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Some Stackelberg Type Location Game

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Abstract—This paper considers a Stackelberg type location game over the unit square $[0,1]\times[0,1]$. There are two chain stores, Players I and II, which sell the same kind of articles. Each store is planning to open a branch in region $[0,1]$. The purpose of each store is to decide the location to open its branch. In such a situation, the demand points, i.e., the customers, distribute continuously over $[0,1]$ in accordance with *cdf* $G(\cdot)$. Each customer wants to buy at a closer store between them, but never moves more than a distance ℓ. We also assume that Player I is forced to behave as the leader of this game and the opponent (Player II) is to be the follower. It is shown that there are various types of Stackelberg equilibriums according to the conditions of $G(\cdot)$ and ℓ.

Keywords—Nonzero sum game, Location, Infinite game, Stackelberg equilibrium.

1. INTRODUCTION

This paper considers a Stackelberg type location game over unit square $[0,1]\times[0,1]$. The model is described as follows.

There are two chain stores which deal in the same kind of articles. Each of the two stores is planning to open a branch at some region where there are no such stores now. The region takes the shape of a line segment, so that we represent it as the unit interval $[0,1]$. The purpose of each store is to decide the location to open its branch in $[0,1]$. Though both stores duopolize this new market over $[0,1]$, it is natural to assume the possibility to open at the same time by both stores is negligible. Thus, one of the two stores is forced to behave as the leader of this game, while on the other hand the opponent is to be the follower. We have to consider a Stackelberg

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type location game. We call the leader Player I and the follower Player II, respectively. Under the above situations, the demand points, that is customers, distribute continuously over $[0,1]$ in accordance with *cdf* $G(\cdot)$. Each of the customers wants to buy at the closer store between them, but never moves more than a certain distance ℓ $(0 < \ell < 1)$. Each player has to decide the optimal location from the viewpoint of Stackelberg equilibrium.

Here, we summarize the assumptions and notations to show our model explicitly as follows.

(i) The customers (demand points) distribute continuously over the line segment $[0,1]$ in accordance with *cdf* $G(\cdot)$ which has its *pdf* $g(\cdot)$.
(ii) When a store locates at point $z \in [0,1]$, a customer who lives at point $t \in [0,1]$ goes to buy at this store with probability $u(t \mid z)$.
(iii) Each customer usually wants to buy at the closest store in $[0,1]$.

Under the assumptions mentioned above, Player I locates first his branch at point $x \in [0,1]$. Then Player II decides the location $y \in [0,1]$ of his branch after observing the location x of his opponent. That is, Player I is the leader and Player II is the follower in this game. Each player has to decide the location of his branch which maximizes to obtain the number of customers in the interval $[0,1]$ on the steady state from the viewpoint of Stackelberg equilibrium, at their planning stage. When both players locate their branches at the same position in $[0,1]$, Players I and II share the market between them with even ratio.

Related to this game, Hotelling first pointed out and considered the location problem from the viewpoint of stability in competition between two players in 1929 [1]. After that, much research extended his work from a game theoretical viewpoint (for example, [2–7]). Gabszewicz and Thisse [8] summarized an excellent survey in *Handbook of Game Theory*. But, they analyzed and considered Nash equilibrium for the location game but not Stackelberg type. Osumi *et al.* proposed and analyzed competitive facility location models but not exactly game theoretical [9,10].

2. GENERAL FORMULATION

Let $M_i(x,y)$ be the expected payoff to Player i $(i = 1,2)$ when Players I and II locate their branches at points x and y in $[0,1]$, respectively. We have

$$M_1(x,y) = \begin{cases} \int_0^{(x+y)/2} u(t \mid x)\, g(t)\, dt, & x < y, \\ \dfrac{1}{2} \int_0^1 u(t \mid x)\, g(t)\, dt, & x = y, \\ \int_{(x+y)/2}^1 u(t \mid x)\, g(t)\, dt, & x > y, \end{cases} \tag{1}$$

$$M_2(x,y) = \begin{cases} \int_0^{(x+y)/2} u(t \mid y)\, g(t)\, dt, & y < x, \\ \dfrac{1}{2} \int_0^1 u(t \mid y)\, g(t)\, dt, & y = x, \\ \int_{(x+y)/2}^1 u(t \mid y)\, g(t)\, dt, & y > x. \end{cases} \tag{2}$$

