Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 154 (2007) 749–757
Topology and its Applications
www.elsevier.com/locate/topol

Linearly ordered compacta and Banach spaces with a projectional resolution of the identity

Wiesław Kubiś

Instytut Matematyki, Akademia Świętokrzyska, ul. Świętokrzyska 15, 25-406 Kielce, Poland

Received 28 February 2006; accepted 23 August 2006

Abstract

We construct a compact linearly ordered space K_{ω_1} of weight $\aleph_1$, such that the space $C(K_{\omega_1})$ is not isomorphic to a Banach space with a projectional resolution of the identity, while on the other hand, K_{ω_1} is a continuous image of a Valdivia compact and every separable subspace of $C(K_{\omega_1})$ is contained in a 1-complemented separable subspace. This answers two questions due to O. Kalenda and V. Montesinos.

MSC: primary 46B03, 46B26; secondary 54F05, 46E15, 54C35

Keywords: Banach space; Projection; Separable complementation property; Plichko space; Linearly ordered compact space

1. Introduction

A subspace F of a Banach space E is *complemented* in E if there exists a bounded linear projection $P : E \to E$ such that $F = PE$. More precisely, we say that F is *k-complemented* if $\|P\| \leqslant k$. A Banach space E has the *separable complementation property* if every separable subspace of E is contained in a complemented separable one. Typical examples of such spaces are Banach spaces with a countably norming Markushevich basis, which are called *Plichko spaces* (see Kalenda [6]). In case of Banach spaces of density $\aleph_1$, the property of being Plichko is equivalent to the existence of a *bounded projectional resolution of the identity*, i.e. a transfinite sequence of projections on separable subspaces satisfying some continuity and compatibility conditions (the precise definition is given below). In particular, every Plichko space of density $\aleph_1$ is the union of a continuous chain of complemented separable subspaces.

A question of Ondřej Kalenda [6, Question 4.5.10] asks whether every closed subspace of a Plichko space is again a Plichko space. We describe a compact connected linearly ordered space K_{ω_1} of weight $\aleph_1$ which is an order preserving image of a linearly ordered Valdivia compact constructed in [7] and whose space of continuous functions is not Plichko. This answers Kalenda's question in the negative.

During the 34th Winter School on Abstract Analysis (Lhota and Rohanovem, Czech Republic, 14–21 January 2006), Vicente Montesinos raised the question whether every Banach space with the separable complementation

E-mail address: wkubis@pu.kielce.pl (W. Kubiś).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.08.010

property is isomorphic to a space with a projectional resolution of the identity. We show that every separable subspace of $C(K_{\omega_1})$ is contained in a 1-complemented separable subspace, answering the above question in the negative.

On the other hand, we show that a Banach space of density $\aleph_1$ has a projectional resolution of the identity, provided it can be represented as the union of a continuous increasing sequence of separable subspaces $\{E_\alpha\}_{\alpha<\omega_1}$ such that each E_α is 1-complemented in $E_{\alpha+1}$. We apply this result for proving that every 1-complemented subspace of a 1-Plichko space of density $\aleph_1$ is again a 1-Plichko space. This gives a partial positive answer to a question of Kalenda [6, Question 4.5.10].

2. Preliminaries

We use standard notation and symbols concerning topology and set theory. For example, $\mathrm{w}(X)$ denotes the weight of a topological space X. A compact space K is called $\aleph_0$-*monolithic* if every separable subspace of K is second countable. Given a surjection $f: X \to Y$, the sets $f^{-1}(y)$, where $y \in Y$, are called the *fibers* of f or f-*fibers*. The letter ω denotes the set of natural numbers $\{0, 1, \ldots\}$. We denote by ω_1 the first uncountable ordinal and we shall write $\aleph_1$ instead of ω_1, when having in mind its cardinality, not its order type. We denote by $|A|$ the cardinality of the set A. A set $C \subseteq \omega_1$ is *closed* if $\sup_{n\in\omega} \xi_n \in C$ whenever $\{\xi_n\colon n \in \omega\} \subseteq C$ is increasing; C is *unbounded* if $\sup C = \omega_1$. Given an ordinal λ, a sequence of sets $\{A_\alpha\}_{\alpha<\lambda}$ will be called *increasing* if $A_\alpha \subseteq A_\beta$ for every $\alpha < \beta < \lambda$, and *continuous* if $A_\delta = \bigcup_{\xi<\delta} A_\xi$ for every limit ordinal $\delta < \lambda$.

In this note we deal with Banach spaces of density $\leqslant \aleph_1$. Recall that a Banach space E has the *separable complementation property* if every separable subspace of E is contained in a complemented separable one. Fix a Banach space E of density $\aleph_1$. A *projectional resolution of the identity* (briefly: *PRI*) in E is a sequence $\{P_\alpha\}_{\alpha<\omega_1}$ of projections of E onto separable subspaces, satisfying the following conditions:

(1) $\|P_\alpha\| = 1$.
(2) $\alpha \leqslant \beta \Longrightarrow P_\alpha = P_\alpha P_\beta = P_\beta P_\alpha$.
(3) $E = \bigcup_{\alpha<\omega_1} P_\alpha E$ and $P_\delta E = \mathrm{cl}(\bigcup_{\xi<\delta} P_\xi E)$ for every limit ordinal $\delta < \omega_1$.

