F-subspace $\tilde{A}_0 = \{x \in A: x^J = -x$ and $T(x) = 0\}$, with the (well defined) multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{2\omega + 1}{3} T(xy)1.$$

Then $(\tilde{A}_0, *)$ is a composition algebra relative to $S|_{\tilde{A}_0}$. Moreover, we have

THEOREM 1.1 [E-M93]. *Let C be an algebra over a field F of characteristic $\neq 3$ with $\dim_F C > 1$.*

(i) *If F contains the cube roots of 1 (type I), then C is a composition algebra with associative bilinear form if and only if there exists a separable alternative algebra A of degree 3 over F such that $C \cong (A_0, *)$.*

Two such algebras C's are isomorphic if and only if so are their corresponding alternative algebras.

(ii) *If F does not contain the cube roots of 1 (type II) and $K = F[\omega]$ with $\omega \neq 1 = \omega^2$, then C is a composition algebras with associative bilinear form if and only if there exists a separable alternative algebra A over K with an involution of the second kind such that $C \cong (\tilde{A}_0, *)$.*

Two such algebras C's are isomorphic if and only if so are their corresponding alternative algebras as algebras with involution.

Actually, in [E-M93] the ground field is assumed to have characteristic $\neq 2, 3$, but everything is valid also in characteristic 2. The only necessary changes affect the argument in Section 6 of the latter reference for types IIc and Ic. For type IIc and following the notation in that paper, $A = K \times C$ with $K^J = k$ and $C^J = C$, the algebra $D = \{x \in C\colon x^{\varphi} = x\}$ is Hurwitz with the restrictions of the trace t and norm n of C, and, although the decomposition $D = Fe \oplus D_0$ [E-M93, end of p. 2496] and (6.1) in [E-M93] are not valid for characteristic 2, the map (with $x \cdot y = \bar{x}\bar{y}$)

$$\begin{aligned} \psi\colon (D, \cdot\,) &\to (\overline{A}_0, *) \\ a &\mapsto (q^{-1}t(a), q^{-1}\omega t(a) - a) \end{aligned}$$

may be shown to be an isomorphism valid in any characteristic $\neq 3$. Similarly, for type Ic the map

$$\begin{aligned} \varphi\colon (C, \cdot\,) &\to (A_0, *) \\ a &\mapsto (q^{-1}t(a), q^{-1}\omega t(a) - a) \end{aligned}$$

is an isomorphism.

In a slightly different way, this has been very nicely developed in [KMRT, Chap. VIII], where it has been noted too that the results in [O-O181, and E-P96] give a proof of the known classification of the separable alternative algebras of generic degree three over fields of characteristic $\neq 3$. The Okubo algebras with idempotents are the ones that appear related to the algebra of 3×3 matrices over a field [E-M93, Proposition 7.4].

Therefore, the knowledge of the symmetric composition algebras will be complete as long as the symmetric composition algebras without nonzero idempotents over fields of characteristic 3 are classified. This is the objective of this paper, which will be structured as follows. The next section will provide a new construction of some Okubo algebras $C_{\lambda,\mu}$ over fields of characteristic 3, depending on two parameters. Actually this construction provides all the Okubo algebras (Theorem 5.1). Then some properties of the symmetric composition algebras without nonzero idempotents will be given in Section 3. In Section 4, it will be shown that for any Okubo algebra without nonzero idempotents, scalars λ and μ can be found so that A is isomorphic to $C_{\lambda,\mu}$ (characteristic $= 3$). Finally, Section 5 will provide the classification of the symmetric composition algebras over fields of characteristic 3.

Unless otherwise stated, *the characteristic of the ground field F will be assumed from now on to be 3*. Given a subset of an algebra A, the subalgebra (respectively ideal or subspace) generated by S will be denoted by alg$\langle S\rangle$ (respectively ideal$\langle S\rangle$ or span$\langle S\rangle$). If the algebra is equipped with a quadratic form n, $S^{\perp}$ will denote the orthogonal subspace to S relative to the polarization of n: $S^{\perp}=\{x\in A\colon n(x,s)=0 \text{ for any } s\in S\}$.

2. A NEW CONSTRUCTION OF OKUBO ALGEBRAS

Let us consider the algebra $F[X,Y]$ of polynomials in two variables over the ground field F and let us take $0\neq\lambda,\ \mu\in F$, and the ideal $I=$ ideal$\langle X^3-\lambda,\ Y^3-\mu\rangle$.

