Expanding maps, Anosov diffeomorphisms and affine structures on infra-nilmanifolds

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Abstract

In this paper we present a complete description of Lie algebras admitting an expanding and/or a hyperbolic automorphism in terms of $\mathbb{R}$-gradings of these Lie algebras. Using this, we then show that any infra-nilmanifold which admits an expanding map or an Anosov diffeomorphism also has a complete, affinely flat structure.

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1. Introduction

Let $G$ be a connected and simply connected nilpotent Lie group and $C$ a compact subgroup of $\text{Aut}(G)$. A discrete cocompact subgroup $E$ of $G \rtimes C$ is called an almost crystallographic group. Note that $G \rtimes \text{Aut}(G)$, and thus also any almost crystallographic group built on $G$, acts on $G$ via

$$\forall g, h \in G, \ \forall \alpha \in \text{Aut}(G): \ (g, \alpha)h = g\alpha(h).$$

A torsion-free almost crystallographic group is said to be an almost Bieberbach group and in this case, the quotient space $E\backslash G$ is said to be an infra-nilmanifold; $(E\backslash G$ is a nilmanifold if $E \subseteq G)$. It is well known that the natural projection of an almost
crystallographic group $E$ onto $C$ (or $\text{Aut}(G)$) has a finite image $F$, which is called the holonomy group of $E$. We refer to [2] and the references therein for more information on almost crystallographic and almost Bieberbach groups.

This paper is concerned with the class of infra-nilmanifolds which admit an expanding map or an Anosov diffeomorphism.

Recall that a $C^1$-map $f: M \to M$ on a compact differentiable manifold $M$ is expanding if for some Riemannian metric on $M$, there exists constants $C > 0$ and $\lambda > 1$ such that $\| (Df)^m v \| \geq C\lambda^m \| v \|$, for all $v \in TM$ and all positive integers $m > 0$. We refer the reader to [4,7,11,12,17,24] for examples of papers dealing with expanding maps. One way of constructing expanding maps on an infra-nilmanifold goes as follows. Let $\phi$ be an automorphism of $G$ and let $d\phi \in \text{Aut}(g)$ be the corresponding automorphism on the Lie algebra $g$ of $G$. Such an automorphism $\phi$ is said to be expanding if all eigenvalues of $d\phi$ are of modulus $> 1$. In this case $d\phi$ is also called expanding. If $E \subseteq G \rtimes C$ is an almost Bieberbach group and $g \in G$, for which $(g, \phi)E(g, \phi)^{-1} \subseteq E$, then $(g, \phi)$ induces an expanding map on the infra-nilmanifold $M = E \setminus G$. Such an expanding map is called an expanding infra-nilmanifold endomorphism. M. Gromov [11] proved that any expanding map of an arbitrary compact manifold $M$ is topologically conjugate to an expanding infra-nilmanifold endomorphism.

The situation for Anosov diffeomorphisms is quite analogous. A diffeomorphism $f: M \to M$ on a manifold $M$ is called Anosov if the tangent bundle $TM$ splits continuously into an expanding and a contracting part, so $TM = E^s \oplus E^u$ with $Df(E^s) = E^s$ and $Df(E^u) = E^u$ and there exists constants $C > 0$ and $1 < \lambda$ such that for all positive integers $m$:

\[
\| (Df)^m (v) \| \geq C\lambda^m \| v \|, \quad \text{if } v \in E^u \quad \text{and} \\
\| (Df)^m (v) \| \leq C^{-1}\lambda^{-m} \| v \|, \quad \text{if } v \in E^s
\]

(such maps are studied in, e.g., [8,13,18,19],...).