Weakening condition (1) to $\sup_{\alpha<\omega_1} \|P_\alpha\| < +\infty$, we obtain the notion of a *bounded projectional resolution of the identity*. For a survey on the use of PRI's in nonseparable Banach spaces and for a historical background we refer to Chapter 6 of Fabian's book [3]. A Banach space of density $\aleph_1$ is a 1-*Plichko space* if it has a projectional resolution of the identity. This is different from (although equivalent to) the original definition: see Definition 4.2.1 and Theorem 4.2.5 in [6]. A space isomorphic to a 1-Plichko space is called a *Plichko space* or, more precisely, a k-*Plichko space*, where $k \geqslant 1$ is the constant coming from the isomorphism to a 1-Plichko space. In fact, a k-Plichko space of density $\aleph_1$ can be characterized as a space having a bounded PRI $\{P_\alpha\}_{\alpha<\omega_1}$ such that $k \geqslant \sup_{\alpha<\omega_1} \|P_\alpha\|$ (see the proof of Theorem 4.2.4(ii) in [6]). Of course, every Plichko space has the separable complementation property.

Recall that a compact space K is called *Valdivia compact* (see [6]) if there exists an embedding $j: K \to [0,1]^\kappa$ such that $j^{-1}[\Sigma(\kappa)]$ is dense in K, where $\Sigma(\kappa) = \{x \in [0,1]^\kappa\colon |\{\alpha\colon x(\alpha) \neq 0\}| \leqslant \aleph_0\}$. Compact spaces embeddable into $\Sigma(\kappa)$ are called *Corson compacta*. By the result of [8], a space of weight $\aleph_1$ is Valdivia compact if and only if it can be represented as the limit of a continuous inverse sequence of metric compacta with all bonding mappings being retractions—a property analogous to the existence of a PRI in a Banach space. Valdivia compacta are dual to 1-Plichko spaces in the following sense: if K is a Valdivia compact then $C(K)$ is 1-Plichko and if E is a 1-Plichko space then the closed unit ball of E^* endowed with the *weak** topology is Valdivia compact. See [6, Chapter 5] for details.

Fix a Banach space E of density $\aleph_1$. A *skeleton* in E is a chain $\mathcal{C}$ of closed separable subspaces of E such that $\bigcup \mathcal{C} = E$ and $\mathrm{cl}(\bigcup_{n\in\omega} C_n) \in \mathcal{C}$, whenever $C_0 \subseteq C_1 \subseteq \cdots$ is a sequence in $\mathcal{C}$. Given a skeleton $\mathcal{C}$, one can always choose an increasing sequence $\{E_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{C}$ so that $E = \bigcup_{\alpha<\omega_1} E_\alpha$ and $E_\delta = \mathrm{cl}(\bigcup_{\alpha<\delta} E_\alpha)$ for every limit ordinal $\alpha < \omega_1$. In particular, every $C \in \mathcal{C}$ is contained in some E_α. We shall consider skeletons indexed by ω_1, assuming implicitly that the enumeration is increasing and continuous.

Lemma 2.1. *Let E be a Banach space of density $\aleph_1$ and assume $\mathcal{C}, \mathcal{D}$ are skeletons in E. Then $\mathcal{C} \cap \mathcal{D}$ is a skeleton in E. More precisely, if $\mathcal{C} = \{C_\alpha\}_{\alpha<\omega_1}$ and $\mathcal{D} = \{D_\alpha\}_{\alpha<\omega_1}$ then there exists a closed and unbounded set $\Gamma \subseteq \omega_1$ such that $C_\alpha = D_\alpha$ for $\alpha \in \Gamma$.*

Proof. Let $\Gamma = \{\alpha < \omega_1\colon C_\alpha = D_\alpha\}$. It is clear that Γ is closed in ω_1. Fix $\xi < \omega_1$. Since C_ξ is separable, we can find α_1 such that $C_\xi \subseteq D_{\alpha_1}$. Similarly, we can find $\alpha_2 > \alpha_1$ such that $D_{\alpha_1} \subseteq C_{\alpha_2}$. Continuing this way, we obtain a sequence $\alpha_1 < \alpha_2 < \cdots$ such that $C_{\alpha_{2n-1}} \subseteq D_{\alpha_{2n}} \subseteq C_{\alpha_{2n+1}}$ for every $n \in \omega$. Let $\delta = \sup_{n\in\omega} \alpha_n$. Then $C_\delta = D_\delta$, i.e. $\delta \in \Gamma$. Thus Γ is unbounded. $\square$

Corollary 2.2. *Assume $\{E_\alpha\}_{\alpha<\omega_1}$ is a skeleton in a Plichko space E of density $\aleph_1$. Then there exists a closed cofinal set $C \subseteq \omega_1$ such that $\{E_\alpha\}_{\alpha\in C}$ consists of complemented subspaces of E.*

Proof. Apply Lemma 2.1 to $\mathcal{C} = \{E_\alpha\}_{\alpha<\omega_1}$ and $\mathcal{D} = \{F_\alpha\}_{\alpha<\omega_1}$, where $F_\alpha = P_\alpha E$ and $\{P_\alpha\}_{\alpha<\omega_1}$ is a bounded PRI on E. $\square$

Note that the skeleton $\{E_\alpha\}_{\alpha\in C}$ from the above corollary consists of k-complemented subspaces, where k is the constant coming from a renorming of E to a 1-Plichko space.

3. Linearly ordered compacta

We shall consider linearly ordered compact spaces endowed with the order topology. Given such a space K, we denote by 0_K and 1_K the minimal and the maximal element of K respectively. As usual, we denote by $[a,b]$ and (a,b) the closed and the open interval with end-points $a,b \in K$. Given two linearly ordered compacta K, L, a map $f : K \to L$ will be called *increasing* if $x \leqslant y \Longrightarrow f(x) \leqslant f(y)$ holds for every $x, y \in K$. It is straight to see that every increasing surjection is continuous. In particular, an order isomorphism is a homeomorphism. We denote by $\mathbb{I}$ the closed unit interval of the reals.

