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Vertex Algebra Approach to Fusion Rules for $N = 2$ Superconformal Minimal Models

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Let L_{c_m} be the vertex operator superalgebra associated to the unitary vacuum module for the $N = 2$ superconformal algebra with the central charge $c_m = \frac{3m}{m+2}$, $m \in \mathbb{N}$. Then the unitary $N = 2$-modules give all irreducible modules for the vertex operator superalgebra L_{c_m}. In this paper, we determine all fusion rules for L_{c_m}-modules from the vertex algebra point of view. These fusion rules coincide with the fusion rules obtained by M. Wakimoto (Fusion rules for $N = 2$ superconformal modules, hep-th/9807144) using a modified Verlinde formula.

1. INTRODUCTION

In the theory of vertex operator (super)algebras fusion rules are defined as the dimension of the vector space of intertwining operators for three (irreducible) modules (cf. [DL, FHL, FZ]). In studying representation theory of certain concrete vertex operator (super)algebras, the determination of fusion rules is one of the basic problems. One can calculate the fusion rules using the Frenkel–Zhu formula (cf. [DLM3, FZ, KWn, Li1, Xu, Wn]) or some explicit determination of intertwining operators (cf. [DL, La, M]).

In the present paper, we will study the fusion rules for modules with vertex operator superalgebras (SVOAs) associated to the minimal models for the $N = 2$ superconformal algebra. Let L_{c_m} be the SVOA associated to the vacuum module for the $N = 2$ superconformal algebra with the central charge $c_m = \frac{3m}{m+2}$. The irreducible L_{c_m}-modules were classified in [A]. It was shown that the SVOA L_{c_m} for $m \in \mathbb{N}$ has finitely many

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irreducible modules which coincide with all unitary $N = 2$-minimal modules.

In the non-unitary case, the SVOA L_{c_m} has uncountably many irreducible modules (cf. [A, EG]).

In this paper, we calculate the fusion rules for L_{c_m}-modules in the case $m \in \mathbb{N}$. Another approach to fusion rules for minimal $N = 2$ superconformal models was made by Wakimoto [W]. He calculated the fusion rules using a modified Verlinde formula. Our result from the present paper coincides with the fusion rules for Neveu–Schwarz sectors obtained in [W]. We should also mention that the modular properties of characters of $N = 2$ minimal models were investigated in [KW].

Let us describe the main concepts of this paper. Let $L(m,0)$ be the VOA associated to the irreducible vacuum $\hat{sl_2}$-module of level $m \in \mathbb{N}$. Let F_n be the lattice vertex superalgebra associated to the lattice $\mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = n$ $(n \in \mathbb{Z})$. Following [A, FST], we investigate the vertex superalgebra $L_{c_m} \otimes F_{-1}$. Using the lattice construction of $\hat{sl_2}$ and $N = 2$-modules we prove that the vertex superalgebra $L_{c_m} \otimes F_{-1}$ is isomorphic to a certain extension of the vertex algebra $L(m,0) \otimes F_{-2(m+2)}$. Then we use the knowledge of the fusion rules for $L(m,0)$ and $F_{-2(m+2)}$ to find the fusion rules for $L_{c_m} \otimes F_{-1}$, which immediately gives the fusion rules for the SVOA L_{c_m}. As a consequence of our construction, we prove that the SVOA L_{c_m} is regular in the sense of [DLM1].

There is some overlap between the present paper and the paper by Li [Li5]. In [Li5] Li studies a certain larger class of extended vertex operator (super)algebras of similar type to our vertex superalgebra $\overline{B}_m$ from Section 5.

2. EXTENSIONS OF VERTEX ALGEBRAS AND INTERTWINING OPERATORS

In this section, we will recall some results from [Li3] on extension of regular vertex algebras by a simple current module. Another approach to the extension problem was made in [DLM2].

Our main goal is the study of extensions of vertex algebras realized as tensor products of affine vertex algebras with vertex algebras associated to negative definite lattices (cf. Section 5). Therefore we shall formulate the results from [Li3] in a slightly different setting.

Let $(V = V^{\text{even}} + V^{\text{odd}}, Y, \mathbf{1}, D)$ be a vertex superalgebra where $\mathbf{1}$ is the vacuum vector and D is the derivation (cf. [Li2, K]). If $V^{\text{odd}} = 0$, we say that V is a vertex algebra.

We shall always assume that vertex superalgebra V contains the Virasoro vector ω such that $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, $D = L(-1)$, and

$$V^{\text{even}} = \bigoplus_{n \in \mathbb{Z}} V_{(n)}, \qquad V^{\text{odd}} = \bigoplus_{n \in 1/2 + \mathbb{Z}} V_{(n)},$$

where $V_{(n)} = \{v \in V,\ L(0)v = nv\}$.

If there is $N \in \mathbb{Z}$ such that $V_{(n)} = 0$ for $n \leq N$, and if $\dim V_{(n)} < \infty$ for every $n \in \frac{1}{2}\mathbb{Z}$, we say that V is a vertex operator superalgebra (SVOA).

We will say that the vertex superalgebra $(V, Y, \mathbf{1}, L(-1))$ is regular, if every (weak) V-module is completely reducible (cf. [DLM1, Li3]).

For any three V-modules M^1, M^2, M^3, let

$$I_V \binom{M^3}{M^1 M^2}$$

be the vector space of all intertwining operators of type

$$\binom{M^3}{M^1 M^2}$$

(cf. [DL, KWn]).

Let $H \in V$ such that $Y(H, z) = \sum_{n \in \mathbb{Z}} H_n z^{-n-1}$. Assume that

$$L(n)H = \delta_{n,0} H, \qquad H_n H = \delta_{n,1} \gamma \mathbf{1}, \tag{2.1}$$

where n is a nonnegative integer and γ is a fixed integer. Assume also that H_0 semisimply acts on V with integral eigenvalues. Then from the commutator formula for vertex algebras we get

$$[H_m, H_n] = m\gamma\delta_{m+n,0} \qquad \text{for } m, n \in \mathbb{Z}.$$

Set

$$\Delta(H, z) = z^{H_0} \exp\left(\sum_{n=1}^{\infty} \frac{H_n}{-n} (-z)^{-n} \right).$$

In this section, we will always assume that vertex algebra V has the property:

(P1) The operators H_n, $n \geq 1$, act locally nilpotently on any V module M.

Proposition 2.1 [Li4]. *Let $\Delta(H, z)$ be defined as above. Assume that V is a vertex algebra with property* (P1). *Then*:

(1) *For any V-module $(M, Y_M(\cdot, z))$, $(\tilde{M}, Y_{\tilde{M}}(\cdot, z)) := (M, Y_M(\Delta(H, z)\cdot, z))$ is a V-module.*

(2) *Let* M^i $(i = 1, 2, 3)$ *be three* V*-modules and let* $I(\cdot, z)$ *be an intertwining operator of type*

$$\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}.$$

Then $\tilde{I}(\cdot, z) = I(\Delta(H, z)\cdot, z)$ *is an intertwining operator of type*

$$\begin{pmatrix} \tilde{M}^3 \\ M^1 \tilde{M}^2 \end{pmatrix}.$$

Remark 2.1. A version of Proposition 2.1 under the assumption (P1) appeared in [Xu, Theorem 3.3.8]. The author also used the proof from [Li4]. It is clear that the assumption (P1) is satisfied for every regular vertex operator superalgebra and for every lattice vertex superalgebra.

We shall also assume that

(P2) $\tilde{\tilde{V}} \cong V$.