In $F[X,Y]$ a new multiplication is defined by

$$X^iY^j\diamond X^{i'}Y^{j'}=\left(1-\left|\begin{vmatrix} i & j\\ i' & j'\end{vmatrix}\right|\right)X^{i+i'}Y^{j+j'}. \tag{6}$$

Then I is still an ideal of the algebra $(F[X,Y],\diamond)$. Let a and b denote the classes of X and Y modulo I. The quotient algebra $(F[X,Y]/I,\diamond)=(F[a,b],\diamond)$ has dimension 9 and decomposes as the direct sum

$$F[a,b]=F\oplus C_{\lambda,\mu},$$

with

$$C_{\lambda,\mu}=\text{span}\langle a^ib^j\colon 0\leq i,j\leq 2,(i,j)\neq(0,0)\rangle.$$

For $u,v\in C_{\lambda,\mu}$, the product $u\diamond v$ decomposes accordingly as

$$u\diamond v=n(u,v)+u*v,$$

with $n(u,v)\in F$ and $u*v\in C_{\lambda,\mu}$. Then $n(a^ib^j,a^{i'}b^{j'})=0$ unless both $i+i'$ and $j+j'$ are multiples of 3. In that case, for $i+i'=3r$ and $j+j'=3s$, $n(a^ib^j,a^{i'}b^{j'})=a^{i+i'}b^{j+j'}=\lambda^r\mu^s$, so $n(,)$ is a nondegenerate symmetric bilinear form on $C_{\lambda,\mu}$, which is the polarization of the quadratic form given by

$$n(u)=\tfrac{1}{2}n(u,u)=-n(u,u).$$

It is clear that the Witt index of n if 4.

It will be shown later on (Theorems 4.5 and 5.1) that the algebras $(C_{\lambda,\mu},*)$ (which will be denoted just by $C_{\lambda,\mu}$) constitute precisely the class of Okubo algebras over F. Also necessary and sufficient conditions will be given for two such algebras to be isomorphic, thus obtaining a classification of the Okubo algebras.

Let us obtain first a suitable basis and multiplication table of $C_{\lambda,\mu}$. First, in $F[a,b]$ the elements a and b are invertible with inverses $\lambda^{-1}a^2$ and $\mu^{-1}b^2$, respectively. Hence it makes sense to consider the "monomials" a^ib^j with $i,j\in\mathbb{Z}$. The formula, which follows from (6),

$$a^ib^j * a^{i'}b^{j'} = \left(1-\delta_{\overline{i+i'},\bar{0}}\delta_{\overline{j+j'},\bar{0}}\right)\left(1-\left|\begin{vmatrix} i & j \\ i' & j' \end{vmatrix}\right|\right)a^{i+i'}b^{j+j'}, \qquad (7)$$

where $\bar{i}$ denotes the class of i modulo 3 and δ is the usual Kronecker delta, is valid for $i,i',j,j'\in\mathbb{Z}$ with $(\bar{i},\bar{j})\neq(\bar{0},\bar{0})\neq(\overline{i'},\overline{j'})$. As a basis for $C_{\lambda,\mu}$ we take the elements $x_{i,j}=-a^ib^j$, with $-1\leq i,j\leq 1$ and $(i,j)\neq(0,0)$. Then

$$n(x_{i,j},x_{i',j'}) = \delta_{i+i',0}\delta_{j+j',0}, \qquad (8)$$

and by using (7), the corresponding multiplication table is easily shown in Table I.

A simple inspection shows that with $\lambda=1$ and $\mu=\alpha$, this multiplication table is exactly Table 2 in [E-P96] (with different notation for the elements in the basis). Therefore, the algebras $C(\alpha)$ in [E-P96] are exactly (up to isomorphism) the algebras $C_{1,\alpha}$ and, as a consequence, the class of algebras $C_{\lambda,\mu}$ contains the class of Okubo algebras with nonzero idempotents.

PROPOSITION 2.1. *For any $0\neq\lambda,\ \mu\in F$, the algebra $C_{\lambda,\mu}$ is an Okubo algebra.*

Proof. By extending scalars to $K=F(\lambda^{1/3},\mu^{1/3})$, it can be assumed that $\lambda=\mu=1$. But $C_{1,1}(=C(1))$ is exactly the pseudo-octonion algebra. ∎

Remark. It can be proved in an elementary way that $C_{\lambda,\mu}$ is a symmetric composition algebra. To do this, one has to check the validity of (5) for the monomials a^ib^j with $0\leq i,\ j\leq 2$, $(i,j)\neq(0,0)$, and this is easily proved by working with vectors in $\mathbb{F}_3^2$, where $\mathbb{F}_3$ is the field of three elements.

Given an algebra A, with multiplication xy, the opposite algebra A^{op} is the new algebra defined on A, but with the new multiplication $x\cdot y=yx$. Interchanging the roles of a and b in $C_{\lambda,\mu}$ and using that $\begin{vmatrix} i & j \\ i' & j' \end{vmatrix}=\begin{vmatrix} j' & i' \\ j & i \end{vmatrix}$