Here, we establish the pure strategy for each player. Since this game is a nonzero sum infinite game between the leader (Player I) and the follower (Player II), it is natural to define $x \in [0,1]$ as the pure strategy for Player I and $y(x) \in [0,1]$ as the pure strategy for Player II. And the purposes of two players have to decide $y^*(x)$ and x^* which satisfy the following two stage maximization process.

Since Player II is the follower of this game, he can maximize his playoff $M_2(x,y(x))$ by selecting strategy $y(x)$ after observing x of Player I. On the other hand, Player I is the leader and knows the payoff functions of both players $M_1(x,y)$, $M_2(x,y)$, and hence, he learns Player II's set of best responses $\{y^*(x) \mid M_2(y^*(x)) = \sup_y M_2(x,y)\}$ to any strategy x of Player I. Having this information he then maximizes his payoff by choosing x^* from condition $M_1(x^*) = \sup_x M_1(x,y^*(x))$. Thus, the situation $(x^*, y^*(x^*))$ is an equilibrium point which gives the Stackelberg equilibrium.

Before discussing our main problem, we examine the case where the customers do not select a store depending on its position, that is,

$$u(t \mid z) = u(t), \qquad \text{for all } z \in [0,1].$$

Then this game is a constant-sum game because of (1) and (2). Putting $V = \int_0^1 u(t)g(t)\,dt$, $K(z) = \int_0^z u(t)g(t)\,dt$, we obtain

$$M_1(x,y) = \begin{cases} K\left(\dfrac{x+y}{2}\right), & x<y, \\ \dfrac{1}{2}V, & x=y, \\ V - K\left(\dfrac{x+y}{2}\right), & x>y, \end{cases} \tag{3}$$

$$M_2(x,y) = \begin{cases} K\left(\dfrac{x+y}{2}\right), & y<x, \\ \dfrac{1}{2}V, & y=x, \\ V - K\left(\dfrac{x+y}{2}\right), & y>x. \end{cases} \tag{4}$$

Let t_0 be the unique root of equation $K(t) = (1/2)\,V$ in $[0,1]$. Then we obtain three kinds of payoff to Player II according to x of Player I as follows:

$$M_2(x,y \mid x<t_0) = \begin{cases} K\left(\dfrac{x+y}{2}\right) < K(x) < \dfrac{1}{2}V, & y<x<t_0, \\ \dfrac{1}{2}V, & y=x, \\ V - K\left(\dfrac{x+y}{2}\right) < V - K(x), & x<\min(y,t_0), \end{cases}$$

$$M_2(x,y \mid x=t_0) = \begin{cases} K\left(\dfrac{t_0+y}{2}\right) < K(t_0) < \dfrac{1}{2}V, & y<x=t_0, \\ \dfrac{1}{2}V, & y=x=t_0, \\ V - K\left(\dfrac{t_0+y}{2}\right) < V - K(t_0) = \dfrac{1}{2}V, & y>x=t_0, \end{cases}$$

and

$$M_2(x,y \mid x>t_0) = \begin{cases} K\left(\dfrac{x+y}{2}\right) < K(x) < \dfrac{1}{2}V, & y<x, \\ \dfrac{1}{2}V, & y=x, \\ V - K\left(\dfrac{x+y}{2}\right) < V - K(x) < \dfrac{1}{2}V, & y>x. \end{cases}$$

Hence, Player II should choose strategy $y^*(x)$ such that for sufficiently small $\epsilon > 0$

$$y^*(x) = \begin{cases} x+\epsilon, & \text{for } x<t_0, \\ t_0, & \text{for } x=t_0, \\ x-\epsilon, & \text{for } x>t_0. \end{cases}$$