Examples of Anosov diffeomorphisms of an infra-nilmanifold $M$ are created as follows. Let $\phi$ be an automorphism of a Lie group $G$, which is hyperbolic, i.e., such that its differential $d\phi \in \text{Aut}(g)$ has no eigenvalues of modulus 1 (we also say that $d\phi$ is hyperbolic). If $E \subseteq G \rtimes \text{Aut}(G)$ is an almost Bieberbach group and $g \in G$ for which $(g, \phi)E(g, \phi)^{-1} \subseteq E$, then $(g, \phi)$ induces an Anosov diffeomorphism on the infra-nilmanifold $E \setminus G$. Such Anosov maps on infra-nilmanifolds are called hyperbolic infra-nilmanifold automorphisms. In [19] A. Manning proved that any Anosov diffeomorphism on an infra-nilmanifold is topologically conjugate to a hyperbolic infra-nilmanifold automorphism. Note that a hyperbolic infra-nilmanifold automorphism is induced by a hyperbolic automorphism $\phi$ such that $\phi$ has at least one eigenvalue $\lambda$ with $|\lambda| > 1$ and at least one eigenvalue $\mu$ with $|\mu| < 1$. Although, it does not really make any difference to the results of this paper we will always refer to this situation when talking about a hyperbolic automorphism (so we do not speak about a hyperbolic automorphism, in case that automorphism is (globally) expanding or (globally) contracting):

**Convention.** In this paper, the term *hyperbolic automorphism* is used to denote an automorphism without eigenvalues of modulus 1 and with at least one eigenvalue $\lambda$ with $|\lambda| > 1$ and at least one eigenvalue $\mu$ with $|\mu| < 1$. 
From the above it follows that to understand the class of (infra-)nilmanifolds admitting an expanding map or an Anosov diffeomorphism, we should first of all understand the class of nilpotent Lie groups/algebras admitting an expanding or a hyperbolic automorphism. This problem is completely solved in this paper in terms of $\mathbb{R}$-gradings of the Lie algebra. In fact, we show that a Lie group $G$ admits an expanding or a hyperbolic automorphism if and only if the corresponding Lie algebra $\mathfrak{g}$ has a grading $\mathfrak{g} = \bigoplus_{r \in \mathbb{R}} \mathfrak{g}_r$, with $\mathfrak{g}_0 = 0$ (see Section 2 for an exact definition).

Thereafter we also show that infra-nilmanifolds admitting an Anosov diffeomorphism or an expanding map, also carry a complete, affinely flat structure. This means that they are diffeomorphic to a quotient manifold $E \setminus \mathbb{R}^n$, where $E$ is acting properly discontinuously and via affine motions on $\mathbb{R}^n$. More details about these affine structures can be found in Section 3 and the references given there.

As all Lie algebras in this paper are connected with finite dimensional Lie groups, we will only consider finite dimensional Lie algebras over the real numbers, even if not explicitly stated that way.

2. Lie algebra automorphisms and $\mathbb{R}$-gradings

In this section we shall show that any automorphism of a real Lie algebra induces an $\mathbb{R}$-grading on this Lie algebra and conversely, that any $\mathbb{R}$-grading of a Lie algebra also induces a (semi-simple) automorphism of this Lie algebra.

The basis for this is the following lemma, which is a refinement of the Jordan-decomposition of a Lie algebra-automorphism. Although this lemma seems to be well known, we present a full proof here, as the whole paper is based on it. (A partial proof can be found in [21].)

**Lemma 2.1.** Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and let $\psi \in \text{Aut}(\mathfrak{g})$. Then there exist unique automorphisms $\psi_u, \psi_1, \psi_+ \in \text{Aut}(\mathfrak{g})$ for which

1. $\psi = \psi_u \psi_1 \psi_+$.
2. $\psi_u$ is unipotent, while $\psi_1$ and $\psi_+$ are semi-simple.
3. All eigenvalues of $\psi_1$ are of modulus 1.
4. All eigenvalues of $\psi_+$ are positive real numbers.
5. $\psi_u, \psi_1$ and $\psi_+$ commute pairwise:

$$[\psi_u, \psi_1] = [\psi_u, \psi_+] = [\psi_1, \psi_+] = 1.$$

**Proof.** First we show the uniqueness. Let $\mathfrak{g}_C$ be the complexified Lie algebra and consider $\varphi, \varphi_u, \varphi_1$ and $\varphi_+$ as automorphisms of $\mathfrak{g}_C$. Recall that any automorphism $\psi$ of a finite dimensional vector space can be uniquely decomposed as a product $\psi = \psi_u \psi_1 \psi_+$, with $\psi_u$ unipotent, $\psi_1$ semi-simple and $[\psi_u, \psi_1] = 1$. This is the so-called multiplicative Jordan decomposition of $\psi$ and $\psi_u$ (respectively $\psi_1$) is called the unipotent (respectively semi-simple) part of $\psi$. From the properties of $\psi_u, \psi_1$ and $\psi_+$ we deduce that $\psi_u$ is the unipotent part of $\psi$ and the semi-simple part is $\varphi_1 = \varphi_1 \varphi_+$. So the uniqueness of $\psi_u$...
is already guaranteed. As the automorphisms $\varphi_+, \varphi_-$ and $\varphi_1$ are all semi-simple and commute pairwise, there exists a basis $X_1, X_2, \ldots, X_n$, with respect to which all three automorphisms are diagonal. This means that