The following lemma belongs to the folklore. The argument given below can be found, for example, in [4, Lemma 2.1] and [7, Proposition 5.7].

Lemma 3.1. *Let K be a linearly ordered space and let $a, b \in K$ be such that $a < b$. Then there exists a continuous increasing function $f : K \to \mathbb{I}$ such that $f(a) = 0$ and $f(b) = 1$.*

Proof. Since X is a normal space, by the Urysohn Lemma, we can find a continuous function $h : X \to \mathbb{I}$ such that $h(a) = 0$ and $h(b) = 1$. Modifying h, without losing the continuity, we may assume that $f(x) = 0$ for $x \leqslant a$ and $f(x) = 1$ for $x \geqslant b$. Define $f(x) = \sup\{h(t)\colon t \leqslant x\}$. Then f is increasing, $f(a) = 0$ and $f(b) = 1$. It is straight to check that f is continuous. $\square$

Proposition 3.2. *Let K be a linearly ordered compact. Then the set of all increasing functions is linearly dense in $C(K)$.*

Proof. Fix $f \in C(K)$ and $\varepsilon > 0$. Then $K = J_0 \cup \cdots \cup J_{k-1}$, where each J_i is an open interval such that the oscillation of f on J_i is $< \varepsilon$. Choose $0_K = a_0 < a_1 < \cdots < a_n = 1_K$ such that for every $i < n$ either $[a_i, a_{i+1}]$ is contained in some J_j or else $|[a_i, a_{i+1}]| = 2$. By Lemma 3.1, for each $i < n$ there exists an increasing function $h_i : [a_i, a_{i+1}] \to \mathbb{I}$ such that $h_i(a_i) = 0$ and $h_i(a_{i+1}) = 1$. Define

$$g(t) = f(a_i) + h_i(t)\big(f(a_{i+1}) - f(a_i)\big) \quad \text{for } t \in [a_i, a_{i+1}].$$

Then $g : K \to \mathbb{R}$ is a piece-wise monotone continuous function such that $\|f - g\| < \varepsilon$. Finally, piece-wise monotone functions are linear combinations of increasing functions. $\square$

Given a continuous surjection of compact spaces $f : X \to Y$, we shall say that $C(Y)$ is *identified with a subspace of $C(X)$ via f*, having in mind the space $\{\varphi f\colon \varphi \in C(Y)\}$, which is linearly isometric to $C(Y)$. In other words, $\varphi \in C(X)$ is regarded to be a member of $C(Y)$ if and only if φ is constant on the fibers of f. The next statement is in fact a reformulation of [7, Proposition 5.7], which says that a linearly ordered compact is the inverse limit of a sequence of "smaller" linearly ordered compacta.

Proposition 3.3. *Assume K is a compact linearly ordered space of weight $\aleph_1$. Then there exist metrizable linearly ordered compacta K_α, $\alpha < \omega_1$ and increasing quotients $q_\alpha : K \to K_\alpha$ such that $\{C(K_\alpha)\}_{\alpha<\omega_1}$ is a skeleton in $C(K)$, where $C(K_\alpha)$ is identified with a subspace of $C(K)$ via q_α. Moreover, for each $\alpha < \beta < \omega_1$ there exists a unique increasing quotient $q_\alpha^\beta : K_\beta \to K_\alpha$ such that $q_\alpha = q_\alpha^\beta q_\beta$.*

Proof. In view of Proposition 3.2, we can choose a linearly dense set $\mathcal{F} \subseteq C(K)$ consisting of increasing functions and such that $|\mathcal{F}| = \aleph_1$. Let $\{\mathcal{F}_\alpha\}_{\alpha<\omega_1}$ be a continuous increasing sequence of countable sets such that $\mathcal{F} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$. Let E_α be the closed linear span of $\mathcal{F}_\alpha$. Clearly, $\{E_\alpha\}_{\alpha<\omega_1}$ is a skeleton in $C(K)$.

Fix $\alpha < \omega_1$. Define the relation $\sim_\alpha$ on K as follows:

$$x \sim_\alpha y \iff (\forall f \in \mathcal{F}_\alpha)\ f(x) = f(y).$$

It is easy to check that $\sim_\alpha$ is an equivalence relation on K whose equivalence classes are closed and convex. Let $K_\alpha = K/{\sim_\alpha}$. Then K_α is a second countable linearly ordered compact space and $\sim_\alpha$ induces an increasing quotient map $q_\alpha : K \to K_\alpha$. If $\alpha < \beta$ then $\sim_\alpha \supseteq \sim_\beta$, therefore there exists a unique (necessarily increasing and continuous) map q_α^β satisfying $q_\alpha = q_\alpha^\beta q_\beta$.

Finally, observe that $f \in E_\alpha$ if and only if f is constant on the equivalence classes of $\sim_\alpha$. This shows that q_α identifies $C(K_\alpha)$ with E_α. □

Fix a linearly ordered compact K. We denote by $\mathbb{L}(K)$ the set of all $p \in K$ such that $|[x, p]| = 2$ for some $x < p$ in K. Such a point x will be denoted by p^-. A point $p \in K$ will be called *internal* if it is not isolated from either of its sides, i.e. $p \notin \{0_K, 1_K\}$ and intervals $[x, p]$, $[p, y]$ are infinite for every $x < p < y$. A point $p \in K$ will be called *external* in K if it is not internal. In other words, $p \in K$ is external iff either $p \in \{0_K, 1_K\}$ or $p \in \mathbb{L}(K)$ or $p = q^-$ for some $q \in \mathbb{L}(K)$. Observe that if K is connected then the only external points are 0_K and 1_K. If K is second countable then the set of all external points of K is countable.