TABLE I

	$x_{1,0}$	$x_{-1,0}$	$x_{0,1}$	$x_{0,-1}$	$x_{1,1}$	$x_{-1,-1}$	$x_{-1,1}$	$x_{1,-1}$
$x_{1,0}$	$-\lambda x_{-1,0}$	0	0	$x_{1,-1}$	0	$x_{0,-1}$	0	$\lambda x_{-1,-1}$
$x_{-1,0}$	0	$-\lambda^{-1}x_{1,0}$	$x_{-1,1}$	0	$x_{0,1}$	0	$\lambda^{-1}x_{1,1}$	0
$x_{0,1}$	$x_{1,1}$	0	$-\mu x_{0,-1}$	0	$\mu x_{1,-1}$	0	0	$x_{1,0}$
$x_{0,-1}$	0	$x_{-1,-1}$	0	$-\mu^{-1}x_{0,1}$	0	$\mu^{-1}x_{-1,1}$	$x_{-1,0}$	0
$x_{1,1}$	$\lambda x_{-1,1}$	0	0	$x_{1,0}$	$-(\lambda\mu)x_{-1,-1}$	0	$\mu x_{0,-1}$	0
$x_{-1,-1}$	0	$\lambda^{-1}x_{1,-1}$	$x_{-1,0}$	0	0	$-(\lambda\mu)^{-1}x_{1,1}$	0	$\mu^{-1}x_{0,1}$
$x_{-1,1}$	$x_{0,1}$	0	$\mu x_{-1,-1}$	0	0	$\lambda^{-1}x_{1,0}$	$-\lambda^{-1}\mu x_{1,-1}$	0
$x_{1,-1}$	0	$x_{0,-1}$	0	$\mu^{-1}x_{1,1}$	$\lambda x_{-1,0}$	0	0	$-\lambda\mu^{-1}x_{-1,1}$

we arrive at:

PROPOSITION 2.2. *For any* $0 \neq \lambda, \mu \in F$, $C_{\mu,\lambda}$ *is isomorphic to* $C_{\lambda,\mu}^{op}$.

3. SYMMETRIC COMPOSITION ALGEBRAS WITHOUT IDEMPOTENTS

Let A be a symmetric composition algebra over F with multiplication denoted by juxtaposition and norm n. As in [E-P96, Sect. 5] or [KMRT, (36.11)], let us consider the cubic form $g\colon A \to F$ given by $g(x) = n(x, x^2)$.

Our hypotheses on the characteristic of F being 3 and on the associativity of n imply that g is a semilinear map of the vector space A over F on the vector space F over its subfield F^3, with $g(\alpha x) = \alpha^3 g(x)$ for any $\alpha \in F$ and $x \in A$.

LEMMA 3.1. *For any* $x \in A$, $g(x^2) = g(x)^2 + n(x)^3$.

Proof. By (4), for any $x \in A$, $x^2x^2 + (x^2x)x = n(x, x^2)x$ and $x^2x = xx^2 = n(x)x$. Therefore

$$x^2x^2 = g(x)x - n(x)x^2. \tag{9}$$

Hence,

$$\begin{aligned} g(x)^2 &= n(x^2, x^2x^2) = n\big(x^2, g(x)x - n(x)x^2\big) \qquad \text{by (9)} \\ &= g(x)n(x^2, x) - n(x)n(x^2, x^2) \\ &= g(x)^2 - n(x)^2 n(x, x) = g(x^2) + n(x)^3. \end{aligned}$$ ∎

COROLLARY 3.2. $F^3 + g(A)$ *is a subfield of* F (*so it is a purely inseparable extension of exponent* 1 *of* F^3).

Proof. Linearizing the assertion in Lemma 3.1, we obtain for any $x, y \in A$ that

$$2g(x)g(y) = g(xy + yx) - n(x, y)^3, \tag{10}$$

so $F^3 + g(A)$ is an F^3-subalgebra of F of finite dimension, hence a subfield of F. ∎

PROPOSITION 3.3. *Let* A *be a symmetric composition algebra over* F. *Then the following conditions are equivalent*:

(i) *A contains some nonzero idempotent,*

(ii) *there is an element $x \in A$ with $g(x) = 1$,*

(iii) *F^3 is contained in $g(A)$,*

(iv) *there is a nonzero element $x \in A$ with $g(x) = 0$.*

Proof. Assume first that there is a nonzero idempotent $e \in A$. Then $e = e^2e = n(e)e$, so $n(e) = 1$ and $g(-e) = n(-e, e^2) = -n(e,e) = -2n(e) = n(e) = 1$ and (ii) is satisfied.

Obviously, assertion (ii) implies (iii). Assume now that F^3 is contained in $g(A)$. Then, since the field extension $F^3 + g(A)/F^3$ is purely inseparable, its degree is a power of 3; but since $\dim_F A = 2, 4$, or 8, $\dim_{F^3}(F^3 + g(A)) = \dim_{F^3} g(A) \leq 8$ and we get that $\dim_{F^3} g(A) = 1$ or 3. Thus g is not one-to-one, so (iv) follows.