Since Player I should maximize

$$M_1(x, y^*(x)) = \begin{cases} K\left(x + \frac{\epsilon}{2}\right) < \frac{1}{2} V, & x < t_0, \\ \frac{1}{2} V, & x = t_0, \\ K\left(x - \frac{\epsilon}{2}\right) < \frac{1}{2} V, & x > t_0, \end{cases}$$

he selects $x^* = t_0$ as the leader's optimal strategy. As a result Player II is forced to locate at $y^*(x^*) = t_0$ as his optimal strategy. By the way, (t_0, t_0) is the saddle point of $M_1(x, y)$ given by (3) and $M_2(x, y)$ given by (4). Thus, we have Proposition 1.

PROPOSITION 1. *Let t_0 be the unique root of equation $K(t) = (1/2) V$ in the interval $[0, 1]$. Then (t_0, t_0) is a pair of optimal strategies for constant-sum game (3) and (4), that is, Nash equilibrium point. Furthermore, (t_0, t_0) also gives Stackelberg equilibrium as a result for the leader-follower game on (3) and (4). The equilibrium values are given by*

$$M_1(t_0, t_0) = M_2(t_0, t_0) = \frac{1}{2} V.$$

3. MAIN PROBLEM

We consider our main problem in this section. Here, we assume that each customer goes to a store within distance ℓ, but never moves more than distance ℓ, that is,

$$u(t \mid z) = \begin{cases} 1, & z - \ell \le t \le z + \ell, \quad z \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Then we get

$$M_1(x, y) = \begin{cases} G\left(\min\left(\frac{x+y}{2}, x + \ell\right)\right) - G(\max(0, x - \ell)), & x < y, \\ \frac{1}{2}\{G(\min(1, x + \ell)) - G(\max(0, x - \ell))\}, & x = y, \\ G(\min(1, x + \ell)) - G\left(\max\left(\frac{x+y}{2}, x - \ell\right)\right), & x > y, \end{cases} \tag{5}$$

$$M_2(x, y) = \begin{cases} G\left(\min\left(\frac{x+y}{2}, y + \ell\right)\right) - G(\max(0, y - \ell)), & y < x, \\ \frac{1}{2}\{G(\min(1, y + \ell)) - G(\max(0, y - \ell))\}, & y = x, \\ G(\min(1, y + \ell)) - G\left(\max\left(\frac{x+y}{2}, y - \ell\right)\right), & y > x. \end{cases} \tag{6}$$

Let t_0 be the unique root of equation $G(t) = 1/2$ in the interval $[0, 1]$ and assume that

$$\ell < \min(t_0, 1 - t_0) < \frac{1}{2},$$

for the sake of argument. If not, our main problem is not different from one in the latter half of Section 2, essentially.

Now, we show the expected payoffs to Player II $M_2(y(x) \mid x)$, given that he has learned x of Player I, according to the case where $0 \le x \le \ell$, $\ell < x < 1-\ell$, and $1-\ell \le x \le 1$ as follows:

$$M_2(y(x) \mid 0 \le x \le \ell) = \begin{cases} G(x+\ell), & y < x-2\ell < x < \ell, \\ G\left(\dfrac{x+y}{2}\right), & x-2\ell \le y < x \le \ell, \\ \dfrac{1}{2}G(x+\ell), & 0 < y = x \le \ell, \\ G(y+\ell) - G\left(\dfrac{x+y}{2}\right), & x < y \le \ell, \\ G(y+\ell) - G\left(\dfrac{x+y}{2}\right), & x \le \ell < y \le 1-\ell, \quad y \le x+2\ell, \\ G(y+\ell) - G(y-\ell), & x \le \ell < y \le 1-\ell, \quad y > x+2\ell, \\ 1 - \dfrac{1}{2}G(x+\ell), & x \le \ell < 1-\ell < y, \quad y \le x+2\ell, \\ 1 - G(y+\ell), & x \le \ell < 1-\ell < y, \quad y > x+2\ell, \end{cases} \tag{7}$$