The next theorem was essentially proved in [Li3]:

THEOREM 2.1 [Li3]. *Let* $H \in V$ *satisfy the condition* (2.1) *with an odd integer* γ. *Assume also that the vertex algebra* V *is regular with the properties* (P1) *and* (P2). *We have*:

(1) *The space* $\overline{V} = V \oplus \tilde{V}$ *has a natural structure of a vertex superalgebra.*

(2) *The vertex superalgebra* $\overline{V}$ *is regular.*

(3) *Assume that* M *is an irreducible* V*-module such that* H_0 *semisimply acts on* M *with integral eigenvalues. Then* $\overline{M} = M \oplus \tilde{M}$ *is a* $\overline{V}$*-module.*

Proposition 2.1 gives a linear isomorphism (the identity map) ψ_M from $\tilde{M}$ onto M such that

$$\psi_M\big(Y_{\tilde{M}}(a, z)u\big) = Y_M\big(\Delta(H, z)a, z\big)\psi_M(u) \qquad \text{for } a \in V, u \in \tilde{M}. \quad (2.2)$$

Since $\Delta(H, z)^{-1} = \Delta(-H, z)$, (2.2) is equivalent to

$$\psi_M\big(Y_{\tilde{M}}(\Delta(-H, z)a, z)u\big) = Y_M(a, z)\psi_M(u). \quad (2.3)$$

For every V-module M, we set $\overline{M} = M \oplus \tilde{M}$. If $\tilde{\tilde{M}} \cong M$, let π_M be an isomorphism from M onto $\tilde{\tilde{M}}$.

Assume now that M^i, $i = 1, 2, 3$ are irreducible V-modules such that $\tilde{\tilde{M}}^i \cong M^i$ and that H_0 semisimply acts on M^i with integral eigenvalues. In particular, $\overline{M}^i$ $i = 1, 2, 3$ are $\overline{V}$-modules.

Assume next that $I_0(\cdot, z)$ is an intertwining operator of type

$$\begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix}.$$

We shall extend I_0 to an intertwining operator from

$$I_{\overline{V}}\begin{pmatrix} \overline{M}^3 \\ \overline{M}^1 \overline{M}^2 \end{pmatrix}.$$

In order to do this we need intertwining operators of types

$$\begin{pmatrix} \tilde{M}^3 \\ M^1 \tilde{M}^1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{M}^3 \\ \tilde{M}^1 M^2 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} M^3 \\ \tilde{M}^1 \tilde{M}^2 \end{pmatrix}.$$

For every $v^1 \in M^1$ and $\tilde{v}^2 \in \tilde{M}^2$, we define

$$I_1(v_1, z)\tilde{v}^2 = I_0(\Delta(H, z)v_1, z)\psi_{M^2}. \tag{2.4}$$

Proposition 2.1 (2) gives that

$$I_1(\cdot, z) \in I_V\begin{pmatrix} \tilde{M}^3 \\ M^1 \tilde{M}^1 \end{pmatrix}.$$

Following [DL], for any $\alpha \in \mathbb{Q}$ we define $(-z)^\alpha = e^{\alpha\pi i}$. For any $v_1 \in M^1$, $v_2 \in M^2$, define $F(v_2, z)v_1 = e^{zL(-1)}I_0(v_1, -z)v_2$. Then $F(\cdot, z)$ is an intertwining operator of type

$$\begin{pmatrix} M^3 \\ M^2 M^1 \end{pmatrix}.$$

For any $\tilde{v}_1 \in \tilde{M}^1$, $v_2 \in M^2$, define

$$F_1(v_2, z)\tilde{v}_1 = \psi_{M^3}^{-1} F(\Delta(H, z)v_2, z)\psi_{M^1}(\tilde{v}_1).$$

Then from Proposition 2.1 (2), $F_1(\cdot, z)$ is an intertwining operator of type

$$\begin{pmatrix} \tilde{M}^3 \\ M^2 \tilde{M}^1 \end{pmatrix}.$$

For any $\tilde{v}_1 \in \tilde{M}^1$, $v_2 \in M^2$ we define

$$I_2(\tilde{v}_1, z)v_2 = e^{zL(-1)}F_1(v_2, -z)\tilde{v}_1. \tag{2.5}$$

Then $I_2(\cdot, z)$ is an intertwining operator of type

$$\begin{pmatrix} \tilde{M}^2 \\ \tilde{M}^1 M^2 \end{pmatrix}.$$

From Proposition 2.1 (2) we obtain an intertwining operator $F_2(\cdot, z) = \tilde{I}_1(\cdot, z)$ of type

$$\begin{pmatrix} \tilde{M}^3 \\ \tilde{M}^1\tilde{M}^2 \end{pmatrix}.$$

Then

$$I_3(\cdot, z) := \pi_{M^3}^{-1}\tilde{F}_2(\cdot, z) \in I_V\begin{pmatrix} M^3 \\ \tilde{M}^1\tilde{M}^2 \end{pmatrix}. \tag{2.6}$$

DEFINITION 2.1. Let

$$I_0 \in I_V\begin{pmatrix} M^3 \\ M^1M^2 \end{pmatrix}.$$

Define intertwining operators I_1, I_2, and I_3 with relations (2.4), (2.5), and (2.6), respectively. For every $v_1 \in M^1$, $v_2 \in M^2$, $\tilde{v}_1 \in \tilde{M}^1$, and $\tilde{v}_1 \in \tilde{M}^1$ define

$$\begin{aligned} \bar{I}(v_1 + \tilde{v}_1, z)(v_2 + \tilde{v}_2) &= I_0(v_1, z)v_2 + I_1(v_1, z)\tilde{v}_2 \\ &\quad + I_2(\tilde{v}_1, z)v_2 + I_3(\tilde{v}_1,)\tilde{v}_2. \end{aligned}$$

THEOREM 2.2. *Let*

$$I_0(\cdot, z) \in I_V\begin{pmatrix} M^3 \\ M^1M^2 \end{pmatrix},$$

and define $\bar{I}(\cdot, z)$ as above. Then

$$\bar{I}(\cdot, z) \in I_{\overline{V}}\begin{pmatrix} \overline{M}^3 \\ \overline{M}^1\overline{M}^2 \end{pmatrix}.$$

The proof of the theorem is completely analogous to the proofs of Theorems 3.7 and 4.3 in [Li3], and it is omitted.

Theorem 2.2 shows that if

$$\dim I_V\begin{pmatrix} M^3 \\ M^1M^2 \end{pmatrix} \geq 1,$$

then

$$\dim I_{\overline{V}}\begin{pmatrix} \overline{M}^3 \\ \overline{M}^1\overline{M}^2 \end{pmatrix} \geq 1.$$

In what follows we shall investigate one special case when

$$\dim I_{\overline{V}} \begin{pmatrix} \overline{M}^3 \\ \overline{M}^1 \overline{M}^2 \end{pmatrix} = 1.$$

First we recall the following result obtained in [DL].

LEMMA 2.1 [DL]. *Assume that W_1, W_2, W_3 are irreducible $\overline{V}$-modules, and let*

$$I(\cdot, z) \in I_{\overline{V}} \begin{pmatrix} W_3 \\ W_1 W_2 \end{pmatrix}.$$

Let $v_1 \in W_1$, $v_2 \in W_2$ be nonzero vectors such that $I(v_1, z)v_2 = 0$. Then $I(\cdot, z) = 0$.