Finally, assume that g is isotropic (condition (iv)) or, equivalently, that $\ker g \neq 0$. Then $g\colon A \to F$ is not one-to-one, so $\dim_{F^3}(F^3 + g(A)) < 9$. Hence $\dim_{F^3}(F^3 + g(A)) = 1$ or 3 and $\dim_{F^3} g(A) \leq 3$. If there is a nonzero element $x \in \ker g$ with $n(x) \neq 0$, then by (9) the element $(-1/n(x))x^2$ is a nonzero idempotent. This is always the case if the dimension of A is 8, since in this case $\dim_F \ker g \geq 5$. So assume that for any $x \in \ker g$, $n(x) = 0$ and take $0 \neq a \in \ker g$. By (9), $a^2a^2 = 0$ so we can assume too that $a^2 = 0$ (otherwise a can be substituted by a^2). Substitute now x by $x + y$ and $x - y$ in (9) and add to obtain

$$\begin{aligned} x^2y^2 + y^2x^2 + (xy + yx)^2 \\ = -n(x,y)(xy + yx) - n(x)y^2 - n(y)x^2. \end{aligned}$$

Take $b \in A$ with $n(a,b) = -1$ and put $x = a$ and $y = b$ in the last equation to get that $e = ab + ba$ is an idempotent, and it is nonzero by (10). ∎

The fact that condition (iv) implies condition (i) is valid for the symmetric composition algebras over any field, as shown in [KMRT, (36.11)].

From the proof above, it follows that if A is a symmetric composition algebra without nonzero idempotents, then $g\colon A \to F$ is a one-to-one semilinear map (this will be very important in the sequel), $g(A) \cap F^3 = 0$, and $F^3 + g(A)$ is a purely inseparable extension of F^3 of exponent 1, so its degree is 3 or 9. Hence the dimension of A is either 2 or 8. This result was already known, since the symmetric composition algebras of dimension 4 are para-Hurwitz algebras [E-P96, Lemma 3.3], so they contain nonzero idempotents.

It follows too from Proposition 3.3 that if A is any symmetric composition algebra without nonzero idempotents, then there is a purely inseparable field extention K of F of degree 3 such that $A_K = K \otimes_F A$ does contain nonzero idempotents (it is enough to take $K = F(\lambda)$ with λ a root of the polynomial $g(a + Xb) = g(a) + X^3g(b)$, for linearly independent elements a and b of A). This result, with a different proof, had been noticed in [O-O281].

COROLLARY 3.4. *The norm of any Okubo algebra without nonzero idempotents represents* 0. *Hence its Witt index is* 4.

Proof. Let A be an Okubo algebra with nonzero idempotents and let n be its norm. Let K/F be a purely inseparable field extension of degree 3 such that $A_K = K \otimes_F A$ contains nonzero idempotents. By [E-P96, Lemma 3.7], the norm in A_K represents 0 and so does n by a theorem of Springer [S52]. Moreover, since n is also the norm of a Hurwitz algebra, and it represents 0, the Witt index has to be maximum. ∎

We close this section with a result on subalgebras generated by one element.

PROPOSITION 3.5. *Let A be a symmetric composition algebra without nonzero idempotents and let x be any nonzero element of A. The* $\text{alg}\langle x \rangle$ *is a two-dimensional composition subalgebra of A.*

Proof. It cannot be $x^2 = 0$, since $g(x) = n(x, x^2) \neq 0$ by Proposition 3.3. By hypothesis, x and x^2 are linearly independent and (4) and (9) show that $\text{alg}\langle x \rangle = Fx + Fx^2$, which is two-dimensional. The coordinate matrix of the restriction of $n(,)$ to $\text{alg}\langle x \rangle$ in the basis $\{x, x^2\}$ is

$$\begin{pmatrix} n(x,x) & n(x,x^2) \\ n(x^2,x) & n(x^2,x^2) \end{pmatrix} = \begin{pmatrix} -n(x) & g(x) \\ g(x) & -n(x)^2 \end{pmatrix},$$

with determinant $\Delta = n(x)^3 - g(x)^2 = -(g(x^2) + n(x)^3)$ by Lemma 3.1. Thus if $\Delta = 0$, $g(x^2) = -n(x)^3 \in g(A) \cap F^3$, which is zero by Proposition 3.3(iii). This forces $x^2 = 0$, a contradiction. ∎

4. OKUBO ALGEBRAS WITHOUT IDEMPOTENTS

In this section the symmetric composition algebras without nonzero idempotents of dimension 8 will be studied. As mentioned in the Introduction, they are necessarily Okubo algebras. By Proposition 3.5, if A is any such Okubo algebra, for any $0 \neq x \in A$, $\text{alg}\langle x \rangle$ is a two-dimensional composition subalgebra of A.

PROPOSITION 4.1. *Let A be an Okubo algebra without nonzero idempotents and norm n. Then for any nonzero isotropic element $x \in A$ there is another element $0 \neq y \in A$ with $n(y) = 0$ and* $n(\text{alg}\langle x \rangle, \text{alg}\langle y \rangle) = 0$.