$$M_2(y(x) \mid \ell < x < 1-\ell) = \begin{cases} G(x+\ell), & y < y < \ell < x < 1-\ell, \quad y < x-2\ell, \\ G\left(\dfrac{x+y}{2}\right), & y < \ell < x < 1-\ell, \quad y \ge x-2\ell, \\ G(y+\ell) - G(y-\ell), & \ell \le y < x < 1-\ell, \quad y < x-2\ell, \\ G\left(\dfrac{x+y}{2}\right) - G(y-\ell), & \ell \le y < x < 1-\ell, \quad y \ge x-2\ell, \\ \dfrac{1}{2}\{G(x+\ell) + G(x-\ell)\}, & \ell \le y = x < 1-\ell, \\ G(y+\ell) - G\left(\dfrac{x+y}{2}\right), & \ell < x < y < 1-\ell, \quad y < x+2\ell, \\ G(y+\ell) - G(y-\ell), & \ell < x < y < 1-\ell, \quad y \ge x+2\ell, \\ 1 - G\left(\dfrac{x+y}{2}\right), & \ell < x < 1-\ell < y, \quad y < x+2\ell, \\ 1 - G(y-\ell), & \ell < x < 1-\ell < y, \quad y \ge x+2\ell, \end{cases} \tag{8}$$

and $M_2(y(x) \mid 1-\ell \le x \le 1)$ has a symmetrically similar form with $M_2(y(x) \mid 0 \le x \le \ell)$.

We shall derive the optimal strategy $y^*(x)$ which maximizes $M_2(y(x) \mid 0 \le x \le \ell)$ and $M_2(y(x) \mid \ell < x < 1-\ell)$ given by (7) and (8), respectively.

3.1. Case Where $g(z)$ Decreases with z

We examine the case where $g(z)$ is a decreasing function with respect to z here. $G(z+\ell) - G(z-\ell)$ has its maximum $G(2\ell)$ at $z = \ell$. When Player I selects x in $[0,\ell]$, Player II should choose $y(x) = x - \epsilon$ which gives him $G(x-\epsilon)$, or $y(x)$ which maximizes $G(y+\ell) - G((x+y)/2)$ for $y \in (x, 2\ell]$, where ϵ is a sufficiently small positive number. For $x \in (\ell, 2\ell]$, Player II should select $y(x) = \ell$ which gives the maximum value $G((x+\ell)/2)$ for $y \in (0,x)$, or $y(x)$ which maximizes $G(y+\ell) - G((x+y)/2)$ for $y \in (x, \min(1, x+2\ell)]$. If $x > 2\ell$, we get $y(x) = \ell$ immediately.

On the other hand, Player I knows Player II's consideration. Hence, he should select $x = \ell$, or x which maximizes $G(x+\ell) - G((x+\ell)/2)$ for $x \in (\ell, 1)$. These considerations lead us to the following procedure.

PROPOSITION 2. *We can lead an optimal Stackelberg strategy of the follower (Player II) for nonzero sum game (7) and (8) in case that $g(z)$ is a decreasing function with respect to z, according to the following procedure.*

(1) *Let y_1 be the value $y \in [0, x]$ that maximizes $G(\min(((x+y)/2), 2\ell))$, and let V_1^{II} be the maximum.*
(2) *Let y_2 be the value $y \in [x, \min(1, x+2\ell)]$ that maximizes $G(x+\ell) - G((x+y)/2)$, and let V_2^{II} be the maximum.*
(3) *An optimal strategy $y^*(x)$ of Player II given x is obtained from*

$$y^*(x) = \begin{cases} y_1, & \text{for } V_1^{\mathrm{II}} \geq V_2^{\mathrm{II}}, \\ y_2, & \text{for } V_1^{\mathrm{II}} < V_2^{\mathrm{II}}. \end{cases}$$

