Soft ideals and soft filters of soft ordered semigroups

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It is shown that soft ordered semigroups over an ordered semigroup are actually soft ordered subsemigroups of \((S, A)\). Soft idealistic semigroups over an ordered semigroup are soft ideals of soft ordered semigroup \((S, A)\). The concept of soft filters of a soft ordered semigroup is introduced. It is shown that restricted complement of a soft filter is a soft prime ideal of \((S, A)\).

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1. Introduction

The theory of soft sets is introduced by Molodtsov in [1]. The basic aim of this theory is to introduce a new tool to discuss uncertainty. In soft set theory, we have enough number of parameters to deal with uncertainty. This quality makes it prominent among its predecessor theories such as probability theory, interval mathematics, fuzzy sets and rough sets. Hybrid soft set theories such as fuzzy soft sets and vague soft sets have been studied in [2,3]. Theories of fuzzy sets and rough sets are quite different in their nature from soft sets, but there is a strong link among these three theories. During recent years, efforts have been made to establish links among these three theories [4,5].

Initially, Maji et al. established the theoretical base for soft sets. They defined several operations on soft sets [6]. Later on it was felt that some of the operations defined for soft sets suffer many difficulties. In order to make these operations sensible, Ali et al. defined some new operations on soft sets [7]. Algebras, which appear as a natural consequence of these new operations have been studied in [8].

The theory of soft sets has emerged as a new tool to discuss algebraic structures. Sometimes a soft algebraic structure behaves differently from the original algebraic structure. For example, a soft semigroup over a regular semigroup may not be regular; see [9]. This behavior of soft algebraic structures makes their study interesting. Study of soft algebraic structures was initiated by Aktas and Cagman. They studied soft groups in [10]. For application of soft sets in algebraic structures, see [9–13].

There is a classical book by Fuchs where ordered algebraic structures including groups and semigroups are discussed [14]. Ordered semigroups have been extensively by Kehayopulu, for example, see [15–17]. The concept of filters in ordered semigroups is introduced by Kehayopulu in [15]. Later on some properties of filters in ordered semigroups were studied in [18,19]. The theory of fuzzy sets was introduced by Zadeh [20]. Fuzzy ordered semigroups are studied in [21,17,22]. Kehayopulu and Tsingelis studied the concept of fuzzy filters in [17]. Study of soft ordered semigroups is initiated by Jun et al. [23]. They introduced the concepts of a soft ordered semigroup and soft ideals of the soft ordered semigroups.

In the present paper, first of all we examine some results which exist in the literature on soft ordered semigroups and point out difficulties associated with these results. Then these results are improved with the help of new operations. The
concept of a soft filter of a soft ordered semigroup is introduced. Finally, it is shown that restricted complement of a soft filter is a soft prime ideal of the soft ordered semigroup.

2. Preliminaries

An ordered semigroup (or po-semigroup) is defined to be an ordered set $S$ and the same time a semigroup such that

for all $a, b, x \in S (a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb$).

An ordered semigroup having a greatest element denoted by $T$, is called a poe-semigroup. If $A$ and $B$ are subsets of an ordered semigroup $S$, then $AB = \{ab : a \in A, b \in B\}$.

A nonempty subset $G$ of an ordered semigroup $(S, \leq)$ is called a subsemigroup of $S$ if $(G, \leq)$ is a semigroup, that is if $G^2 \subseteq G$.

A nonempty subset $I$ of an ordered semigroup $(S, \leq)$ is called a left (resp. right) ideal of $S$ denoted by $I \triangleleft S$ (resp. $I \triangleright S$), if it satisfies

$$SI \subseteq I \text{ (resp. IS} \subseteq I)$$

for all $a \in I, b \in S (b \leq a \Rightarrow b \in I)$.

It is mentioned here that intersection of two left (resp. right) ideals of $S$ is either empty or a left (resp. right) ideal of $S$. A subset $I$ of $S$ which is a left ideal and a right ideal of $S$ is called an ideal of $S$.

A subset $T$ of $S$ is called prime if $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$ for subsets $A, B$ of $S$. $T$ is called a prime left (right) ideal if $T$ is prime as a left (right) ideal. $T$ is called a prime ideal if $T$ is prime as an ideal.

A subsemigroup $F$ of an ordered semigroup $S$ is called a left (resp. right) filter of $S$, if

$$ab \in F \text{ for } a, b \in S \text{ implies } a \in F \text{ (resp. } b \in F),$$

$$a \in F \text{ and } a \leq b \text{ for } b \in S \text{ implies } b \in F.$$

A subsemigroup $F$ is called a filter of $S$ if $F$ is a left and a right filter.

Molodtsov defined soft sets in the following way.