PROPOSITION 2.2. *Let M^1, M^2, M^3 be irreducible V-modules such that $\overline{M}^1, \overline{M}^2, \overline{M}^3$ are irreducible $\overline{V}$-modules. Assume that*

$$\dim I_V \begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix} = 1 \qquad \text{and} \qquad \dim I_V \begin{pmatrix} \tilde{M}^3 \\ M^1 M^2 \end{pmatrix} = 0.$$

Then

$$\dim I_{\overline{V}} \begin{pmatrix} \overline{M}^3 \\ \overline{M}^1 \overline{M}^2 \end{pmatrix} = 1.$$

Proof. Let

$$I_V \begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix} = \mathbb{C} I_0.$$

Let

$$\bar{I} \in I_{\overline{V}} \begin{pmatrix} \overline{M}^3 \\ \overline{M}^1 \overline{M}^2 \end{pmatrix}$$

be the extended intertwining operator from Definition 2.1 and Theorem 2.2.

Assume that

$$F \in I_{\overline{V}} \begin{pmatrix} \overline{M}^3 \\ \overline{M}^1 \overline{M}^2 \end{pmatrix}, \qquad F \neq 0.$$

Let P_3 and $\tilde{P}_3$ be the projection maps from $\overline{M}^3$ onto M^3 and $\tilde{M}^3$, respectively. Then for every $v_1 \in M^1$, $v_2 \in M^2$, we define

$$F_0(v_1, z) = P_3 F(v_1, z) v_2, \qquad F_1(v_1, z) v_2 = \tilde{P}_3 F(v_1, z) v_2.$$

Then we have that

$$F_0 \in I_V \begin{pmatrix} M^3 \\ M^1 M^2 \end{pmatrix} \qquad \text{and} \qquad F_1 \in I_V \begin{pmatrix} \tilde{M}^3 \\ M^1 M^2 \end{pmatrix}.$$

Since

$$\dim I_V \begin{pmatrix} \tilde{M}^3 \\ M^1 M^2 \end{pmatrix} = 0,$$

we get $F_1 = 0$. Since $F \neq 0$, Lemma 2.1 easily implies that $F_0 \neq 0$. Then there is $k \in \mathbb{C}$, $k \neq 0$, such that $F_0 = kI_0$.

Since

$$F(v_1, z)v_2 = F(v_1, z)v_2 = kI_0(v_1, z)v_2 = k\bar{I}(v_1, z)v_2,$$

for every $v_1 \in M^1$, $v_2 \in M^2$, Lemma 2.1 implies that $F = k\bar{I}$. In this way we have proved that

$$\dim I_{\overline{V}} \begin{pmatrix} \overline{M}^3 \\ \overline{M}^1 \overline{M}^2 \end{pmatrix} = 1.$$

∎

3. $N = 2$ SUPERCONFORMAL VERTEX ALGEBRAS

In this section, we recall the results from [A] on the representation theory of SVOAs associated to the $N = 2$ superconformal algebra. We should also mention that the study of these SVOAs was initiated in [EG].

$N = 2$ superconformal algebra $\mathscr{A}$ is the infinite-dimensional Lie superalgebra with basis $L(n), T(n), G^{\pm}(r), C, n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}$ and (anti)commutation relations given by

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta_{m+n,0},$$

$$[L(m), G^{\pm}(r)] = \left(\frac{1}{2}m - r\right)G^{\pm}(m+r),$$

$$[L(m), T(n)] = -nT(n+m),$$

$$[T(m), T(n)] = \frac{C}{3}m\delta_{m+n,0},$$

$$[T(m), G^{\pm}(r)] = \pm G^{\pm}(m+r),$$

$$\{G^+(r), G^-(s)\} = 2L(r+s) + (r-s)T(r+s) + \frac{C}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+3,0},$$

$$[L(m), C] = [T(n), C] = [G^{\pm}(r), C] = 0,$$

$$\{G^{+}(r), G^{+}(s)\} = \{G^{-}(r), G^{-}(s)\} = 0$$

for all $m, n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

We denote the Verma module generated from a highest weight vector $|h, q, c\rangle$ with $L(0)$ eigenvalue h, $T(0)$ eigenvalue q, and central charge c by $M_{h,q,c}$. An element $v \in M_{h,q,c}$ is called a singular vector if

$$L(n)v = T(n)v = G^{\pm}(r)v = 0 \qquad \text{for } n, r > 0,$$

and if v is an eigenvalue of $L(0)$ and $T(0)$. Let $J_{h,q,c}$ be the maximal $U(\mathscr{A})$-submodule in $M_{h,q,c}$. Then

$$L_{h,q,c} = \frac{M_{h,q,c}}{J_{h,q,c}}$$

is an irreducible highest weight module.

Now we will consider the Verma module $M_{0,0,c}$. One easily sees that for every $c \in \mathbb{C}$

$$G^{\pm}\left(-\tfrac{1}{2}\right)|0, 0, c\ \rangle$$

is a singular vector in $M_{0,0,c}$. Set

$$V_c = \frac{M_{0,0,c}}{U(\mathscr{A})G^{\pm}\left(-\frac{1}{2}\right)|0,0,c\ \rangle + U(\mathscr{A})G^{-}\left(-\frac{1}{2}\right)|0,0,c\ \rangle}.$$

Then V_c is a highest weight $\mathscr{A}$-module. Let $\mathbf{1}_c$ denote the highest weight vector. Let $L_c = L_{0,0,c}$ be the corresponding simple module. Define four vectors in V_c,

$$\tau^{\pm} = G^{\pm}\left(-\tfrac{3}{2}\right)\mathbf{1}_c, \qquad j = T(-1)\mathbf{1}_c, \qquad \nu = L(-2)\mathbf{1}_c,$$

and set

$$\begin{aligned}
G^{+}(z) &= Y(\tau^{+}, z) = \sum_{n\in\mathbb{Z}} G^{+}\left(n + \tfrac{1}{2}\right)z^{-n-2},\\
G^{-}(z) &= Y(\tau^{-}, z) = \sum_{n\in\mathbb{Z}} G^{-}\left(n + \tfrac{1}{2}\right)z^{-n-2},\\
L(z) &= Y(\nu, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2},\\
T(z) &= Y(j, z) = \sum_{n\in\mathbb{Z}} T(n)z^{-n-1}.
\end{aligned} \tag{3.1}$$

It is easy to see that the fields $G^{+}(z), G^{-}(z), L(z), T(z)$ are mutually local and the theory of local fields (cf. [K, Li2]) implies the following result.

PROPOSITION 3.1. *There is a unique extension of the fields* (3.1) *such that* V_c *becomes a vertex operator superalgebra* (*SVOA*). *Moreover*, L_c *is a simple SVOA*.

We will now present the classification result from [A].

For $m \in \mathbb{N}$ set $c_m = \frac{3m}{m+2}$. Define

$$h_{j,k} = \frac{jk - \frac{1}{4}}{m+2}, \qquad q_{j,k} = \frac{j-k}{m+2}, \qquad L_{c_m}^{j,k} = L_{h_{j,k}, q_{j,k}, c_m}.$$

THEOREM 3.1 [A]. *Let* $m \in \mathbb{N}$. *Then the set*

$$\left\{ L_{c_m}^{j,k} \,|\, j,k \in \mathbb{N}_{1/2}, 0 < j,k, j+k < m+2 \right\}$$

provides all irreducible modules for the SVOA L_{c_m}. *So*, *irreducible* L_{c_m}*-modules are exactly all unitary modules for the* $N = 2$ *superconformal algebra with the central charge* c_m.