$$\forall i = 1, 2, \ldots, n: \varphi_+(X_i) = \lambda_i X_i, \quad \varphi_-(X_i) = \lambda_i^+ X_i, \quad \varphi_1(X_i) = \lambda_i^\bot X_i$$

for some complex numbers $\lambda_i, \lambda_i^+ \text{ and } \lambda_i^\bot$, where $\lambda_i^+ \in \mathbb{R}_0^+$ and $|\lambda_i^\bot| = 1$. Together with the fact that $\varphi_+ = \varphi_+ \varphi_1$, this implies that

$$\lambda_i^+ = |\lambda_i| \quad \text{and} \quad \lambda_i^\bot = \frac{1}{|\lambda_i|} \lambda_i.$$  

This shows that $\varphi_+$ and $\varphi_1$ are uniquely determined on the eigenspaces of $\varphi_+$. However, as these eigenspaces span the whole space $g_C$, this finishes the uniqueness part.

We now show the existence part. As $\text{Aut}(g)$ is an algebraic group, it contains the unipotent and the semi-simple part of all of its elements. This implies that we can write $\varphi = \varphi_u \varphi_s$, where $\varphi_u$ is a unipotent automorphism of $g$ and $\varphi_s$ is a semi-simple automorphism of $g$, commuting with $\varphi_u$. To see how we can decompose $\varphi_s$, we again look at the complexified Lie algebra $g_C$ and we regard $\varphi_s$ as being an automorphism of $g_C$.

As $\varphi_s$ is semi-simple, $g_C$ is, as a vector space, the direct sum of the eigenspaces of $\varphi_s$. If $V_\lambda$ is the eigenspace corresponding to the eigenvalue $\lambda$, we define $\varphi_+$ on $V_\lambda$ as being scalar multiplication by $|\lambda|$. Doing this for any eigenspace, we can define $\varphi_+$ on the whole vector space $g_C$. Note that by its definition, $\varphi_+$ commutes with $\varphi_s$. Moreover, any eigenspace $V_\lambda$ of $\varphi_s$ is mapped onto itself by $\varphi_u$, and therefore $\varphi_+$ also commutes with $\varphi_u$.

We claim that $\varphi_+$ is a Lie algebra automorphism. Indeed, let $X$ and $Y$ be two eigenvectors (for $\varphi_s$) of $g_C$, then

$$\varphi_+(X) = \lambda X, \quad \varphi_+(Y) = \mu Y \quad \text{and} \quad \varphi_+([X, Y]) = \lambda \mu [X, Y]$$

for some $\lambda, \mu \in \mathbb{C}$. But then we also have that

$$\varphi_+(X) = |\lambda| X, \quad \varphi_+(Y) = |\mu| Y \quad \text{and} \quad \varphi_+([X, Y]) = |\lambda \mu| [X, Y] = [\varphi_+(X), \varphi_+(Y)].$$

This shows that $\varphi_+$ respects the Lie bracket of eigenvectors of $\varphi_s$. However, since these eigenvectors span the whole space $g_C$, $\varphi_+$ is a Lie algebra automorphism of $g_C$.

Note that any element $Z$ of $g_C$ is of the form $Z = X + iY$, where $X, Y \in g$ and we can use $\bar{Z} = X - iY$. We want to show that $\varphi_+$ restricts to an automorphism of $g$. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the different real eigenvalues of $\varphi_s$ and let $\mu_1, \mu_2, \mu_3, \ldots, \mu_q, \mu_\lambda$ be the different non-real eigenvalues of $\varphi_s$ (where we used $\mu_\lambda$ to denote the complex conjugate of $\mu_\lambda$). If we denote the eigenspace corresponding to the eigenvalue $\lambda$ by $V_\lambda$, we have that

$$g_C = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_p} \oplus V_{\mu_1} \oplus V_{\mu_2} \oplus \cdots \oplus V_{\mu_q} \oplus V_{\mu_\lambda}.$$  