**Definition 1** ([11]). Let $U$ be a universal set and let $E$ be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of $U$ and let $A \subseteq E$. A pair $(F, A)$ is called a soft set (over $U$) if $F$ is a mapping $F : A \rightarrow \mathcal{P}(U)$.

In other words, a soft set over $U$ is a parametrized family of subsets of the universe $U$. Let for $\epsilon \in A, F(\epsilon)$ may be considered as the set of $\epsilon$-approximate element in the soft set $(F, A)$. Clearly a soft set is not a set.

**Definition 2** ([6]). Let $(F, A)$ and $(G, B)$ be soft sets over $U$, then $(G, B)$ is called a soft subset of $(F, A)$ if $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in B$.

**Definition 3** ([6]). The intersection of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft set $(H, C)$, where $C = A \cap B$, and for all $\epsilon \in C, H(\epsilon) = F(\epsilon)$ or $G(\epsilon)$ (as both are same set). This intersection is denoted as $(F, A) \cap (G, B) = (H, C)$.

**Definition 4** ([6]). The union of two soft sets $(F, A)$ and $(G, B)$ over a common universe $U$, is defined as the soft set $(H, C)$, where $C = A \cup B$ and for all $c \in C$

$$H(c) = \begin{cases} F(c), & \text{if } c \in A \setminus B \\ G(c), & \text{if } c \in B \setminus A \\ F(c) \cup G(c), & \text{if } c \in A \cap B \end{cases}$$

this is denoted as $(F, A) \cup (G, B) = (H, C)$.

**Definition 5** ([6]). If $(F, A)$ and $(G, B)$ are soft sets over a common universe $U$. Then “$(F, A)$ AND $(G, B)$” denoted by $(F, A) \wedge (G, B)$, is defined as $(F, A) \wedge (G, B) = (H, A \times B)$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

**Definition 6** ([6]). If $(F, A)$ and $(G, B)$ are soft sets over a common universe $U$. Then “$(F, A)$ OR $(G, B)$” denoted by $(F, A) \vee (G, B)$, is defined as $(F, A) \vee (G, B) = (K, A \times B)$, where $K(\alpha, \beta) = F(\alpha) \cup G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

3. Some results on soft ordered semigroups

In this section, we critically examine some results about soft ordered semigroups which exist in the literature. In the next section, these results will be strengthened.

**Definition 7** ([23]). Let $(F, A)$ be a soft set over $S$. Then $(F, A)$ is called a soft ordered semigroup over $S$ if it satisfies

for all $x \in A(F(x) \neq \emptyset \Rightarrow F(x)$ is a subsemigroup of $S$).
Theorem 8 ([23]). Let $(F, A)$ and $(G, B)$ be soft ordered semigroups over $S$. If $A \cap B \neq \emptyset$, then any intersection $(F, A) \cap (G, B)$ is a soft ordered semigroup over $S$.

We mention here, the above theorem is not true in general. To show this we consider the following example.

Example 9. Let $S = \{a, b, c, d, e\}$ be a semigroup with following multiplication given by

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Then $S$ is an ordered semigroup. Let $(F, A)$ be a soft set over $S$, where $A = \{x, y, z\}$ is the set of parameters. If $(F, A)$ is a soft ordered semigroup over $S$, then $F(x) = \{a, b, d\}$, $F(y) = \{c, e\}$, $F(z) = S$. We define another soft set $(G, B)$ over $S$ such that $B = \{x, y\}$ is the parameter set and $G(x) = \{c, e\}$, $G(y) = \{a, b\}$. Clearly, $A \cap B \neq \emptyset$, $(F, A)$ and $(G, B)$ are soft ordered semigroups but $F(x) \neq G(x)$ and $F(y) \neq G(y)$, so $(F, A) \cap (G, B)$ turns out redundant.

Definition 10 ([23]). Let $(F, A)$ and $(G, B)$ be soft ordered semigroups over $S$. Then $(F, A)$ is called a soft ordered subsemigroup of $(G, B)$, denoted by $(F, A) \subseteq (G, B)$ if it satisfies

i. $A \subseteq B$,
ii. $F(x)$ is a subsemigroup of $G(x)$ for all $x \in A$.

Theorem 11 ([23]). Let $(F, A)$ be a soft ordered semigroup over $S$ and let $(G_1, B_1)$ and $(G_2, B_2)$ be soft ordered subsemigroups of $(F, A).$ Then we have

1. $(G_1, B_1) \cap (G_2, B_2) \subseteq (F, A)$.
2. If $B_1 \cap B_2 = \emptyset$, then $(G_1, B_1) \cap (G_2, B_2) \subseteq (F, A)$.

We mention here that assertion 1 of the above theorem is not true in general. To show this we have the following.