Now fix $f \in C(K)$. We say that $p \in K$ is *irrelevant* for f if one of the following conditions holds:

(1) $p = 0_K$ and $f \restriction [p, b]$ is constant for some $b > p$,
(2) $p = 1_K$ and $f \restriction [a, p]$ is constant for some $a < p$,
(3) $0_K < p < 1_K$ and $f \restriction [a, b]$ is constant for some $a < p < b$.

We say that p is *essential* for f if p is not irrelevant for f. We denote by $\operatorname{ess}(f)$ the set of all essential points of f. Observe that $\operatorname{ess}(f)$ is closed in K and f is constant on every interval contained in $K \setminus \operatorname{ess}(f)$. In fact, it is not hard to see that f is constant on every interval of the form $[a, b]$ where a, b are such that $(a, b) \cap \operatorname{ess}(f) = \emptyset$ and $(a, b) \neq \emptyset$.

Now assume that X is a closed subset of a linearly ordered compact K. Define

$$\mathrm{E}(K, X) = \{ f \in C(K) \colon \operatorname{ess}(f) \subseteq X \}.$$

In other words, $\mathrm{E}(K, X)$ is the set of all $f \in C(K)$ which are constant on every interval of K which has one of the following form: $[0_K, 0_X]$, $[1_X, 1_K]$, $[p^-, p]$, where $p \in \mathbb{L}(X) \setminus \mathbb{L}(K)$ and p^- denotes the predecessor of p in X. Clearly, $\mathrm{E}(K, X)$ is a closed linear subspace of K.

Proposition 3.4. *Assume K is a linearly ordered compact and $E \subseteq C(K)$ is separable. Then there exists a closed separable subspace X of K such that $E \subseteq \mathrm{E}(K, X)$.*

Proof. By Proposition 3.2, there exists a countable set $F \subseteq C(K)$ consisting of increasing functions such that E is contained in the closed linear span of F. Let

$$X = \operatorname{cl}\left(\bigcup_{f \in F} \operatorname{ess}(f) \right).$$

Then $F \subseteq \mathrm{E}(K, X)$, therefore also $E \subseteq \mathrm{E}(K, X)$. It remains to show that $\operatorname{ess}(f)$ is separable for every $f \in F$.

Fix an increasing function $f \in C(K)$ and let $Y = \operatorname{ess}(f)$, $Z = f[Y] = f[K]$. Observe that $f \restriction Y$ is an increasing two-to-one map onto a second countable linearly ordered space $Z \subseteq \mathbb{R}$. It is well known that in this case Y is separable. For completeness, we give the proof.

Fix a countable dense set $D \subseteq f[Y]$ which contains all external points of $f[Y]$. Then $D' = Y \cap f^{-1}[D]$ is countable. We claim that D' is dense in Y. For fix a nonempty open interval $U \subseteq Y$.

If $f \restriction U$ is not constant, there are $x < y$ in U with $f(x) < f(y)$. If $(f(x), f(y))$ is nonempty, it contains some $r \in D$. Thus $f^{-1}(r) \subseteq D' \cap U$, therefore $D' \cap U \neq \emptyset$. If $(f(x), f(y))$ is empty, then $f(y) \in D$; hence $y \in D' \cap U$.

If $f \restriction U$ is constant, then $U = \{x, y\}$ with $x \leqslant y$. If $y = 1_Y$, then $y \in D'$; if $x = 0_Y$ then $x \in D'$. Suppose that $0_Y < x$ and $y < 1_Y$. Then $x \in \mathbb{L}(Y)$ and $y = p^-$ for some $p \in \mathbb{L}(Y)$. If $x < y$ then $f(y) < f(p)$ and hence $y \in D'$. If $x = y$ then either $f(x^-) < f(x)$ or $f(x) = f(y) < f(p)$. In both cases $x \in D'$. □

Proposition 3.5. *Let K be a linearly ordered space which is a continuous image of a Valdivia compact. Then K is $\aleph_0$-monolithic, i.e. every separable subspace of K is second countable.*

Proof. Suppose $X \subseteq K$ is closed, separable and not second countable. Then $\mathbb{L}(X)$ is uncountable. Indeed, otherwise there would exist a countable set D, dense in X, containing both $\mathbb{L}(X)$ and $\{p^-\colon p \in \mathbb{L}(X)\}$, where p^- denotes the predecessor of p in X. Then $\{(a,b)\colon a, b \in D,\ a < b\}$ would be a countable base for X.