Remark 3.1. Theorem 3.1 shows that the SVOA L_{c_m} for $m \in \mathbb{N}$ has exactly $\frac{(m+2)(m+1)}{2}$ non-isomorphic irreducible modules. In the non-unitary case, the SVOA L_{c_m} has uncountably many irreducible modules (see [A, Theorem 7.2]).

4. LATTICE VERTEX SUPERALGEBRAS

In this section, we shall recall the lattice construction of vertex superalgebras from [DL, K, Xu].

Let L be a lattice. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-form $\langle \cdot , \cdot \rangle$ on L to $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c$ be the affinization of $\mathfrak{h}$. We also use the notation $h(n) = t^n \otimes h$ for $h \in \mathfrak{h}$, $n \in \mathbb{Z}$.

Set $\hat{\mathfrak{h}}^+ = t\mathbb{C}[t] \otimes \mathfrak{h}$, $\hat{h}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$. Then $\hat{\mathfrak{h}}^+$ and $\hat{\mathfrak{h}}^-$ are abelian superalgebras of $\hat{\mathfrak{h}}$. Let $U(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-)$ be the universal enveloping algebra of $\hat{\mathfrak{h}}^-$. Let $\lambda \in \mathfrak{h}$. Consider the induced $\hat{\mathfrak{h}}$-module

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C}_\lambda \simeq S(\hat{\mathfrak{h}}^-) \qquad \text{(linearly)},$$

where $t\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on $\mathbb{C}$, $\mathfrak{h}$ acting as $\langle h, \lambda \rangle$ for $h \in \mathfrak{h}$, and c acts on $\mathbb{C}$ as multiplication by 1. We shall write $M(1)$ for $M(1,0)$. For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$ write $h(n) = t^n \otimes h$. Set $h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$.

Then $M(1)$ is a VOA which is generated by the fields $h(z)$, $h \in \mathfrak{h}$ and $M(1, \lambda)$, of $\lambda \in \mathfrak{h}$, are irreducible modules for $M(1)$.

Let $\hat{L}$ be the canonical central extension of L by the cyclic group $\langle \pm 1 \rangle$

$$1 \to \langle \pm 1 \rangle \to \hat{L} \xrightarrow{-} L \to 1 \tag{4.1}$$

with the commutator map $c(\alpha,\beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$. Let e: $L \to \hat{L}$ be a section such that $e_0 = 1$ and let ϵ: $L \times L \to \langle \pm 1 \rangle$ be the corresponding 2-cocycle. Then $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha) = (-1)^{\langle \alpha, \beta \rangle}$,

$$\epsilon(\alpha,\beta)\epsilon(\alpha+\beta,\gamma) = \epsilon(\beta,\gamma)\epsilon(\alpha,\beta+\gamma), \tag{4.2}$$

and $e_\alpha e_\beta = \epsilon(\alpha,\beta)e_{\alpha+\beta}$ for $\alpha, \beta, \gamma \in L$. From the induced $\hat{L}$-module

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\langle \pm 1 \rangle} \mathbb{C} \simeq \mathbb{C}[L] \qquad \text{(linearly)},$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and -1 acts on $\mathbb{C}$ as multiplication by -1. For $a \in \hat{L}$, write $\iota(a)$ for $a \otimes 1$ in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by $a \cdot \iota(b) = \iota(ab)$ and $(-1)\cdot\iota(b) = -\iota(b)$ for $a, b \in \hat{L}$.

Furthermore we define an action of $\mathfrak{h}$ on $\mathbb{C}\{L\}$ by $h \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)$ for $h \in \mathfrak{h}$, $a \in \hat{L}$. Define $z^h \cdot \iota(a) = z^{\langle h, \bar{a} \rangle}\iota(a)$.

The untwisted space associated with L is defined to be

$$V_L = \mathbb{C}\{L\} \otimes_{\mathbb{C}} M(1) \simeq \mathbb{C}[L] \otimes S(\hat{\mathfrak{h}}^-) \qquad \text{(linearly)}.$$

Then $\hat{L}, \hat{\mathfrak{h}}, z^h$ $(h \in \mathfrak{h})$ act naturally on V_L by acting on either $\mathbb{C}\{L\}$ or $M(1)$ as indicated above. Define $\mathbf{1} = \iota(e_0) \in V_L$. We use a normal ordering procedure, indicated by open colons, which signify that in the enclosed expression, all creation operators $h(n)$ $(n < 0)$, $a \in \hat{L}$, are to be placed to the left of all annihilation operators $h(n)$, z^h $(h \in \mathfrak{h}, n \geq 0)$. For $a \in \hat{L}$, set

$$Y(\iota(a), z) = {:} e^{\int (\bar{a}(z) - \bar{a}(0)z^{-1})} a z^{\bar{a}} {:}.$$

Let $a \in \hat{L}$; $h_1, \dots, h_k \in \mathfrak{h}$; $n_1, \dots, n_k \in \mathbb{Z}$ $(n_i > 0)$. Set

$$v = \iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k) \in V_L.$$

Define vertex operator $Y(v,z)$ with

$$\begin{aligned} &{:}\left(\frac{1}{(n_1-1)!}\left(\frac{d}{dz}\right)^{n_1-1} h_1(z)\right) \cdots \\ &\qquad \left(\frac{1}{(n_k-1)!}\left(\frac{d}{dz}\right)^{n_k-1} h_k(z)\right) Y(\iota(a), z){:}. \end{aligned} \tag{4.3}$$

This gives us a well-defined linear map

$$Y(\cdot, z) \colon V_L \to (\operatorname{End} V_L)[[z, z^{-1}]]$$

$$v \mapsto Y(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \qquad (v_n \in \operatorname{End} V_L).$$

Let $\{h_i \mid i = 1, \dots, d\}$ be an orthonormal basis of $\mathfrak{h}$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^{d} h_i(-1) h_i(-1) \in V_L.$$

Then $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ gives rise to a representation of the Virasoro algebra on V_L with the central charged d and

$$\begin{aligned} &L(0)(\iota(a) \otimes h_1(-n_1) \cdots h_n(-n_k)) \\ &\quad = \left(\tfrac{1}{2}\langle \bar{a}, \bar{a} \rangle + n_1 + \cdots + n_k\right)(\iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k)). \end{aligned} \tag{4.4}$$

The following theorem was proved in [DL, K].

THEOREM 4.1. *The structure* $(V_L, Y, \mathbf{1}, L(-1))$ *is a vertex superalgebra.*

Let P be the dual lattice of L. Then there is a 1−1 correspondence between the set of equivalence classes of irreducible modules for V_L and the set of coset of P/L [D, Xu].

The following proposition was proved in [Li4, Proposition 2.16].

PROPOSITION 4.1 [Li4]. *Let* $\beta \in P$. *Then as a* V_L-*module*, $(V_L, Y(\Delta(\beta, z)\cdot, z))$ *is isomorphic to the* V_L-*module* $V_{L+\beta}$.

Define the Schur polynomials $p_r(x_1, x_2, \dots)$ in variables $x_1, x_2, \dots$ by the following equation:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{r=0}^{\infty} p_r(x_1, x_2, \dots) y^r. \tag{4.5}$$

For any monomial $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ we have an element $h(-1)^{n_1} h(-2)^{n_2} \cdots h(-r)^{n_r}\mathbf{1}$ in both $M(1)$ and V_L for $h \in \mathfrak{h}$. Then for any polynomial $f(x_1, x_2, \dots)$, $f(h(-1), h(-2), \dots)\mathbf{1}$ is a well-defined element in $M(1)$ and V_L. In particular, $p_r(h(-1), h(-2), \dots)\mathbf{1}$ for $r \in \mathbb{N}$ are elements of $M(1)$ and V_L.