It is obvious that $V_{\lambda_i}$ ($1 \leq i \leq p$) has a basis (as a complex vector space) of vectors of $g$. Also $V_{\mu_j} \oplus V_{\mu_\lambda}$ has a basis (as a complex vector space) of vectors of $g$. Indeed, if $Z_1, Z_2, \ldots, Z_r$ is a basis of $V_{\mu_j}$, then $\overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_r$ is a basis of $V_{\mu_\lambda}$. But then

$$Z_1 + \overline{Z}_1, Z_2 + \overline{Z}_2, \ldots, Z_r + \overline{Z}_r, i(Z_1 - \overline{Z}_1), i(Z_2 - \overline{Z}_2), \ldots, i(Z_r - \overline{Z}_r)$$
is a basis of $V_{\mu j} \oplus V_{\bar{\mu} j}$ of vectors in $g$. This shows that $\varphi_+$ has a basis of eigenvectors in $g$ and therefore restricts to an automorphism of $g$. The proof now finishes by taking $\varphi_1 = \varphi_+ \varphi_+^{-1}$. □

Remark 2.2.

(1) We will refer to the automorphism $\varphi_+$ as the positive part of $\varphi$ (see also [14]). An automorphism $\varphi$ which equals its positive part will be called a positive automorphism.

(2) The above lemma is not only valid for Lie algebra automorphisms, but holds in any finite dimensional real algebra.

(3) In case one regards $\mathbb{R}^n$ as an abelian Lie algebra (i.e., a trivial algebra in which all products are 0), the above lemma gives a unique decomposition $A = A_u A_s A_1$ of any matrix $A \in \text{GL}(n, \mathbb{R})$, into a unipotent factor, a semi-simple factor with positive real eigenvalues and a semi-simple factor with eigenvalues of modulus 1.

The importance of the previous lemma lies in the fact that in the search for expanding or hyperbolic automorphisms, one can restrict its attention to positive automorphisms.

Proposition 2.3. Let $g$ be a finite dimensional real Lie algebra and $\varphi \in \text{Aut}(g)$.

(1) If the eigenvalues of $\varphi$ are $\lambda_1, \lambda_2, \ldots, \lambda_m$, then the eigenvalues of $\varphi_+$ are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_m|.$

(2) $\varphi$ is an expanding automorphism $\iff$ $\varphi_+$ is an expanding automorphism.

(3) $\varphi$ is a hyperbolic automorphism $\iff$ $\varphi_+$ is a hyperbolic automorphism.

Proof. This follows immediately from Lemma 2.1. □

We are now ready to show how an automorphism of a Lie algebra induces a $\mathbb{R}$-grading.

Definition 2.4. Let $g$ be a Lie algebra. An $\mathbb{R}$-grading of $g$ consists of a direct sum decomposition of vector spaces

$$g = \bigoplus_{r \in \mathbb{R}} g_r$$

where each $g_r$ is a subspace of $g$ and such that $\forall r, s \in \mathbb{R}$: $[g_r, g_s] \subseteq g_{r+s}$.

For finite dimensional Lie algebras (as we are always assuming in this paper), all but finitely many of the spaces $g_r$ will be trivial. In literature, one often works with $\mathbb{Z}$-graded Lie algebras, these are of course also $\mathbb{R}$-graded.

Theorem 2.5. Let $g$ be a finite dimensional real Lie algebra.

(1) If $\varphi$ is a positive automorphism of a real Lie algebra $g$, then $g = \bigoplus_{r \in \mathbb{R}} g_r$, where $g_r$ is the eigenspace of $\varphi$ belonging to the eigenvalue $e^r$, is a $\mathbb{R}$-grading of $g$. 

Conversely, if $g = \bigoplus_{r \in \mathbb{R}} g_r$ is a $\mathbb{R}$-grading of $g$, then the linear map defined by

$$\varphi : g \to g, \text{ such that } \forall r \in \mathbb{R}, \forall X \in g_r : \varphi(X) = e^r X$$

is a positive Lie algebra automorphism of $g$.

**Remark 2.6.** It is obvious that the steps in the theorem above are each others opposite. We will speak about the $\mathbb{R}$-grading induced by the automorphism $\varphi$ and about the automorphism induced by a given $\mathbb{R}$-grading.