Let K_1 be the quotient of K obtained by replacing each interval of the form $[a^-, a]$, where $a \in \mathbb{L}(X)$, by a second countable interval I_a such that $[a^-, a]$ has an increasing map onto I_a. For example, one can define $I_a = \mathbb{I}$ if $[a^-, a]$ is connected and $I_a = \{0, 1\}$ otherwise. Then K_1 is a linearly ordered, non-metrizable continuous image of a Valdivia compact. On the other hand, K_1 is first countable. By the result of Kalenda [5], K_1 is Corson compact. Finally, Nakhmanson's theorem [9] (see also [1, Theorem IV.10.1]) says that K_1 is metrizable, a contradiction. □

4. Main lemmas

Let K denote the double arrow space, i.e. the linearly ordered space of the form $(\mathbb{I} \times \{0\}) \cup (0,1) \times \{1\}$ endowed with the lexicographic order. Then K is compact in the order topology and admits a natural two-to-one increasing quotient $q : K \to \mathbb{I}$. Example 2 of Corson [2] shows that $C(\mathbb{I})$ is not complemented in $C(K)$, when embedded via q. Corson's argument can be sketched as follows. Suppose $P : C(K) \to C(\mathbb{I})$ is a projection. Then $C(K)$ is isomorphic to $C(\mathbb{I}) \oplus E$, where $E = C(K)/C(\mathbb{I})$. On the other hand, it is easy to check that E is isomorphic to $c_0(\mathbb{I})$, therefore $C(K)$ is weakly Lindelöf. On the other hand, $C(K)$ is not weakly Lindelöf, because K is a non-metrizable linearly ordered compact (by Nakhmanson's theorem [9]). Taking a separable space $F \subseteq C(\mathbb{I})$ instead of $C(\mathbb{I})$, one can repeat the above argument to show that $C(\mathbb{I})$ is not contained in a separable complemented subspace of $C(K)$.

Corson's argument uses essentially topological properties of nonseparable Banach spaces. Below we prove a more concrete result, which requires a direct argument. We shall apply it in the proof of our main result.

Lemma 4.1. *Assume $\theta : K \to L$ is an increasing surjection of linearly ordered compacta such that the set*

$$Q = \{x \in L\colon x \text{ is internal in } L \text{ and } |\theta^{-1}(x)| > 1\}$$

is somewhere dense in L. Then $C(L)$ is not complemented in $C(K)$, when identified with the subspace of $C(K)$ via θ.

Proof. For each $x \in L$ define $x^- = \min \theta^{-1}(x)$ and $x^+ = \max \theta^{-1}(x)$. Then each fiber of θ is of the form $[x^-, x^+]$, where $x \in L$. Recall that θ identifies $C(L)$ with the set of all $f \in C(K)$ which are constant on every interval $[x^-, x^+]$, where $x \in L$. Suppose $P : C(K) \to C(K)$ is a bounded linear projection onto $C(L)$. Fix $N \in \omega$ such that

$$-1 + N/3 \geqslant \|P\|.$$

Given $p \in Q$, choose an increasing function $\chi_p \in C(K)$ such that $\chi_p(t) = 0$ for $t \leqslant p^-$ and $\chi_p(t) = 1$ for $t \geqslant p^+$. Let $h_p = P\chi_p$. There exists a (unique) function $\bar{h}_p \in C(L)$ such that $h_p = \bar{h}_p\theta$. Define

$$Q^- = \{q \in Q\colon \bar{h}_q(q) < 2/3\} \quad \text{and} \quad Q^+ = \{q \in Q\colon \bar{h}_q(q) > 1/3\}.$$

Then at least one of the above sets is somewhere dense. Further, define

$$U_p^- = (\bar{h}_p)^{-1}(-\infty, 2/3) \quad \text{and} \quad U_p^+ = (\bar{h}_p)^{-1}(1/3, +\infty).$$

Suppose that the set Q^- is dense in the interval (a,b). Choose $p_0 < p_1 < \cdots < p_{N-1}$ in $Q^- \cap (a,b)$ so that $p_i \in U^-_{p_0} \cap \cdots \cap U^-_{p_{i-1}}$ for every $i < N$. This is possible, because each p_i is internal in L. Choose $f \in C(K)$ such that $0 \leqslant f \leqslant 1$ and

$$f \restriction \theta^{-1}(p_i) = \chi_{p_i} \restriction \theta^{-1}(p_i) \quad \text{for } i < N$$

and f is constant on $[p^-, p^+]$ for every $p \in L \setminus \{p_0, \dots, p_{N-1}\}$. The function f can be constructed as follows. For each $i < N-1$ choose a continuous function $\varphi_i : [p_i, p_{i+1}] \to \mathbb{I}$ such that $\varphi_i(p_i) = 1$ and $\varphi_i(p_{i+1}) = 0$. Define

$$f(t) = \begin{cases} 0 & \text{if } t < p_0, \\ \chi_i(t) & \text{if } t \in [p_i^-, p_i^+],\ i < N, \\ \varphi_i \theta(t) & \text{if } t \in [p_i^+, p_{i+1}^-],\ i < N-1, \\ 1 & \text{if } t > p_{N-1}. \end{cases}$$

Let $g = f - \sum_{i<N} \chi_{p_i}$. Then g is constant on each interval of the form $\theta^{-1}(p)$, where $p \in L$. Indeed, if $p \notin \{p_0, \dots, p_{N-1}\}$ then all the functions $f, \chi_{p_0}, \dots, \chi_{p_{N-1}}$ are constant on $\theta^{-1}(p)$. If $t \in [p_j^-, p_j^+]$ then

$$g(t) = f(t) - \sum_{i \leqslant j} \chi_{p_i}(t) = f(t) - (j-1) - \chi_{p_j}(t) = j - 1,$$

because $f(t) = \chi_{p_j}(t)$. It follows that $g \in C(L)$, i.e. $Pg = g$. Hence

$$Pf = Pg + P\left(\sum_{i<N} \chi_{p_i}\right) = g + \sum_{i<N} h_{p_i}.$$

Now choose $t \in U^-_{p_0} \cap \cdots \cap U^-_{p_{N-1}}$ such that $t > p_{N-1}$. Note that $|(Pf)(t)| \leqslant \|P\|$, because $0 \leqslant f \leqslant 1$. On the other hand, $h_{p_i}(t) < 2/3$ for $i < N$ and consequently

$$\begin{aligned} -\|P\| \leqslant (Pf)(t) &= g(t) + \sum_{i<N} h_{p_i}(t) = f(t) - \sum_{i<N} \chi_{p_i}(t) + \sum_{i<N} h_{p_i}(t) = f(t) - N + \sum_{i<N} h_{p_i}(t) \\ &< 1 - \sum_{i<N} (2/3) = 1 - N/3 \leqslant -\|P\|, \end{aligned}$$

which is a contradiction.