Suppose $a, b \in \hat{L}$ such that $\bar{a} = \alpha$, $\bar{b} = \beta$. Then

$$\begin{aligned} Y(\iota(a), z)\iota(b) &= z^{\langle \alpha, \beta \rangle} \exp\left(\sum_{n=1}^{\infty} \frac{\alpha(-n)}{n} z^n\right) \iota(ab) \\ &= \sum_{r=0}^{\infty} p_r(\alpha(-1), \alpha(-2), \dots) \iota(ab) z^{r + \langle \alpha, \beta \rangle}. \end{aligned} \tag{4.6}$$

Thus

$$\iota(a)_i \iota(b) = 0 \qquad \text{for } i \geq -\langle \alpha, \beta \rangle. \tag{4.7}$$

Especially, if $\langle \alpha, \beta \rangle \geq 0$, we have $\iota(a)_i \iota(b) = 0$ for $i \geq 0$, and if $\langle \alpha, \beta \rangle = -n < 0$, we get

$$\iota(a)_{i-1}\iota(b) = p_{n-i}(\alpha(-1), \alpha(-2), \dots)\iota(ab) \qquad \text{for } i \in \{0, \dots, n\}. \tag{4.8}$$

5. THE VERTEX ALGEBRA $L(m,0) \otimes F_{-2(m+2)}$

Let $n \in \mathbb{Z}$ and $\langle \beta, \beta \rangle = n$. Define

$$E_n = \mathbb{Z}\beta, \qquad F_n = V_{E_n}.$$

Then F_n is a vertex algebra if n is even and a vertex superalgebra if n is odd. For $i \in \mathbb{Z}$, let $\bar{i} = i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$. We define $F_n^{\bar{i}} = V_{\mathbb{Z}\beta + (i/n)\beta}$. Clearly $F_n = F_n^{\bar{0}}$. It is well known (cf. [D, DL, Xu]) that the set $\{F_n^{\bar{i}}\}_{i=0,\dots,|n|-1}$ provides all irreducible F_n-modules.

The fusion algebra is (cf. [DL])

$$F_n^{\bar{i}} \times F_n^{\bar{j}} = F_n^{\overline{i+j}}. \tag{5.1}$$

If $n = 2k$ is even, we define $\tilde{E}_{2k} = \frac{\beta}{2} + \mathbb{Z}\beta$, and $MF_{2k} = V_{\tilde{E}_{2k}} = F_{2k}^{\bar{k}}$. Then F_{2k} is a vertex algebra, and MF_{2k} is a F_{2k}-module.

We shall also need the following result from [DLM1].

Proposition 5.1 [DLM1]. *The vertex (super)algebra F_n is regular; i.e., any (weak) F_n-module is completely reducible.*

In what follows, we will use the vertex algebra $F_{-2(m+2)}$ and its irreducible modules

$$F_{-2(m+2)}^{\bar{s}} = V_{\mathbb{Z}\beta - (s/2(m+2))\beta}, \qquad s = 0, \dots, 2m+3.$$

From the construction of lattice vertex algebras it is clear that $F_{-2(m+2)}^{\bar{s}}$ is a weak module for the Heisenberg VOA $M(1)$, and one easily obtains the following decomposition:

$$F_{-2(m+2)}^{\bar{s}} \cong \bigoplus_{k \in \mathbb{Z}} M\left(1, \frac{-s - 2(m+2)k}{\sqrt{-2(m+2)}}\right). \tag{5.2}$$

Let $\mathfrak{g}$ be the Lie algebra sl_2 with generators x, y, h and relations $[x,y] = h$, $[h,x] = 2x$, $[h,y] = -2y$. Let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra of type $A_1^{(1)}$. Let Λ_0, Λ_1 denote the fundamental weights for $\hat{\mathfrak{g}}$. For any complex numbers m, j, set $L(m,j) = L((m-j)\Lambda_0 + j\Lambda_1)$. Then $L(m,0)$ has a natural structure of a VOA.

Let $m \in \mathbb{N}$. Then $L(m,0)$ is a regular VOA, and the set $\{L(m,j)\}_{j=0,\dots,m}$ provides all irreducible $L(m,0)$-modules. The fusion algebra (cf. [FZ]) is given by

$$L(m,j) \times L(m,k) = \sum_{i=\max\{0,j+k-m\}}^{\min\{j,k\}} L(m,j+k-2i). \qquad (5.3)$$

In particular, $L(m,m) \times L(m,j) = L(m,m-j)$.

We will now study the vertex superalgebra $L(m,0) \otimes F_{-2(m+2)}$, and its certain extension by a simple current module. For $r \in \{0,\dots,m\}$ and $\bar{s} \in \frac{\mathbb{Z}}{2(m+2)\mathbb{Z}}$, we define $B_m(r,\bar{s}) = L(m,r) \otimes F^{\bar{s}}_{-2(m+2)}$. Set $B_m = B_m(0,\bar{0})$. Then we have:

PROPOSITION 5.2. *The vertex algebra B_m is regular, and the set*

$$\{B_m(r,\bar{s}) \mid r = 0,\dots,m;\ s = 0,\dots,2m+3\}$$

provides all irreducible B_m-modules.

Define

$$H = \tfrac{1}{2}(h(-1)\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \beta(-1)\mathbf{1}).$$

Then $H \in B_m$ satisfies the condition (2.1) with $\gamma = -1$. Let $Y(\cdot,z)$ be the vertex operator which defines the vertex algebra structure on B_m, and let $(\tilde{B}_m, \tilde{Y}(\cdot,z)) = (B_m, Y(\Delta(H,z)\cdot\,,z))$. It is easy to see that $\tilde{B}_m \cong L(m,m) \otimes MF_{-2(m+2)}$. Moreover, if $r+s$ is even (resp. odd) then H_0 acts semisimply on $B_m(r,\bar{s})$ with integral (resp. half-integral) eigenvalues, and

$$\tilde{B}_m(r,\bar{s}) \cong B_m(m-r, \overline{m+2+s}).$$

Set $\overline{B}_m(r,\bar{s}) = B_m(r,s) \oplus \tilde{B}_m(r,\bar{s})$.

Now, applying Theorem 2.1 to the vertex algebra B_m, we get the following result (see also Proposition 5.2 from [Li3]).

THEOREM 5.1. *We have*:

(1) *The space $\overline{B}_m = B_m \oplus \tilde{B}_m$ has a natural structure of a vertex superalgebra.*

(2) *The vertex superalgebra $\overline{B}_m$ is regular.*

(3) *The set*

$$\{\overline{B}_m(r,\bar{s}) \mid 0 \le r \le m;\ 0 \le s < m+2;\ r+s \in 2\mathbb{Z}\}$$

provides all irreducible $\overline{B}_m$-modules.

In order to calculate the fusion rules for $\overline{B}_m$-modules, we shall first study the fusion rules for B_m-modules. Using (5.1) and (5.3) we get the following lemma.