**Proof.** Assume that $\varphi$ is a positive automorphism of $g$, then $\varphi$ is semi-simple and therefore $g$ decomposes as a direct sum of its eigenspaces, from which is follows that $g = \bigoplus_{r \in \mathbb{R}} g_r$, where $g_r$ denotes the eigenspace belonging to eigenvalue $e^r$ (note that all eigenvalues of $\varphi$ are of the from $e^r$). If $X \in g_r$ and $Y \in g_s$, we have that

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] = [e^r X, e^s Y] = e^{r+s} [X, Y]$$

form which is follows that $[X, Y] \in g_{r+s}$, which shows that the direct sum decomposition is indeed a $\mathbb{R}$-grading.

Conversely, assume that $g = \bigoplus_{r \in \mathbb{R}} g_r$ is a $\mathbb{R}$-grading and define $\varphi$ as in the theorem. As

$$\forall X \in g_r, \forall Y \in g_s : \varphi([X, Y]) = e^{r+s} [X, Y] = [e^r X, e^s Y] = [\varphi(X), \varphi(Y)]$$

(use that $[g_r, g_s] \subseteq g_{r+s}$), we can conclude that $\varphi$ is a Lie algebra automorphism. Moreover, $\varphi$ is positive, since $g$ decomposes into a direct sum of its eigenspaces and all eigenvalues are positive real numbers. $\blacksquare$

As a consequence, we find the following description of Lie algebras admitting an expanding or a hyperbolic automorphism.

**Theorem 2.7.** Let $g$ be a finite dimensional real Lie algebra, then

1. $g$ admits an expanding automorphism if and only if $g$ admits a positive $\mathbb{R}$-grading, i.e., a grading

$$g = \bigoplus_{r \in \mathbb{R}} g_r, \text{ with } g_r = 0, \forall r \leq 0.$$ 

2. $g$ admits a hyperbolic automorphism if and only if $g$ admits a mixed $\mathbb{R}$-grading, i.e., a grading

$$g = \bigoplus_{r \in \mathbb{R}} g_r, \text{ with } g_0 = 0, \text{ and } \exists r_- < 0 : g_{r_-} \neq 0, \exists r_+ > 0 : g_{r_+} \neq 0.$$ 

**Proof.** This follows from Proposition 2.3 and Theorem 2.5. $\blacksquare$

**Remark 2.8.** If a Lie algebra admits a negative $\mathbb{R}$-grading $g = \bigoplus_{r \in \mathbb{R}} g_r$, $g_r = 0$ if $r \geq 0$, then it also admits a positive $\mathbb{R}$-grading $g = \bigoplus_{r \in \mathbb{R}} g'_r$, with $g'_r = g_{-r}$. Therefore, a Lie algebra admits a $\mathbb{R}$-grading $g = \bigoplus_{r \in \mathbb{R}} g_r$, with $g_0 = 0$ if and only if it admits either an expanding or a hyperbolic automorphism or both.
Using the results obtained so far, it is now also easy to obtain the following (known) result.

**Proposition 2.9.** Let \( \mathfrak{g} \) be a finite dimensional real Lie algebra admitting an expanding or a hyperbolic automorphism, then \( \mathfrak{g} \) is nilpotent.

**Proof.** If \( \mathfrak{g} \) admits an expanding or a hyperbolic automorphism, then \( \mathfrak{g} \) has a \( \mathbb{R} \)-grading \( \mathfrak{g} = \bigoplus_{r \in \mathbb{R}} \mathfrak{g}_r \), with \( \mathfrak{g}_0 = 0 \).

Let \( X \in \mathfrak{g}_s \) for some \( s \neq 0 \), then \( \forall Y \in \mathfrak{g}_r : (\text{ad}(X))^k(Y) \in \mathfrak{g}_{ks+r} \). As all but finitely many \( \mathfrak{g}_r \) are trivial, this shows that \( \text{ad}(X) \) is nilpotent for all \( X \) and so \( \mathfrak{g} \) is nilpotent. \( \square \)