Proof. Let us take any nonzero $x \in A$ with $n(x) = 0$ (this is possible by Corollary 3.4). Since $\text{alg}\langle x \rangle$ is a two-dimensional composition subalgebra of A, A decomposes as the direct sum $A = \text{alg}\langle x \rangle \oplus \text{alg}\langle x \rangle^{\perp}$. The kernel of the left multiplication L_x by x has dimension 4, since both its kernel and its image are isotropic subspaces whose dimensions add up to 8, so there is an element $z \in \operatorname{alg}\langle x\rangle^{\perp}$ with $y = xz \neq 0$. Then again $\operatorname{alg}\langle y\rangle = Fy + Fy^2$ is a two-dimensional composition subalgebra of A, and by (2) and (3),

$$\begin{aligned} n(x, y) &= n(x, xz) = n(x^2, z) = 0, \\ n(x^2, y) &= n(x^2, xz) = n(x)n(x, z) = 0, \\ n(x, y^2) &= n(xy, y) = n(xy, xz) = n(x)n(y, z) = 0, \\ n(x^2, y^2) &= -n(yx, xy) + n(x, y)^2 = -n(y, x(xy)) \\ &= -n(xz, x(xy)) = -n(x)n(z, xy) = 0, \end{aligned}$$

so $n(\operatorname{alg}\langle x\rangle, \operatorname{alg}\langle y\rangle) = 0$, as required. ∎

PROPOSITION 4.2. *Let A be any Okubo algebra without nonzero idempotents and norm n, and let x and y be nonzero isotropic vectors of A with $n(\operatorname{alg}\langle x\rangle, \operatorname{alg}\langle y\rangle) = 0$. Then either $xy = 0$ or $yx = 0$, but not both. Moreover, $\operatorname{alg}\langle xy + yx\rangle \subseteq (\operatorname{alg}\langle x\rangle \oplus \operatorname{alg}\langle y\rangle)^{\perp}$.*

Proof. Because of (10), Proposition 3.3, and the hypothesis of the proposition,

$$g(xy + yx) = 2g(x)g(y) \neq 0,$$

so at least one of xy and yx is not 0.

Now by (3), (4), and (5)

$$(xy)(yx) = -x(y(xy)) + n(xy, x)y = -n(y)x^2 + n(x^2, y)y = 0,$$

and similarly $(yx)(xy) = 0$. Therefore $(xy)(yx) + (yx)(xy) = 0$. Also $n(xy, yx) = -n(y^2, x^2) + n(x, y)^2 = 0$ by (2), so by (10) again

$$2g(xy)g(yx) = g((xy)(yx) + (yx)(xy)) + n(xy, yx)^3 = 0,$$

and either $g(xy) = 0$ and $g(yx) = 0$. Proposition 3.3 finishes the proof with the exception of the last assertion, which now follows quickly from (2) and (3). ∎

THEOREM 4.3. *Given scalars $\lambda \in F \setminus F^3$ and $\mu \in F \setminus F^3(\lambda)$, there is a unique, up to isomorphism, Okubo algebra A containing elements x, y with $g(x) = \lambda$, $g(y) = \mu$, $n(\operatorname{alg}\langle x\rangle, \operatorname{alg}\langle y\rangle) = 0$ and $xy = 0$. Moreover, A does not contain nonzero idempotents and it is (isomorphic to) the Okubo algebra $C_{\lambda, \mu}$.*

Proof. In case an Okubo algebra A satisfying these conditions exists, then λ, $\mu \in F^3 + g(A)$, so $F^3 \subset F^3(\lambda) \subset F^3 + g(A)$ and the degree of the field extension $F^3 + g(A)/F^3$ is necessarily 9. By Proposition 3.3, A does not contain any nonzero idempotent. Besides, by repeated use of (3) and (5)

$$\begin{aligned} yx^2 &= -x(xy) + n(x,y)x = 0, \\ x^2y &= -(yx)x + n(x,y)x = -(yx)x, \\ (yx)^2 &= (yx)(yx) = -((yx)x)y + n(y,yx)x \\ &= (x^2y)y - n(x,y)xy + n(y^2,x)x \\ &= -y^2x^2 + n(x^2,y)y = -y^2x^2, \\ (x^2y)^2 &= (x^2y)(x^2y) = -((x^2y)y)x^2 + n(x^2,x^2y)y \\ &= (y^2x^2)x^2 - n(x^2,y)yx^2 + n(x^2x^2,y)y \\ &= (y^2x^2)x^2 = -(x^2x^2)y^2 + n(x^2,y^2)x^2 \\ &= -(g(x)x - n(x)x^2)y^2 = -\lambda xy^2 \qquad (\text{by } (9)). \end{aligned}$$

From Proposition 3.5 it follows that

$$0 \neq \operatorname{alg}\langle yx\rangle = \operatorname{span}\langle yx, (yx)^2\rangle = \operatorname{span}\langle yx, y^2x^2\rangle,$$

so $y^2x^2 \neq 0$. Also

$$\operatorname{alg}\langle x^2y\rangle = \operatorname{span}\langle x^2y, xy^2\rangle.$$