LEMMA 5.1. *Let $B_m(r_1, \overline{s_1})$, $B_m(r_2, \overline{s_2})$, $B_m(r_3, \overline{s_3})$ be three irreducible B_m-modules. Then we have*:

(1)

$$\dim I_{B_m}\begin{pmatrix} B_m(r_3, \overline{s_3}) \\ B_m(r_1, \overline{s_1}) B_m(r_2, \overline{s_2}) \end{pmatrix} \leq 1.$$

(2)

$$\dim I_{B_m}\begin{pmatrix} B_m(r_3, \overline{s_3}) \\ B_m(r_1, \overline{s_1}) B_m(r_2, \overline{s_2}) \end{pmatrix} = 1$$

if and only if

$$\overline{s_3} = \overline{s_1 + s_2}, \qquad r_1 + r_2 + r_3 \in 2\mathbb{Z},$$
$$|r_2 - r_1| \leq r_3 \leq \min\{r_1 + r_2, 2m - r_1 - r_2\}.$$

(3) *If*

$$\dim I_{B_m}\begin{pmatrix} B_m(r_3, \overline{s_3}) \\ B_m(r_1, \overline{s_1}) B_m(r_2, \overline{s_2}) \end{pmatrix} = 1,$$

then

$$\dim I_{B_m}\begin{pmatrix} \tilde{B}_m(r_3, \overline{s_3}) \\ B_m(r_1, \overline{s_1}) B_m(r_2, \overline{s_2}) \end{pmatrix} = 0.$$

Finally, Lemma 5.1 and Proposition 2.2 enable us to calculate the fusion rules for modules for the vertex superalgebra $\overline{B}_m$.

THEOREM 5.2. *Let $\overline{B}_m(r_1, \overline{s_1})$, $\overline{B}_m(r_2, \overline{s_2})$, $\overline{B}_m(r_3, \overline{s_3})$ be three irreducible $\overline{B}_m$-modules. Then we have*:

(1)

$$\dim I_{\overline{B}_m}\begin{pmatrix} \overline{B}_m(r_3, \overline{s_3}) \\ \overline{B}_m(r_1, \overline{s_1}) \overline{B}_m(r_2, \overline{s_2}) \end{pmatrix} \leq 1.$$

(2)

$$\dim I_{\overline{B}_m}\begin{pmatrix} \overline{B}_m(r_3, \overline{s_3}) \\ \overline{B}_m(r_1, \overline{s_1}) \overline{B}_m(r_2, \overline{s_2}) \end{pmatrix} = 1$$

if and only if one of the conditions $(*)$ *and* $(**)$ *holds, where*

$(*)$ $\overline{s_3} = \overline{s_1 + s_2}$, $r_1 + r_2 + r_3 \in 2\mathbb{Z}$, $|r_2 - r_1| \le r_3 \le \min\{r_1 + r_2, 2m - r_1 - r_2\}$,

$(**)$ $\overline{s_3 + m + 2} = \overline{s_1 + s_2}$, $r_1 + r_2 + m - r_3 \in 2\mathbb{Z}$, $|r_2 - r_1| \le m - r_3 \le \min\{r_1 + r_2, 2m - r_1 - r_2\}$.

Remark 5.1. In [FM], the authors constructed a family of extended VOAs A_m. Some properties of the VOA A_m are similar to the properties of our vertex superalgebra $\overline{B}_m$. Moreover, it was proved in [Li5] that the VOA A_m is a certain extension of the VOA $L(m,0) \otimes F_{2m}$.

6. LATTICE CONSTRUCTIONS OF MODULES FOR AFFINE $\hat{sl_2}$ AND $N = 2$ SUPERCONFORMAL ALGEBRA

Define the lattice

$$A_{1,m} = \bigoplus_{i=1}^{m} \mathbb{Z}\alpha_i,$$

where

$$\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j},$$

for every $i \in \{1, \dots, m\}$. We also define

$$\tilde{A}_{1,m} = \frac{\alpha_1 + \cdots \alpha_m}{2} + A_{1,m}.$$

Then $V_{A_{1,m}}$ is a VOA, and $V_{\tilde{A}_{1,m}}$ is a $V_{A_{1,m}}$-module.

Define now the following three vectors in $V_{A_{1,m}}$:

$$x = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m}),$$

$$y = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m}),$$

$$h = \alpha_1(-1)\mathbf{1} + \cdots + \alpha_m(-1)\mathbf{1},$$

and the fields $x(z) = Y(x,z)$, $y(z) = Y(y,z)$, $h(z) = Y(h,z)$. Then the results from [DL, Chap. 13] imply the following proposition.

PROPOSITION 6.1. *The components of the fields* $x(z), y(z), h(z)$ *provide a structure of level* m $\hat{\mathfrak{g}}$*-modules on* $V_{A_{1,m}}$ *and* $V_{\tilde{A}_{1,m}}$. *Moreover, the VOA* $L(m,0)$ *is isomorphic to a subalgebra of* $V_{A_{1,m}}$, *and* $L(m,m)$ *is a* $L(m,0)$*-submodule of* $V_{\tilde{A}_{1,m}}$.

Let $m \in \mathbb{N}$. Define the lattice

$$N_{2,m} = \bigoplus_{i=1}^{m} \mathbb{Z}\gamma_i,$$

where

$$\langle \gamma_i, \gamma_j \rangle = 2\delta_{i,j} + 1,$$

for every $i, j \in \{1, \dots, m\}$.

Then $V_{N_{2,m}}$ is a SVOA. Define the four vectors:

$$\tau^{+} = \sqrt{\frac{2}{m+2}}\left(\iota(e_{\gamma_1}) + \cdots + \iota(e_{\gamma_m})\right),$$

$$\tau^{-} = \sqrt{\frac{2}{m+2}}\left(\iota(e_{-\gamma_1}) + \cdots + \iota(e_{-\gamma_m})\right),$$

$$j = \frac{1}{m+2}\left(\gamma_1(-1)\mathbf{1} + \cdots + \gamma_m(-1)\mathbf{1}\right),$$

$$\omega = \frac{1}{2(m+2)}\left(\gamma_1(-1)^2\mathbf{1} + \cdots + \gamma_m(-1)^2\mathbf{1}\right) + \frac{1}{m+2}\sum_{i \neq j} \iota(e_{\gamma_i - \gamma_j})$$

and the fields $G^{\pm}(z) = Y(\tau^{\pm}, z)$, $T(z) = Y(j, z)$, $L(z) = Y(\omega, z)$.

PROPOSITION 6.2. *The components of the fields $G^{\pm}(z), T(z), L(z)$ provide on $V_{N_{2,m}}$ a structure of a $U(\mathscr{A})$-module of central charge c_m. Moreover, the SVOA $L_{c_m} \cong U(\mathscr{A}).\mathbf{1}$ is a subalgebra of $V_{N_{2,m}}$.*

Proof. It is straightforward to check using relations (4.7) and (4.8) that the components of the fields $G^{\pm}(z), T(z), L(z)$ satisfy the $N = 2$ commutation relations with $C = c_m$. Thus, $V_{N_{2,m}}$ is an $U(\mathscr{A})$-module of central charge c_m. Moreover, U($\mathscr{A}$).**1** is a vertex subalgebra of $V_{N_{2,m}}$ isomorphic to a certain quotient of the SVOA V_{c_m}. Let $\mathbf{1}_{c_m}$ be the vacuum vector in V_{c_m}.