3. \( \mathbb{R} \)-graded Lie algebras and affine structures

A manifold \( M \) is affinely flat if it admits a system of coordinate charts \( f_i : U_i \rightarrow \mathbb{R}^n \) on an open cover \( U_i, \ i \in I, \) of \( M \), for which the transition functions \( f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \rightarrow f_j(U_i \cap U_j) \) are (restrictions of) affine maps of \( \mathbb{R}^n \). An affinely flat manifold \( M \) inherits the usual affine connection of \( \mathbb{R}^n \). Using this connection, we have the notion of a geodesic on \( M \), which are curves that in local coordinates are straight lines which are traversed at constant speed. An affinely flat manifold is said to be complete if every partial geodesic \( \gamma : [a, b] \rightarrow M \) can be extended to a full geodesic \( \gamma : [0, \infty) \rightarrow M \). By [1], we know that all connected, complete, affinely flat manifolds are obtained as a quotient \( E \backslash \mathbb{R}^n \), where \( E \) acts freely, properly discontinuously and via affine motions on \( \mathbb{R}^n \) (see also [9,10,16,20]).

It follows that an infra-nilmanifold \( E \backslash G \) admits a complete, affinely flat structure if and only if we can find a morphism \( \rho : E \rightarrow \text{Aff}(\mathbb{R}^n) \) letting the almost Bieberbach group \( E \) act properly discontinuously and cocompactly on \( \mathbb{R}^n \). We will refer to such a morphism as an affine structure on \( E \).

For nilmanifolds \( N \backslash G \) (i.e., \( N \) is a uniform, discrete subgroup of a connected and simply connected nilpotent Lie group \( G \)) it is known that the following are equivalent [3,5,16]:

1. The discrete group \( N \) admits an affine structure.
2. The Lie group \( G \) acts simply transitive and via affine motions on \( \mathbb{R}^n \), for some \( n \).
3. The Lie algebra \( \mathfrak{g} \) of \( G \) admits a complete left symmetric structure.

Recall that a left symmetric structure on a Lie algebra \( \mathfrak{g} \) is a bilinear product \( \cdot : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \) satisfying

\[ \forall X, Y \in \mathfrak{g} : [X, Y] = X \cdot Y - Y \cdot X \quad \text{and} \]
\[ \forall X, Y, Z \in \mathfrak{g} : [X, Y] \cdot Z = X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z). \]

A left symmetric algebra is said to be complete if the linear transformation \( t_Y : \mathfrak{g} \rightarrow \mathfrak{g} : X \mapsto X + X \cdot Y \) is bijective for all \( Y \in \mathfrak{g} \). In case \( \mathfrak{g} \) is a nilpotent Lie algebra, the completeness
of a left symmetric structure on $\mathfrak{g}$ is equivalent to the fact that the left multiplication map $\lambda_X : \mathfrak{g} \to \mathfrak{g} : Y \mapsto X \cdot Y$ is nilpotent for all $X$ in $\mathfrak{g}$ [23].

**Theorem 3.1.** Let $\mathfrak{g}$ be a finite dimensional real Lie algebra admitting a $\mathbb{R}$-grading

$$\mathfrak{g} = \bigoplus_{r \in \mathbb{R}} \mathfrak{g}_r, \quad \text{with } \mathfrak{g}_0 = 0.$$  

Then $\mathfrak{g}$ also admits a complete left symmetric structure.

**Proof.** The proof is a careful copy of the situation of positively $\mathbb{Z}$-graded Lie algebras ([3] and [15]). We define a bilinear product $\cdot$ on $\mathfrak{g}$ as follows:

$$\forall X \in \mathfrak{g}_r, \forall Y \in \mathfrak{g}_s : \quad X \cdot Y = \begin{cases} \frac{s}{r+s} [X,Y] & \text{if } r+s \neq 0, \\ 0 & \text{if } r+s = 0. \end{cases}$$

It is obvious that the left multiplication maps $\lambda_X$ are nilpotent for all $X \in \mathfrak{g}$. So, it remains to prove that this product defines a left symmetric structure. In the sequel we will always write $X \cdot Y = \frac{s}{r+s} [X,Y]$, for $X \in \mathfrak{g}_r$ and $Y \in \mathfrak{g}_s$, even if $r+s = 0$. Note that when $r+s = 0$, then $[X,Y] = 0$ also (since $[X,Y] \in \mathfrak{g}_0 = 0$). The reader should keep this abuse of notation in mind, when following the proof.