By the last assertion in Proposition 4.2,

$$\begin{aligned} \operatorname{alg}\langle yx\rangle &\subseteq (\operatorname{alg}\langle x\rangle \oplus \operatorname{alg}\langle y\rangle)^{\perp}, \\ \operatorname{alg}\langle x^2y\rangle &\subseteq (\operatorname{alg}\langle x^2\rangle \oplus \operatorname{alg}\langle y\rangle)^{\perp} = (\operatorname{alg}\langle x\rangle \oplus \operatorname{alg}\langle y\rangle)^{\perp}, \end{aligned}$$

and since $x^2y = -(yx)x$,

$$\operatorname{alg}\langle x^2y\rangle \subseteq (\operatorname{alg}\langle yx\rangle \oplus \operatorname{alg}\langle y\rangle)^{\perp} \subseteq \operatorname{alg}\langle yx\rangle^{\perp}.$$

Therefore A is the orthogonal direct sum

$$A = \operatorname{alg}\langle x\rangle \oplus \operatorname{alg}\langle y\rangle \oplus \operatorname{alg}\langle yx\rangle \oplus \operatorname{alg}\langle x^2y\rangle,$$

and $\{x, x^2, y, y^2, yx, y^2x^2, x^2y, xy^2\}$ is a basis of A. By means of (4) and (5) it is easy to check that all the products of elements of this basis are completely determined. For instance,

$$(yx)(x^2y) = -y(x^2(yx)) = -y\big(n(x^2,x)y - x(yx^2)\big) = -\lambda y^2,$$

because $g(x) = \lambda$ and $yx^2 = 0$, or

$$\big((x^2y)(yx)\big) = -((yx)y)x^2 = -n(y)xx^2 = 0,$$

in accordance with Proposition 4.2.

Therefore there is at most one Okubo algebra, up to isomorphism, satisfying the hypotheses. But $C_{\lambda,\mu}$ is an Okubo algebra, and it verifies

$$g(-x_{1,0}) = n\big(-x_{1,0}, x_{1,0}^2\big) = n(-x_{1,0}, -\lambda x_{-1,0}) = \lambda$$

by (8). In the same way $g(-x_{0,1}) = \mu$, and also $n(\mathrm{alg}\langle x_{1,0}\rangle, \mathrm{alg}\langle x_{0,1}\rangle) = 0$ and $x_{1,0} * x_{0,1} = 0$. This concludes the proof. ∎

Remarks. (i) Actually one can check in a straightforward way that, in the proof above, the basis $\{-x, -\lambda^{-1}x^2, -y, -\mu^{-1}y^2, yx, (\lambda\mu)^{-1}y^2x^2, \lambda^{-1}x^2y, \mu^{-1}xy^2\}$ presents exactly the same multiplication table as in Table I.

(ii) If the equality $yx = 0$ instead of $xy = 0$ were required in the theorem above, a unique Okubo algebra without idempotents would have been obtained and, by uniqueness, this algebra is the opposite algebra of the one obtained in the theorem. This agrees with Proposition 2.2.

Given an algebra A, the symmetrized algebra A^+ is the new algebra defined on A, but with the new multiplication defined by $x \circ y = xy + yx$.

THEOREM 4.4. *Let A and B be two Okubo algebras without nonzero idempotents, with respective norms n_A and n_B and associated cubic forms g_A and g_B. Then the following conditions are equivalent*:

(i) $g_A(A) = g_B(B)$,

(ii) *A^+ is isomorphic to B^+*,

(iii) *A and B are either isomorphic or anti-isomorphic*.

Moreover, if these conditions are satisfied, there is a unique isomorphism ϕ: $A^+ \to B^+$, so A is isomorphic (respectively anti-isomorphic) to B if and only if this unique ϕ is an isomorphism (respectively an anti-isomorphism) from A to B.

In particular the group of automorphisms of A and of A^+ is trivial and no Okubo algebra without idempotents is isomorphic to its opposite algebra.

Proof. Assume first that (i) is satisfied. Since g_A and g_B are one-to-one by Proposition 3.3, there is a unique map ϕ: $A \to B$ such that $g_A(x) = g_B(\phi(x))$ for any $x \in A$. This map ϕ is clearly linear and invertible. Besides, by Lemma 3.1, for any $x \in A$

$$g_B\left(\phi(x)^2\right) = g_B(\phi(x))^2 + n_B(\phi(x))^3$$

$$g_A(x^2) = g_A(x)^2 + n_A(x)^3$$

and subtracting and using that $g_A(x) = g_B(\phi(x))$ we obtain

$$g_B\left(\phi(x)^2 - \phi(x^2)\right) = \left(n_B(\phi(x)) - n_A(x)\right)^3 \in g_B(B) \cap F^3 = 0$$

by Proposition 3.3. Therefore $\phi(x^2) = \phi(x)^2$ and ϕ is an isomorphism form A^+ to B^+.