In case where the set Q^- is nowhere dense, we use the fact that Q^+ must be somewhere dense and we choose a decreasing sequence $p_0 > p_1 > \cdots > p_{N-1}$ in Q^+ so that $p_i \in U^+_{p_0} \cap \cdots \cap U^+_{p_{i-1}}$ for $i < N$. Taking $t \in U^+_{p_0} \cap \cdots \cap U^+_{p_{N-1}}$ with $t < p_{N-1}$ and considering a similar function f, we obtain

$$\begin{aligned} \|P\| \geqslant (Pf)(t) &= f(t) - \sum_{i<N} \chi_{p_i}(t) + \sum_{i<N} h_{p_i}(t) = f(t) + \sum_{i<N} h_{p_i}(t) \\ &> -1 + \sum_{i<N} (1/3) = -1 + N/3 \geqslant \|P\|, \end{aligned}$$

which again is a contradiction. □

Recall that, given a compact space X and its closed subspace Y, a *regular extension operator* is a linear operator $T : C(Y) \to C(X)$ such that T is positive (i.e. $Tf \geqslant 0$ whenever $f \geqslant 0$), $T1 = 1$ and $(Tf) \restriction Y = f$ for every $f \in C(Y)$. Observe that in this case $\|T\| = 1$. The operator T provides an isometric embedding of $C(Y)$ into $C(X)$ such that the image is a 1-complemented subspace.

Lemma 4.2. *Assume X is a closed subset of a linearly ordered compact K. Then there exists a regular extension operator $T : C(X) \to C(K)$ such that* $\mathrm{E}(K, X) \subseteq TC(X)$.

Proof. For each $a \in \mathbb{L}(X)$ choose a continuous increasing function $h_a : [a^-, a] \to \mathbb{I}$ such that $h_a(a^-) = 0$ and $h_a(a) = 1$. Define

$$(Tf)(p) = \begin{cases} f(p) & \text{if } p \in X, \\ f(0_X) & \text{if } p < 0_X, \\ f(1_X) & \text{if } p > 1_X, \\ (1 - h_a(p))f(a^-) + h_a(p)f(a) & \text{if } p \in (a^-, a) \text{ for some } a \in \mathbb{L}(X). \end{cases}$$

It is straight to check that $Tf \in C(K)$ for every $f \in C(X)$. Further, $(Tf) \restriction X = f$, $T1 = 1$ and T is positive, therefore it is a regular extension operator. Note that Tf is constant both on $[0_K, 0_X]$ and $[1_X, 1_K]$. Finally, if $f \in \mathrm{E}(K, X)$ then $T(f \restriction X)$ is constant on each interval of the form $[a^-, a]$ where $a \in \mathbb{L}(X) \setminus \mathbb{L}(K)$, therefore $f = T(f \restriction X)$. This shows that $\mathrm{E}(K, X)$ is contained in the range of T. □

The above lemma implies that $C(K)$ has the separable complementation property, whenever K is a linearly ordered $\aleph_0$-monolithic compact space.

5. The space K_{ω_1}

Let $\langle K, \leqslant \rangle$ be a linearly ordered compact space. Define the following relation on K:

$$x \sim y \iff [x, y] \text{ is scattered.}$$

It is clear that $\sim$ is an equivalence relation and its equivalence classes are closed and convex, therefore $K/{\sim}$ is a linearly ordered compact space, endowed with the quotient topology and with the quotient ordering (i.e. $[x]_\sim \leqslant [y]_\sim \iff x \leqslant y$). In case where K is dense-in-itself, the $\sim$-equivalence classes are at most two-element sets. Let $q : K \to K/{\sim}$ denote the quotient map. We call q the *connectification* of K. In fact, $K/{\sim}$ is a connected space, because if $[x]_\sim < [y]_\sim$, then setting $a = \max[x]_\sim$ and $b = \min[y]_\sim$, we have that $a < b$ and $a \not\sim b$, therefore $(a, b) \neq \emptyset$ and $[x]_\sim < [z]_\sim < [y]_\sim$ for any $z \in (a, b)$.

In [7], a linearly ordered Valdivia compact space V_{ω_1} has been constructed, which has an increasing map onto every linearly ordered Valdivia compact. The space V_{ω_1} is 0-dimensional, dense-in-itself and has weight $\aleph_1$. Moreover, every clopen interval of V_{ω_1} is order isomorphic to V_{ω_1}. In particular, every nonempty open subset of V_{ω_1} contains both an increasing and a decreasing copy of ω_1.