THEOREM 1. *Let t^* be the unique root of equation $G((t+\ell)/2) = (1/2)\,G(2\ell)$ in the interval $[0, \ell]$. Then an optimal Stackelberg strategy x^* of Player I is given by $x^* = t^*$. Thus, the resulting optimal Stackelberg strategy $y^*(t^*)$ of Player II is the value $y \in (t^*, 1)$ which maximizes $G(y+\ell) - G((y+t^*)/2)$. And the Stackelberg equilibrium values v_1^* for I and v_2^* for II are given by*

$$v_1^* = \frac{1}{2}\,G(2\ell); \qquad v_2^* = \max_{y>t^*}\left\{G(y+\ell) - G\left(\frac{y+t^*}{2}\right)\right\}.$$

3.2. Cases Where $-g(z)$ is Unimodal

We also consider the case where $-g(z)$ is a unimodal function with respect to z, that is, there exists $t_m \in (0,1)$ such that

$$g(z)\begin{cases} \text{decreases for } z \leq t_m, \\ \text{increases for } z > t_m. \end{cases}$$

For the sake of an interesting argument, we also suppose that

$$\ell < \frac{1}{2}\,\min(t_m, 1 - t_m)$$

is this case.

A similar argument as in Section 3.1 holds on interval $[0, t_m]$, and the symmetrical argument holds on interval $[t_m, 1]$ because of our assumptions mentioned above. Thus, we have a procedure described in Propositions 3 and 4.

PROPOSITION 3. *We can get an optimal Stackelberg strategy of the follower (Player II) for games (7) and (8) under the above assumptions, according to the following procedure.*

In Case 1, Player I selects his location x in $[0, t_m]$.

(1) *Let y_1 be the value $y \in [0, \ell]$ which maximizes $G(\min((x+y)/2, 2\ell))$, and let V_1^{II} be the maximum.*
(2) *Let y_2 be the value $y \in [x, \min(1, x+2\ell)]$ which maximizes $G(y+\ell) - G((x+y)/2)$, and let V_2^{II} be the maximum.*
(3) *Let y_3 be the value $y \in [\max(1-\ell, x), 1]$ which maximizes $1 - G(\max(1-2\ell, (x+y)/2))$, and let V_3^{II} be the maximum.*
(4) *Let j^* be the number j which gives $\max(V_1^{\mathrm{II}}, V_2^{\mathrm{II}}, V_3^{\mathrm{II}})$.*

Then, $y^*(x) = y_{j^*}$ is a candidate of optimal Stackelberg strategy for Player II given that $0 \leq x \leq t_m$.

In Case 2, Player I selects his location x in $[t_m, 1]$.

(1) *Let y_1' be the value $y \in [0, \min(\ell, x)]$ which maximizes $G(\min(2\ell, (x+y)/2))$, and let $V_1^{\mathrm{II}\prime}$ be the maximum.*

(2) *Let y'_2 be the value $y \in [\max(0, x-2\ell), x]$ which maximizes $G((x+y)/2) - G(y-\ell)$, and let $V_2^{\mathrm{II}\prime}$ be the maximum.*
(3) *Let y'_3 be the value $y \in [1-\ell, 1]$ which maximizes $1 - G(\max(1-2\ell, (x+y)/2))$, and let $V_3^{\mathrm{II}\prime}$ be the maximum.*
(4) *Let j^* be the number which gives $\max(V_1^{\mathrm{II}\prime}, V_2^{\mathrm{II}\prime}, V_3^{\mathrm{II}\prime})$.*

Then, $y^*(x) = y_{j^*}$ is a candidate of optimal Stackelberg strategy for Player II given that $t_m < x \leq 1$.

The proposition leads us to Theorem 2, easily.