By definition, $[X,Y] = X \cdot Y - Y \cdot X$, for all $X \in \mathfrak{g}_r$ and $Y \in \mathfrak{g}_s$ ($\forall r,s \in \mathbb{R}$). It follows that $[X,Y] = X \cdot Y - Y \cdot X, \forall X,Y \in \mathfrak{g}$.

To finish the proof, it suffices to show that

$$\forall r,s,t \in \mathbb{R}, \forall X \in \mathfrak{g}_r, \forall Y \in \mathfrak{g}_s, \forall Z \in \mathfrak{g}_t : \quad [X,Y] \cdot Z = X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z).$$

But this is just an easy calculation:

$$X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = X \cdot \frac{t}{s+t} [Y,Z] - \frac{t}{r+t} [X,Z] = \frac{t}{s+t} \frac{r+t}{r+s+t} [X,[Y,Z]] - \frac{t}{r+t} \frac{r+t}{s+r+t} [Y,[X,Z]]$$

$$= \frac{t}{r+s+t} ([X,[Y,Z]] - [Y,[X,Z]])$$

$$= [X,Y] \cdot Z,$$

where the last equality comes from the Jacobi identity. (The reader is invited to check that the proof goes through, when some of the denominators are 0.) □

**Corollary 3.2.** Let $M$ be a nilmanifold admitting an expanding map or an Anosov diffeomorphism, then $M$ has a complete, affinely flat structure.

**Example 3.3.** Let us illustrate the notions seen thus far by means of the (free) 2-step nilpotent Lie algebra $\mathfrak{g}$, with basis $X_1$, $X_2$, $X_3$, $Y_1$, $Y_2$ and $Y_3$ and where the non-zero brackets between basis vectors are given by

$$[X_1, X_2] = Y_1, \quad [X_1, X_3] = Y_2, \quad [X_2, X_3] = Y_3.$$
A first trivial grading of this Lie algebra is obtained by taking \( g_1 = \langle X_1, X_2, X_3 \rangle \) and \( g_2 = \langle Y_1, Y_2, Y_3 \rangle \). This grading gives rise to an expanding automorphism on \( g \) and to the standard left symmetric structure on a 2-step nilpotent Lie algebra given by \( A \cdot B = \frac{1}{2} [A, B], \forall A, B \in g \).

The following is a non-trivial grading of \( g \):

\[
g_{-1} = \langle X_1 \rangle, \quad g_1 = \langle Y_1, Y_2 \rangle, \quad g_2 = \langle X_2, X_3 \rangle \quad \text{and} \quad g_4 = \langle Y_3 \rangle.
\]

The (positive) hyperbolic automorphism \( \varphi \) corresponding to this grading satisfies

\[
\varphi(X_1) = \frac{1}{e} X_1, \quad \varphi(Y_1) = eY_1, \quad \varphi(Y_2) = eY_2,
\]

\[
\varphi(X_2) = e^2 X_2, \quad \varphi(X_3) = e^2 X_3 \quad \text{and} \quad \varphi(Y_3) = e^4 Y_3.
\]

The non-zero products between the basis vectors for the corresponding left symmetric structures are given by:

\[
X_1 \cdot X_2 = 2Y_1, \quad X_2 \cdot X_1 = Y_1, \quad X_1 \cdot X_3 = 2Y_2,
\]

\[
X_3 \cdot X_1 = Y_2, \quad X_2 \cdot X_3 = \frac{1}{2} Y_3, \quad X_3 \cdot X_2 = -\frac{1}{2} Y_3.
\]

4. Affine structures for infra-nilmanifolds with expanding maps or Anosov diffeomorphisms

In this section, we will extend the result of the previous section to the case of infra-nilmanifolds. This extension is based on Section 4 of [5].

Before we can deal with this case of infra-nilmanifolds, we need the following lemma.