Theorem 5.1. *Let $K_{\omega_1} = V_{\omega_1}/{\sim}$, where $q : V_{\omega_1} \to K_{\omega_1}$ is the connectification of V_{ω_1}. Then*

(a) *K_{ω_1} is a connected linearly ordered compact of weight $\aleph_1$.*
(b) *K_{ω_1} is a two-to-one increasing image of a linearly ordered Valdivia compact.*
(c) *Every separable subspace of $C(K_{\omega_1})$ is contained in a separable 1-complemented subspace.*
(d) *$C(K_{\omega_1})$ does not have a skeleton of complemented subspaces; in particular it is not a Plichko space.*

Proof. Clearly, K_{ω_1} satisfies (a) and (b). For the proof of (c), fix a separable space $E_0 \subseteq C(K_{\omega_1})$. By Proposition 3.4, there exists a closed separable subspace X of K_{ω_1} such that $E_0 \subseteq \mathrm{E}(K_{\omega_1}, X)$. By Lemma 4.2, $\mathrm{E}(K_{\omega_1}, X)$ is contained in a 1-complemented subspace of $C(K_{\omega_1})$, isometric to $C(X)$. By Proposition 3.5, K_{ω_1} is $\aleph_0$-monolithic, therefore X is second countable. This shows (c).

Suppose now that $C(K_{\omega_1})$ has a skeleton $\mathcal{F}$ consisting of complemented subspaces. By Proposition 3.3 and by Lemma 2.1, there exists an increasing surjection $h : K_{\omega_1} \to L$ such that L is metrizable and $C(L) \in \mathcal{F}$, when identified with a subspace of $C(K_{\omega_1})$ via h. Then L is order isomorphic to the unit interval $\mathbb{I}$, being a connected separable linearly ordered compact. The set

$$A = \{x \in L \colon 0_L < x < 1_L \text{ and } |h^{-1}(x)| > 1\}$$

is dense in L, because every non-degenerate interval of K_{ω_1} contains a copy of ω_1. By Lemma 4.1, $C(L)$ is not complemented in $C(K_{\omega_1})$. This shows (d) and completes the proof. □

6. Constructing compatible projections

In this section we show that a Banach space of density $\aleph_1$ is 1-Plichko if (and only if) it has a skeleton consisting of 1-complemented subspaces.

Lemma 6.1. *Assume E is a Banach space which is the union of a continuous chain $\{E_\alpha\}_{\alpha<\kappa}$ of closed subspaces such that for every $\alpha<\kappa$, E_α is 1-complemented in $E_{\alpha+1}$. Then there exist projections $S_\alpha : E \to E$, $\alpha<\kappa$, such that $\|S_\alpha\|=1$, $S_\alpha E = E_\alpha$ and $S_\alpha S_\beta = S_\alpha = S_\beta S_\alpha$ whenever $\alpha \leqslant \beta < \kappa$.*

Proof. We construct inductively projections $\{S_\alpha^\beta\colon \alpha \leqslant \beta \leqslant \kappa\}$ with the following properties:

(a) $S_\alpha^\beta : E_\beta \to E_\alpha$ has norm 1 for every $\alpha \leqslant \beta$,
(b) $S_\alpha^\beta S_\beta^\gamma = S_\alpha^\gamma$ whenever $\alpha \leqslant \beta \leqslant \gamma$.

We start with $S_0^0 = \mathrm{id}_{E_0}$. Fix $0<\delta\leqslant\kappa$ and assume S_α^β have been constructed for each $\alpha \leqslant \beta < \delta$ and they satisfy conditions (a), (b). There are two cases:

Case 1: $\delta = \gamma+1$. Using the assumption, fix a projection $T : E_{\gamma+1} \to E_\gamma$ with $\|T\|=1$ and define $S_\alpha^\delta = S_\alpha^\gamma T$ for every $\alpha<\delta$. Clearly both (a) and (b) are satisfied.

Case 2: δ is a limit ordinal. Let $G=\bigcup_{\alpha<\delta} E_\alpha$. Then G is a dense linear subspace of E_δ. Define $h_\alpha : G \to E_\alpha$ by setting

$$h_\alpha(x) = S_\alpha^\beta x, \quad \text{where } \beta = \min\bigl\{\xi \in [\alpha,\delta)\colon x \in E_\xi\bigr\}.$$

Note that if $\alpha<\beta$, $x\in E_\beta$ and $\gamma\in[\beta,\delta)$ then

$$S_\alpha^\gamma x = S_\alpha^\beta S_\beta^\gamma x = S_\alpha^\beta x, \tag{$*$}$$

because of (b) and by the fact that $S_\beta^\gamma \restriction E_\beta = \mathrm{id}_{E_\beta}$. Using ($*$), it is easy to see that h_α is a linear operator. Clearly $\|h_\alpha\|=1$, thus it can be uniquely extended to a linear operator $S_\alpha^\delta : E_\delta \to E_\alpha$. Finally, $\|S_\alpha^\delta\| = \|h_\alpha\| = 1$ and S_α^δ is a projection onto E_α. Thus (a) holds.

It remains to show (b). Fix $\alpha<\beta<\delta$. By continuity, it suffices to check that $S_\alpha^\delta x = S_\alpha^\beta S_\beta^\delta x$ holds for every $x\in G$. Fix $x\in G$ and find $\gamma\in[\beta,\delta)$ such that $x\in E_\gamma$. We have

$$S_\alpha^\beta S_\beta^\delta x = S_\alpha^\beta S_\beta^\gamma x = S_\alpha^\gamma x = S_\alpha^\delta x.$$

Thus both conditions (a) and (b) hold. It follows that the construction can be carried out.

Finally, define $S_\alpha := S_\alpha^\kappa$. Clearly, S_α is a projection of E onto E_α and $\|S_\alpha\|=1$. If $\alpha<\beta<\kappa$ then

$$S_\alpha S_\beta = S_\alpha^\kappa S_\beta^\kappa = S_\alpha^\beta S_\beta^\kappa S_\beta^\kappa = S_\alpha^\beta S_\beta^\kappa = S_\alpha^\kappa = S_\alpha$$

and of course $S_\beta S_\alpha = S_\alpha$, because $E_\alpha \subseteq E_\beta$. This completes the proof. □

Corollary 6.2. *Assume E is a Banach space with a skeleton $\{E_\alpha\}_{\alpha<\omega_1}$ such that E_α is 1-complemented in $E_{\alpha+1}$ for every $\alpha<\omega_1$. Then E is a 1-Plichko space.*

The following application of Lemma 6.1 provides a partial positive answer to a question of Kalenda [6, Question 4.5.10].