Next we notice that in the lattice SVOA $V_{N_{2,m}}$ the following relation holds:

$$\tau^{+}_{-m-1}\tau^{+}_{-m} \cdots \tau^{+}_{-2}\tau^{+} = 0. \tag{6.1}$$

Since it is well known that the maximal submodule of the highest weight $U(\mathscr{A})$-module V_{c_m} is generated by the vector $G^{+}(-m-\frac{3}{2}) \cdots G^{+}(-\frac{3}{2})\mathbf{1}_{c_m}$ (cf. [FS]), the relation (6.1) implies that $L_{c_m} \cong U(\mathscr{A}).\mathbf{1}$. ∎

7. DECOMPOSITION OF THE VERTEX SUPERALGEBRA $L_{c_m} \otimes F_{-1}$

In this section let D be the lattice

$$D = \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_m + \mathbb{Z}\delta,$$

such that

$$\langle \gamma_i, \gamma_j \rangle = 2\delta_{i,j} + 1, \qquad \langle \gamma_i, \delta \rangle = 0, \qquad \langle \delta, \delta \rangle = -1.$$

Then

$$D = D_0 \cup D_1,$$

where

$$D_i = \{ \alpha \in D \,|\, \langle \alpha, \alpha \rangle \in i + 2\mathbb{Z} \}, \qquad i = 1, 2.$$

Then we easily get

$$D_0 = (\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m + \mathbb{Z}\beta),$$

$$D_1 = \left(\frac{\alpha_1 + \cdots + \alpha_m + \beta}{2} + \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m + \mathbb{Z}\beta \right),$$

where

$$\alpha_i = \gamma_i + \delta, \qquad i = 1, \dots, m, \qquad \beta = \gamma_1 + \cdots + \gamma_m + (m+2)\delta.$$

We see that

$$\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}, \qquad \langle \alpha_i, \beta \rangle = 0, \qquad \langle \beta, \beta \rangle = -2(m+2).$$

In this way we have proved the following lemma.

LEMMA 7.1. *We have*

$$D \cong (N_{2,m} + E_{-1}) \cong \left(A_{1,m} + E_{-2(m+2)} \right) \cup \left(\tilde{A}_{1,m} + \tilde{E}_{-2(m+2)} \right).$$

From Lemma 7.1 we obtain the following result.

PROPOSITION 7.1. *The vertex algebra $V_{A_{1,m}} \otimes F_{2(m+2)}$ is isomorphic to a subalgebra (with the same Virasoro vector) of $V_D \cong V_{N_{2,m}} \otimes F_{-1}$. As a $V_{A_{1,m}} \otimes F_{-2(m+2)}$-module, V_D decomposes as follows*:

$$V_D \cong V_{A_{1,m}} \otimes F_{-2(m+2)} \oplus V_{\tilde{A}_{1,m}} \otimes MF_{-2(m+2)}.$$

Moreover,

$$V_{\tilde{A}_{1,m}} \otimes MF_{-2(m+2)} \cong \left(V_{A_{1,m}} \otimes F_{-2(m+2)}\right).\{v_1, v_2\},$$

where $v_1 = \iota(e_{\gamma_1 + \cdots + \gamma_m + (m+1)\delta})$, $v_2 = \iota(e_{-(\gamma_1 + \cdots + \gamma_m + (m+1)\delta)})$.

We are now in the position to find the decomposition of the vertex superalgebra $L_{c_m} \otimes F_{-1}$ as a B_m-module.

THEOREM 7.1. *The vertex algebra $L(m,0) \otimes F_{-2(m+2)}$ is isomorphic to a subalgebra (with the same Virasoro vector) of the vertex superalgebra $L_{c_m} \otimes F_{-1}$. As a $L(m,0) \otimes F_{-2(m+2)}$-module, $L_{c_m} \otimes F_{-1}$ decomposes as follows:*

$$L_{c_m} \otimes F_{-1} \cong L(m,0) \otimes F_{-2(m+2)} \oplus L(m,m) \otimes MF_{-2(m+2)}.$$

In particular, $L_{c_m} \otimes F_{-1} \cong \overline{B}_m$.

Proof. Let V be the subalgebra of the vertex superalgebra V_D generated by the vectors $\tau^1 = \sqrt{\frac{m+2}{2}}\,\tau^+ = \iota(e_{\gamma_1}) + \cdots + \iota(e_{\gamma_m})$, $\tau^2 = \sqrt{\frac{m+2}{2}}\,\tau^- = \iota(e_{-\gamma_1}) + \cdots + \iota(e_{-\gamma_m})$, $e_1 = \iota(e_\delta)$, $e_2 = \iota(e_{-\delta})$.

It is clear that $V \cong L_{c_m} \otimes F_{-1}$.

Let W be the subalgebra of V_D generated by the vectors

$$x = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m}) = (\tau^1)_{-1} e_1, \tag{7.1}$$

$$y = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m}) = (\tau^2)_{-1} e_2, \tag{7.2}$$

$$f_1 = \iota(e_\beta) = \iota(e_{\gamma_1 + \cdots + \gamma_m})_{-1} \iota(e_{(m+2)\delta}), \tag{7.3}$$

$$f_2 = \iota(e_{-\beta}) = \iota(e_{-(\gamma_1 + \cdots + \gamma_m)})_{-1} \iota(c_{-(m+2)\delta}), \tag{7.4}$$

$$v_1 = \iota(e_{(1/2)(\alpha_1 + \cdots + \alpha_m + \beta)}) = \iota(e_{\gamma_1 + \cdots + \gamma_m})_{-1} \iota(e_{(m+1)\delta}), \tag{7.5}$$

$$v_2 = \iota(e_{-(1/2)(\alpha_1 + \cdots + \alpha_m + \beta)}) = \iota(e_{-(\gamma_1 + \cdots + \gamma_m)})_{-1} \iota(e_{-(m+1)\delta}). \tag{7.6}$$

Then we see that $W = W^0 + W^1$, where W^0 is the subalgebra generated by the vectors x, y, f_1, f_2, and $W^1 = W^0.\{v_1, v_2\}$ is a W^0-module. Moreover, we have

$$W^0 \cong L(m,0) \otimes F_{-2(m+2)} \qquad \text{and} \qquad W^1 \cong L(m,m) \otimes MF_{-2(m+2)}.$$

We claim that $W = V$. First we will show that $W \subseteq V$. To see this, it is enough to verify that $x, y, f_1, f_2, v_1, v_2 \in V$. The relations (7.1) and (7.2) immediately give that $x, y \in V$. Since $\iota(e_{\pm(\gamma_1 + \cdots + \gamma_m)}) \in V$, the relations (7.3)–(7.6) give that $f_1, f_2, v_1, v_2 \in V$. This implies that $W \subseteq V$.

Next we notice that in the vertex superalgebra V_D the following formulas hold:

$$e_1 = \iota(e_\delta) = \iota(e_{-(\alpha_1 + \cdots + \alpha_m)})_{m-1} v_1, \tag{7.7}$$

$$e_2 = \iota(e_{-\delta}) = \iota(e_{(\alpha_1 + \cdots + \alpha_m)})_{m-1} v_2, \tag{7.8}$$

$$\tau^1 = x_{-2} e_2, \tag{7.9}$$

$$\tau^2 = y_{-2} e_1. \tag{7.10}$$

Since $\iota(e_{\pm(\alpha_1 + \cdots + \alpha_m)}) \in W$, the relations (7.7)–(7.10) imply that $e_1, e_2, \tau^1, \tau^2 \in W$, which implies that $V \subseteq W$. In this way we prove that $V = W$. ∎

THEOREM 7.2. *As a $L(m,0) \otimes F_{-2(m+2)}$-module, $L_{c_m}^{j,k} \otimes F_{-1}$ decomposes as follows*:

$$L_{c_m}^{j,k} \otimes F_{-1} \cong L(m, j+k-1) \otimes F_{-2(m+2)}^{\overline{k-j}} \oplus L(m, m+1-j-k) \otimes F_{-2(m+2)}^{\overline{m+2+k-j}}.$$

In particular,

$$L_{c_m}^{j,k} \otimes F_{-1} \cong \overline{B}_m(j+k-1, \overline{k-j}).$$

Proof. Let $v_{j,k}$ be the highest weight vector in $L_{c_m}^{j,k}$.