**Lemma 4.1.** Let \( g \) be a finite dimensional real Lie algebra and let \( \varphi, \psi \in \text{Aut}(g) \). If \([\psi, \varphi] = 1\), then also

\[
[\psi, \varphi_u] = 1, \quad [\psi, \varphi_+] = 1 \quad \text{and} \quad [\psi, \varphi_1] = 1.
\]

**Proof.** As \([\psi, \varphi] = 1\), we have that

\[
\varphi = \psi \varphi \psi^{-1} = \psi \varphi_u \psi^{-1} \psi \varphi_1 \psi^{-1} \psi \varphi_+ \psi^{-1},
\]

where

1. \( \psi \varphi_u \psi^{-1} \) is unipotent.
2. \( \psi \varphi_1 \psi^{-1} \) and \( \psi \varphi_+ \psi^{-1} \) are semi-simple.
3. All eigenvalues of \( \psi \varphi_1 \psi^{-1} \) are of modulus 1.
4. All eigenvalues of \( \psi \varphi_+ \psi^{-1} \) are positive real numbers.
5. \( \psi \varphi_u \psi^{-1}, \psi \varphi_1 \psi^{-1} \) and \( \psi \varphi_+ \psi^{-1} \) commute pairwise.

By the uniqueness aspect of Lemma 2.1 it follows that

\[
\varphi_u = \psi \varphi_u \psi^{-1}, \quad \varphi_1 = \psi \varphi_1 \psi^{-1} \quad \text{and} \quad \varphi_+ = \psi \varphi_+ \psi^{-1}
\]

which was to be shown. \( \square \)
Theorem 4.2. Let $M$ be an infra-nilmanifold admitting an expanding map or an Anosov diffeomorphism, then $M$ also admits an affine structure.

Proof. As mentioned in the introduction, $M$ is a quotient manifold $E/G$, where $E$ is a discrete subgroup of $G \rtimes F$ ($F$ being the holonomy group of $E$), where $G$ is a connected and simply connected nilpotent Lie group. As is well known by the results of Gromov and Manning, any expanding map or an Anosov diffeomorphism on $M$ is homotopic to a map induced by an element of $\text{Aff}(G) = G \rtimes \text{Aut}(G)$. Therefore, we assume that our map is induced by $\psi = (g, \varphi) \in \text{Aff}(G) = G \rtimes \text{Aut}(G)$, where $\varphi$ is an expanding or a hyperbolic automorphism of $G$. Thus we have

$$\psi E \psi^{-1} \subseteq E.$$  \hspace{1cm} (1)

It follows from (1) that

$$\forall \alpha \in F: \quad \varphi \alpha \varphi^{-1} \in F.$$

As $F$ is finite, it follows that the map $\mu(\varphi) : F \rightarrow F : \alpha \mapsto \varphi \alpha \varphi^{-1}$ is an automorphism of $F$. As $\text{Aut}(F)$ is also finite, there exists a $k$ such that $\mu(\varphi)^k$ is trivial. Moreover, powers of Anosov diffeomorphisms and expanding maps are again Anosov or expanding, therefore, we can assume that $\mu(\varphi)$ itself is trivial. Thus, $\forall \alpha \in F: \ [\varphi, \alpha] = 1$.

For all $\alpha \in F$, we will use $d\alpha \in \text{Aut}(g)$ to denote the differential of $\alpha$ ($g$ is the Lie algebra of $G$). Analogously, we have $d\varphi$ and we know that

$$\forall \alpha \in F: \quad [\alpha, d \varphi] = 1.$$

But then, by Lemma 4.1 we also have that

$$\forall \alpha \in F: \quad [\alpha, (d \varphi)_+] = 1.$$

Consider the grading of $g = \bigoplus_{r \in \mathbb{R}} g_r$ induced by $(d \varphi)_+$ and where $g_0 = 0$ ($(d \varphi)_+$ is expanding or hyperbolic). We also use $\cdot$ to denote the complete left symmetric structure on $g$, induced by this grading. As $[\alpha, (d \varphi)_+] = 1$ for all $\alpha \in F$, we have that $d\alpha(g_r) = g_r$, $\forall r \in \mathbb{R}, \forall \alpha \in F$. As a consequence, we also have that

$$\forall X, Y \in g, \forall \alpha \in F: \quad (d\alpha)(X \cdot Y) = (d\alpha X) \cdot (d\alpha Y).$$

By Theorem 4.1 of [5], this allows us to conclude that $M$ admits an affine structure.  \hspace{1cm} $\square$

Remark 4.3. Note that the converse of this theorem does not hold. Indeed, all nilmanifolds built on a 3-step nilpotent Lie group admit a complete, affinely flat structure [22], but there are 3-step nilpotent Lie groups (having uniform lattices) [6] not allowing any expanding nor hyperbolic automorphism.

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References