Conversely, let ϕ: $A^+ \to B^+$ be an isomorphism. For any $0 \neq x \in A$, $x \circ x^2 = xx^2 + x^2 x = 2n(x)x$ by (4), so $\phi(x) \circ \phi(x^2) = 2n_A(x)\phi(x)$, but also $\phi(x) \circ \phi(x^2) = \phi(x) \circ \phi(x)^2 = 2n_B(\phi(x))\phi(x)$. Hence $n_B(\phi(x)) = n_A(x)$ for any x and ϕ is orthogonal. Therefore $g_A(x) = g_B(\phi(x))$ for any x and $g_A(A) = g_B(B)$. Thus conditions (i) and (ii) are equivalent.

Evidently condition (iii) implies (ii). Finally, let us assume that conditions (i) and (ii) are satisfied. By Proposition 4.1 there are nonzero isotropic elements $x, y \in A$ with $n_A(\text{alg}\langle x \rangle, \text{alg}\langle y \rangle) = 0$ and we can assume, by Proposition 4.2, that $xy = 0$ (otherwise interchange x and y). Let ϕ be the unique invertible linear map such that $g_A(a) = g_B(\phi(a))$ for any $a \in A$. As above, ϕ is an isomorphism (the only possible one) from A^+ to B^+ and it is an orthogonal map. Hence $\phi(x)$ and $\phi(y)$ are nonzero isotropic vectors of B. But $\text{alg}\langle \phi(x) \rangle = \text{span}\langle \phi(x), \phi(x)^2 \rangle = \text{span}\langle \phi(x), \phi(x^2) \rangle = \phi(\text{alg}\langle x \rangle)$, also $\text{alg}\langle \phi(y) \rangle = \phi(\text{alg}\langle y \rangle)$, so $n_B(\text{alg}\langle \phi(x) \rangle, \text{alg}\langle \phi(y) \rangle) = 0$. By Proposition 4.2 either $\phi(x)\phi(y) = 0$, and the uniqueness of Theorem 4.3 forces that ϕ is an isomorphism, or $\phi(y)\phi(x) = 0$ and ϕ is an anti-isomorphism.

The final assertions of the theorem are now straightforward. ∎

Remark. Notice that even though the automorphism group of any Okubo algebra without nonzero idempotents A is trivial, it is not so with the derivation algebra $\text{Der}\, A$. Since Okubo algebras are flexible and Lie-admissible (see [M86]), $0 \neq \text{ad}\, A \subseteq \text{Der}\, A$, where $\text{ad}\, x(y) = [x, y] = xy - yx$.

If $g_{\lambda\mu}$ denotes the cubic form associated to $C_{\lambda,\mu}$, (8) and Table I imply that $g_{\lambda\mu}(x_{i,j}) = -\lambda^i\mu^j$ for any i, j. Hence $g_{\lambda\mu}(C_{\lambda,\mu}) = \text{span}_{F^3}\langle \lambda^i\mu^j: -1 \leq i, j \leq 1, (i, j) \neq (0,0)\rangle$. Let us denote by $G_{\lambda\mu}$ this set.

The next result summarizes some of the work in this section.

THEOREM 4.5. (a) *The Okubo algebras without nonzero idempotents are exactly, up to isomorphism, the algebras* $C_{\lambda,\mu}$, *with* $\lambda \in F \setminus F^3$ *and* $\mu \in F \setminus F^3(\lambda)$.

(b) *If* $\lambda, \lambda' \in F \setminus F^3$, $\mu \in F \setminus F^3(\lambda)$, *and* $\mu' \in F \setminus F^3(\lambda')$, *then*

(i) $C^+_{\lambda,\mu}$ *is isomorphic to* $C^+_{\lambda',\mu'}$ *if and only if* $G_{\lambda\mu} = G_{\lambda'\mu'}$.

(ii) *If* $G_{\lambda\mu} = G_{\lambda'\mu'}$, *then the algebras* $C_{\lambda,\mu}$ *and* $C_{\lambda',\mu'}$ *are isomorphic if and only if* $g^{-1}_{\lambda\mu}(\lambda')g^{-1}_{\lambda\mu}(\mu') = 0$. *If this is not satisfied, then* $g^{-1}_{\lambda\mu}(\mu')g^{-1}_{\lambda\mu}(\lambda') = 0$ *and* $C_{\lambda,\mu}$ *and* $C_{\lambda',\mu'}$ *are anti-isomorphic*.

5. CLASSIFICATION OF THE SYMMETRIC COMPOSITION ALGEBRAS OVER FIELDS OF CHARACTERISTIC THREE

Assume for a while that the characteristic of the ground field is arbitrary. As mentioned in the Introduction, the symmetric composition algebras without nonzero idempotents are the forms of two-dimensional para-Hurwitz algebras and the Okubo algebras without nonzero idempotents.