First we notice that $L_{c_m}^{j,k} \otimes F_{-1}$ is an irreducible $\overline{B}_m$-module, and in particular

$$L_{c_m}^{j,k} \otimes F_{-1} \cong L(m,r) \otimes F_{-2(m+2)}^{\bar{s}} \oplus L(m, m-r) \otimes F_{-2(m+2)}^{\overline{m+2+s}},$$

for certain $r \in \{0, \dots, m\}$ and $s \in \{0, \dots, 2m+3\}$. Now Lemma 7.1 from [A] gives that

$$\frac{4h_{j,k}}{m+2} + q_{j,k}^2 = \frac{r(r+2)}{(m+2)^2},$$

which implies that $r = j + k - 1$.

Let S_0 be the subalgebra of $L_{c_m} \otimes F_{-1}$ isomorphic to $F_{-2(m+2)}$. Set

$$H = \sqrt{\frac{m+2}{2}}\,(T(-1)\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \delta(-1)\mathbf{1}).$$

Then $H \in S_0$. We have $[H_n, H_m] = n\delta_{n+m,0} (n, m \in \mathbb{Z})$, which implies that the vector $H \in S_0$ spans a subalgebra of S_0 isomorphic to the Heisenberg

VOA $M(1)$. Then for any nonnegative integer n, we have

$$H_n(v_{j,k} \otimes \mathbf{1}) = \delta_{n,0} \frac{j-k}{\sqrt{-2(m+2)}}.$$

This implies that

$$S_0.(v_{j,k} \otimes \mathbf{1}) \cong \bigoplus_{i \in \mathbb{Z}} M\left(1, \frac{j-k-2(m+2)i}{\sqrt{-2(m+2)}}\right),$$

and the decomposition (5.2) gives that $S_0.(v_{j,k} \otimes \mathbf{1}) \cong F^{\overline{k-j}}_{-2(m+2)}$. In this way we get $\bar{s} = \overline{k-j}$. ▮

8. THE MAIN RESULT

In the Section 7 we proved that the vertex superalgebra $L_{c_m} \otimes F_{-1}$ is isomorphic to the vertex superalgebra $\overline{B}_m$ obtained as an extension of the vertex algebra $L(m,0) \otimes F_{-2(m+2)}$. Since the vertex superalgebra $\overline{B}_m$ is regular (see Theorem 5.1(2)), we also have the following result.

THEOREM 8.1. *Let $m, m_1, \ldots, m_k \in \mathbb{N}$.*

(a) *The SVOA L_{c_m} is regular.*

(b) *The SVOA $L_{c_{m_1}} \otimes \cdots \otimes L_{c_{m_k}}$ is regular.*

Proof. First we notice that every irreducible $\overline{B}_m$-module has the form $N \otimes F_{-1}$, where N is an irreducible L_{c_m}-module.

Assume now that M is any weak L_{c_m}-module. Then $M \otimes F_{-1}$ is also a weak $\overline{B}_m$-module. Regularity of the vertex superalgebra $\overline{B}_m$ provides a decomposition

$$M \otimes F_{-1} \cong \bigoplus_i M_i \otimes F_{-1},$$

where M_i is an irreducible L_{c_m}-module. This implies that $M \cong \oplus_i M_i$, and we prove that M is completely reducible. This proves (a). The proof of (b) follows from (a) in the standard way (cf. [DLM1]). ▮

Remark 8.1. Since every regular vertex operator (super)algebra is rational, Theorem 8.1 also gives that the SVOA L_{c_m} is rational.

We are now going to calculate the fusion rules for irreducible L_{c_m}-modules. Since the vertex superalgebras L_{c_m} and F_{-1} are regular, we have the following natural statement on fusion rules (cf. [M]).

LEMMA 8.1. *Assume that M^1, M^2, M^3 are irreducible L_{c_m}-modules. Then we have*

$$I_{L_{c_m}}\begin{pmatrix} & M^3 & \\ M^1 & & M^2 \end{pmatrix} \cong I_{\overline{B}_m}\begin{pmatrix} & M^3 \otimes F_{-1} & \\ M^1 \otimes F_{-1} & & M^2 \otimes F_{-1} \end{pmatrix}.$$

Now we are in the position to find the fusion rules for irreducible modules for the SVOA L_{c_m}.

THEOREM 8.2. *Assume that $L_{c_m}^{j_1,k_1}$, $L_{c_m}^{j_2,k_2}$, and $L_{c_m}^{j_3,k_3}$ are L_{c_m}-modules. Then we have*:

(1)

$$\dim I_{L_{c_m}}\begin{pmatrix} & L_{c_m}^{j_3,k_3} & \\ L_{c_m}^{j_1,k_1} & & L_{c_m}^{j_2,k_2} \end{pmatrix} \le 1.$$

(2)

$$\dim I_{L_{c_m}}\begin{pmatrix} & L_{c_m}^{j_3,k_3} & \\ L_{c_m}^{j_1,k_1} & & L_{c_m}^{j_2,k_2} \end{pmatrix} = 1$$

if and only if one of the conditions (F1) *and* (F2) *holds*, *where*

(F1) $(j_1 + j_2 - j_3) - (k_1 + k_2 - k_3) = 0$, $|j_2 + k_2 - j_1 - k_1| < j_3 + k_3$, $j_3 + k_3 < \min\{j_1 + k_1 + j_2 + k_2, 2m + 4 - (j_1 + k_1 + j_2 + k_2)\}$,

(F2) $(j_1 + j_2 - j_3) - (k_1 + k_2 - k_3) = \pm(m + 2)$, $|j_2 + k_2 - j_1 - k_1| < m + 2 - j_3 - k_3$, $m + 2 - j_3 - k_3 < \min\{j_1 + k_1 + j_2 + k_2, 2m + 4 - (j_1 + k_1 + j_2 + k_2)\}$.

Proof. By using Lemma 8.1 and Theorem 7.2 we see that

$$\dim I_{L_{c_m}}\begin{pmatrix} & L_{c_m}^{j_3,k_3} & \\ L_{c_m}^{j_1,k_1} & & L_{c_m}^{j_2,k_2} \end{pmatrix} = \dim I_{\overline{B}_m}\begin{pmatrix} & \overline{B}_m(r_3,\overline{s_3}) & \\ \overline{B}_m(r_1,\overline{s_1}) & & \overline{B}_m(r_2,\overline{s_2}) \end{pmatrix},$$

where

$$r_i = j_i + k_i - 1, \qquad \overline{s_i} = \overline{k_i - j_i}, \qquad (i = 1,2,3). \tag{8.1}$$

Now we apply Theorem 5.2 to calculate

$$\dim I_{\overline{B}_m}\begin{pmatrix} & \overline{B}_m(r_3,\overline{s_3}) & \\ \overline{B}_m(r_1,\overline{s_1}) & & \overline{B}_m(r_2,\overline{s_2}) \end{pmatrix}.$$

It is easy to see that the conditions $(*)$ and $(**)$ in Theorem 5.2 (after the substitution given by (8.1)) are equivalent to the conditions (F1) and (F2), respectively. This completes the proof. ∎

Remark 8.2. The fusion rules obtained in Theorem 8.2 are completely identical with the fusion rules for the Neveu–Schwarz sector $NS^{(m+2)}$ obtained in [W, Theorem 2.1]. In [W], the author used a modified Verlinde formula, and showed that the corresponding fusion algebra is associative.

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