Theorem 6.3. *Assume E is a 1-Plichko Banach space of density $\aleph_1$. Then every 1-complemented subspace of E is 1-Plichko.*

Proof. Let $\{P_\alpha\colon \alpha<\omega_1\}$ be a PRI on E and let $Q : E\to E$ be a projection with $\|Q\|=1$. Let $F := QE$ and assume F is not separable. Define $E_\alpha = P_\alpha E$, $F_\alpha = F\cap E_\alpha$ and $R_\alpha = QP_\alpha$. Note that $R_\alpha E = Q[E_\alpha]$. We claim that the set $S=\{\delta<\omega_1\colon Q[E_\delta]=F_\delta\}$ is closed and unbounded in ω_1. Indeed, define

$$\varphi(\alpha) = \min\bigl\{\beta<\omega_1\colon Q[E_\alpha]\subseteq E_\beta\bigr\}$$

and observe that φ is well defined, since each $Q[E_\alpha]$ is separable and $E = \bigcup_{\xi<\omega_1} E_\xi$. Now, if $\delta < \omega_1$ is such that $\varphi(\alpha) < \delta$ whenever $\alpha < \delta$, then

$$Q[E_\delta] = Q\left[\operatorname{cl}\left(\bigcup_{\alpha<\delta} E_\alpha\right)\right] \subseteq \operatorname{cl}\bigcup_{\alpha<\delta} Q[E_\alpha] \subseteq \bigcup_{\alpha<\delta} E_{\varphi(\alpha)} \subseteq E_\delta,$$

therefore $\delta \in S$. Thus S is unbounded in ω_1. Without loss of generality we may assume that $S = \omega_1$.

Now observe that $R_\delta \restriction F$ is a projection of F onto F_δ and therefore F_δ is 1-complemented in F. Thus in particular $F = \bigcup_{\xi<\omega_1} F_\xi$, where $\{F_\xi\}_{\xi<\omega_1}$ is a chain of separable subspaces of F satisfying conditions (1), (2) of Lemma 6.1. By this lemma we get a PRI on F, which shows that F is 1-Plichko. □

Notice that in the above proof we did not use any compatibility between the projections P_α (which are assumed in the definition of a PRI). Thus, a slight modification gives the following

Corollary 6.4. *Assume F is a complemented subspace of a Banach space E of density $\aleph_1$. If E has a skeleton consisting of complemented subspaces then so does F.*

7. Final remarks and questions

By Theorem 5.1, the following two properties of a Banach space E of density $\aleph_1$ turn out to be different:

C_k: E is the union of an increasing sequence of separable k-complemented subspaces,
CC_k: E has a skeleton of separable k-complemented subspaces.

Property C_k is equivalent to the fact that every separable subspace of E is contained in a separable k-complemented subspace. Every k-Plichko space has property CC_k and property CC_1 is equivalent to the fact that E is 1-Plichko (Corollary 6.2). If E satisfies CC_k and F is an l-complemented subspace of E then F satisfies CC_{kl}, by the arguments from the proof of Theorem 6.3. We do not know whether CC_k implies k-Plichko, in case where $k > 1$. We also do not know whether a closed subspace of a Plichko space necessarily has the separable complementation property. Finally, we do not know whether a 1-complemented subspace of a Banach space with property C_1 necessarily has the separable complementation property.

Acknowledgements

The author would like to thank the anonymous referee for suggesting several improvements, especially in the proof of Proposition 3.4.

References

[1] A.V. Arkhangel'skiĭ, Topological Function Spaces, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic, Dordrecht, 1992. Translated from the Russian by R.A.M. Hoksbergen.
[2] H.H. Corson, The weak topology of a Banach space, Trans. Amer. Math. Soc. 101 (1961) 1–15.
[3] M. Fabian, Gâteaux Differentiability of Convex Functions and Topology: Weak Asplund Spaces, Wiley–Interscience, New York, 1997.
[4] R. Haydon, J.E. Jayne, I. Namioka, C.A. Rogers, Continuous functions on totally ordered spaces that are compact in their order topologies, J. Funct. Anal. 178 (2000) 23–63.
[5] O. Kalenda, Embedding of the ordinal segment $[0, \omega_1]$ into continuous images of Valdivia compacta, Comment. Math. Univ. Carolin. 40 (1999) 777–783.
[6] O. Kalenda, Valdivia compact spaces in topology and Banach space theory, Extracta Math. 15 (1) (2000) 1–85.
[7] W. Kubiś, Compact spaces generated by retractions, Topology Appl. 153 (18) (2006) 3383–3396. Preprint available at: http://arxiv.org/abs/math.GN/0511567.
[8] W. Kubiś, H. Michalewski, Small Valdivia compact spaces, Topology Appl. 153 (2006) 2560–2573. Preprint available at: http://arxiv.org/abs/math.GN/0507062.
[9] L.B. Nakhmanson, On the tightness of $L_p(X)$ of a linearly ordered compact space X, in: Investigations in the Theory of Approximations, vol. 123, Ural. Gos. Univ., Sverdlovsk, 1988, pp. 71–74 (in Russian).