If A is a two-dimensional symmetric composition algebra without nonzero idempotents and $0 \neq x \in A$ is an element with $n(x) = 1$ (these elements always exist, because $n(u) \neq 0$ implies $n((1/n(u))u^2) = 1$), then $A = \operatorname{alg}\langle x \rangle = \operatorname{span}\langle x, x^2 \rangle$. Let $\lambda = g(x) = n(x, x^2)$. Then the multiplication in A is determined by (4) and (9), which give $xx^2 = x^2x = x$ and $x^2x^2 = \lambda x - x^2$, respectively. The nondegeneracy of n is equivalent to the restriction $\lambda \neq \pm 2$. Besides, for any $\mu \in F$,

$$\left(x^2 + \mu x\right)^2 = \left(\mu^2 - 1\right)x^2 + (\lambda + 2\mu)x,$$

which is $\neq 0$ since $\lambda \neq \pm 2$. But there are no nonzero idempotents, so

$$0 \neq \begin{vmatrix} \mu^2 - 1 & \lambda + 2\mu \\ 1 & \mu \end{vmatrix} = \mu^2 - 3\mu - \lambda$$

and the polynomial $X^3 - 3X - \lambda$ is irreducible (a condition that already implies $\lambda \neq \pm 2$).

Therefore any two-dimensional symmetric composition algebra without nonzero idempotents (that is, not being para-Hurwitz) is isomorphic to the algebra, which will be denoted by A_λ, with a basis $\{u, v\}$ and multiplication given by

$$u^2 = v, \qquad uv = vu = u, \qquad v^2 = \lambda u - v, \tag{11}$$

where $\lambda \in F$ such that $X^3 - 3X - \lambda$ is irreducible. This is the content of [E-M91, Theorem 4.3], which is valid in any characteristic. In particular, if the characteristic is 3, the restriction on λ is equivalent to the condition $\lambda \in F \setminus F^3$.

From now on, we return to our restriction of the characteristic being 3. Hence the polynomial $X^3 - 3X - \lambda$ becomes just $X^3 - \lambda$.

THEOREM 5.1. *Let A be a symmetric composition algebra with norm n and cubic form g. Then one and only one of the following possibilities occurs*:

(1) *There is a scalar $\lambda \in F \setminus F^3$ such that A is isomorphic to A_λ. This possibility happens if and only if* $\dim_{F^3} g(A) = 2$.

(2) *A is a para-Hurwitz algebra and in this case $g(A) = F^3$.*

(3) *A is an Okubo algebra. Then there are nonzero elements $\lambda, \mu \in F$ such that A is isomorphic to $C_{\lambda,\mu}$ and either*:

(a) $g(A) = F^3$, *which occurs if and only if A is isomorphic to the pseudo-octonian algebra $P_8(F)$.*

(b) *$g(A)$ is a subfield of F and a cubic extension of F^3. This happens if and only if there is an element $\alpha \in F \setminus F^3$ such that A is isomorphic to $C_{1,\alpha}$ ($\cong C(\alpha)$ in* [*E-P*96]).

(c) $\dim_{F^3} g(A) = 8$. *This happens if and only if A does not contain any nonzero idempotent and then $\lambda \in F \setminus F^3$ and $\mu \in F \setminus F^3(\lambda)$.*

The cases (2) *and* (3)(*a*) (*in which $g(A) = F^3$*) *are distinguished by the commutative center $K(A) = \{x \in A\colon\ xy = yx\ \forall y \in A\}$, which is* 0 *for any Okubo algebra and nonzero for para-Hurwitz algebras.*

Moreover, two algebras A and B in case (1) *are isomorphic if and only if $g(A) = g(B)$ and the same happens in cases* (3)(*a*) *and* (3)(*b*). *Two para-Hurwitz algebras* (*case* (2)) *are isomorphic if and only if so are the corresponding Hurwitz algebras if and only if their norms are equivalent. Finally the isomorphism conditions for case* 3(*c*) *are shown in Theorem* 4.5.

Proof. Just a few details need to be checked. If A is a para-Hurwitz algebra, then there is a multiplication $\diamond$ on A which makes it a Hurwitz algebra with unity element 1 and the product in A is given by $xy = \bar{x}\diamond\bar{y}$. Then for $x \in A_0 = \{z \in A\colon\ n(1, z) = 0\}$, $g(x) = n(x, x^2) = n(x, \bar{x}\diamond\bar{x}) = n(x, x\diamond x) = n(x, -n(x)1) = 0$, so $g(A) = F^3 g(1) = F^3$. In the algebra A_λ, it follows from (11), (4), and Lemma 3.1 that $g(u) = n(u, u^2) = n(u, v) = \lambda$ and $g(v) = g(u^2) = g(u)^2 + n(u)^3 = \lambda^2 + 1$, so $\dim_{F^3} g(A_\lambda) = 2$.

The condition for isomorphism among the algebras in case (1) follows exactly as in the proof of Theorem 4.4, taking into account that these algebras are commutative. For the algebras in case (3) it follows from

Theorem 4.4 and [E-P96, Theorem 5.1]. Finally, if two composition algebras are isomorphic, their norms are equivalent [P71]; if moreover these are para-Hurwitz algebras this is equivalent to their Hurwitz counterparts being isomorphic (see also [O-O181, Lemma 3]). ∎

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