THEOREM 2. *Let t_1^* and t_2^* be the unique root of equations $G((t+\ell)/2) = (1/2)\,G(2\ell)$ in $[0, \ell]$ and $1 - G(1-2\ell) = (1/2)\,G(1-(t+1-\ell)/2)$ in $[1-\ell, 1]$, respectively. Let y_1^*, y_2^*, y_3^*, and y_4^* be defined as follows:*

$$y_1^* = \arg\max_{y \in (t_1^*, \ell)} \left\{ G(y+\ell) - G\left(\frac{y+t_1^*}{2}\right) \right\},$$
$$y_2^* = \arg\max_{y \in (1-\ell, t_2^*)} \left\{ G\left(1 - \frac{y+t_2^*}{2}\right) - G(1-y-\ell) \right\},$$
$$y_3^* = \ell,$$

and

$$y_4^* = 1 - \ell.$$

And define $v(y_j^*)$, $j = 1, 2, 3, 4$, as follows:

$$v\left(y_1^*\right) = G\left(y_1^* + \ell\right) - G\left(\frac{y_1^* + t_1^*}{2}\right),$$
$$v\left(y_2^*\right) = G\left(1 - \frac{y_2^* + t_2^*}{2}\right) - G\left(1 - y_2^* - \ell\right),$$
$$v\left(y_3^*\right) = G\left(2y_3^*\right) - G(0) = G(2\ell),$$

and

$$v\left(y_4^*\right) = G(1) - G\left(y_4^* - \ell\right) = 1 - G(1-2\ell).$$

Finally, we denote y^* as the maximizer for $\{v(y_j^*) \mid j = 1, 2, 3, 4\}$.

Then a pair of strategies $(x^*, y^*(x^*))$ gives Stackelberg equilibrium as a result for the leader-follower game on (5) and (6), where x^* of Player I and $y^*(x^*)$ of Player II are given as follows:

$$x^* = \left\{ \begin{matrix} t_1^* \\ t_2^* \\ 1-\ell \\ \ell \end{matrix} \right\} \quad \text{and} \quad y^*\left(x^*\right) = \left\{ \begin{matrix} y_1^* \\ y_2^* \\ \ell \\ 1-\ell \end{matrix} \right\}, \qquad \text{if } y^* = \left\{ \begin{matrix} y_1^* \\ y_2^* \\ y_3^* \\ y_4^* \end{matrix} \right\}.$$

4. SIMPLE EXAMPLES

Here, we show simple examples on our main results given in the previous section.

First, we examine the case where $g(z) = 2(1-z)$ and $\ell < 1/2$. Since $G(z) = 2z - z^2$, we easily find

$$t^* = 2 - \ell - 2\sqrt{2\ell^2 - 2\ell + 1}$$

and $y^* = \ell$ is the value $y \in (t^*, 1)$ which maximizes $G(y+\ell) - G((x+t^*)/2)$, and we obtain a pair of optimal Stackelberg strategies $(x^*, y^*(x^*))$ given by

$$\left(x^*, y^*\left(x^*\right)\right) = \left(t^*, \ell\right).$$

The equilibrium values v_1^* for I and v_2^* for II are

$$v_1^* = v_2^* = G\left(\frac{t^*+\ell}{2}\right) = \frac{1}{2}\,G(2\ell).$$

Note: when $\ell = 0.25$, we get $t^* \approx 0.17$.

Next, we also examine the case where

$$g(x) = \frac{4}{3}\begin{cases} 1-z, & \text{for } z \le \dfrac{1}{2}, \\ z, & \text{for } z > \dfrac{1}{2}, \end{cases}$$

and

$$\ell < \frac{1}{4}.$$

We easily have that both of the pairs $(\ell, 1-\ell)$ and $(1-\ell, \ell)$ are Stackelberg equilibrium points. The equilibrium values are

$$v_1^* = v_2^* = G(2\ell).$$

5. CONCLUDING REMARKS

In this report, we formulated our problem under general forms of *cdf* $G(z)$ and selecting probability $u(t \mid z)$, however, analyzed only on two kinds of special types. As we observed in Sections 3 and 5, Stackelberg equilibrium points take on various kinds of aspects according to the types of $G(z)$ and $u(t \mid z)$. There remain a lot of problems to solve.

Finally, we remark that considering Stackelberg equilibrium is more realistic compared to the Nash equilibrium in such a location game.

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