Example 12. Consider the ordered semigroup of Example 9. Define a soft set $(F, A)$ over $S$, where $A = \{a, b, c, d\}$ and $F(a) = F(b) = F(c) = F(d) = S.$ Then $(F, A)$ is a soft ordered semigroup over $S.$ Let $(G_1, B_1)$ and $(G_2, B_2)$ be two soft subsets of $(F, A)$ where $B_1 = \{a, b\}, B_2 = \{c, d\}, G_1(a) = \{a, b, d\}, G_1(b) = \{c, e\}$ and $G_2(b) = \{a, b, d\}, G_2(c) = S.$ Then both $(G_1, B_1)$ and $(G_2, B_2)$ are soft ordered subsemigroups of $(F, A).$ Also $B_1 \cap B_2 \neq \emptyset$ but $G_1(b) \neq G_2(b)$ so $(G_1, B_1) \cap (G_2, B_2)$ is not possible.

Definition 13 ([23]). Let $(F, A)$ be a soft ordered semigroup over $S$. A soft set $(G, I)$ over $S$ is called

1. a soft $l$-ideal of $(F, A)$, denoted by $(G, I) \triangleleft (F, A)$, if it satisfies
   i. $I \subseteq A$,
   ii. $\forall x \in I, G(x) \triangleleft F(x)$.
2. a soft $r$-ideal of $(F, A)$, denoted by $(G, I) \triangleright (F, A)$, if it satisfies
   i. $I \subseteq A$,
   ii. $\forall x \in I, G(x) \triangleright F(x)$.

If $(G, I)$ is both a soft $l$-ideal and a soft $r$-ideal, then we say that $(G, I)$ is a soft ideal of $(F, A)$ and it is denoted by $(G, I) \bowtie (F, A)$.

Theorem 14 ([23]). Let $(F, A)$ be a soft ordered semigroup over $S$. For any soft sets $(G_1, I_1)$ and $(G_2, I_2)$ over $S$, where $I_1 \cap I_2 \neq \emptyset$ we have

1. If $(G_1, I_1) \bowtie (F, A)$ and $(G_2, I_2) \bowtie (F, A)$, then $(G_1, I_1) \bowtie (G_2, I_2) \bowtie (F, A)$.
2. If $(G_1, I_1) \bowtie (F, A)$ and $(G_2, I_2) \bowtie (F, A)$, then $(G_1, I_1) \bowtie (G_2, I_2) \bowtie (F, A)$.

The following example shows that in general above theorem does not hold.

Example 15. Consider the ordered semigroup $S$ of Example 9. Let $(F, A)$ be a soft set over $S$, where $A = \{a, b, c, d\}$ and $F(a) = \{a, b, d\}, F(b) = \{c, e\}, F(c) = S, F(d) = S$. Then $(F, A)$ is a soft ordered semigroup over $S$. Now let $(G_1, I_1)$ and $(G_2, I_2)$ be two soft sets over $S$, where $I_1 = I_2 = \{a, d\}$ and $G_1(a) = \{a\}, G_1(d) = \{d\}$ and $G_2(a) = \{d\}$ and $G_2(d) = \{a\}$. Then clearly $(G_1, I_1) \bowtie (F, A)$ and $(G_2, I_2) \bowtie (F, A)$, but $(G_1, I_1) \bowtie (G_2, I_2)$ cannot be obtained because $G_1(a) \neq G_2(a)$ and $G_1(d) \neq G_2(d)$. Thus assertion 1, of the above theorem is not true in general. Similarly, it can be shown that assertion 2, of the above is also not true.
Definition 16 ([23]). Let \((F, A)\) be a soft set over \(S\). Then \((F, A)\) is called an \(l\)-idealistic (resp. \(r\)-idealistic) soft ordered semigroup over \(S\) if \(F(x)\) is a left (resp. right) ideal of \(S\) for all \(x \in A\).

Theorem 17 ([23]). Let \((F, A)\) and \((G, B)\) be \(l\)-idealistic (resp. \(r\)-idealistic) soft ordered semigroups over \(S\). If \(A \cap B \neq \emptyset\), then any intersection \((F, A) \cap (G, B)\) is an \(l\)-idealistic (resp. \(r\)-idealistic) soft ordered semigroup over \(S\).

With the help of the following example, we show that the above theorem suffers difficulties.

Example 18. Consider the ordered semigroup \(S = \{a, b, c, d, e\} \) with multiplication table given by

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and the partial order

\[ \leq: \{ (a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e) \} \]

Let \((F, A)\) and \((G, B)\) be two soft sets defined over \(S\), where \(A = B = \{a, b, c\}\) such that \(F(a) = \{a, b\}, F(b) = \{a, b, c\}, F(c) = \{a, b, c\}\) and \(G(a) = \{a, b\}, G(b) = \{a, b, c\}, G(c) = \{a\}\). Then clearly \((F, A)\) and \((G, B)\) are \(l\)-idealistic soft ordered semigroups over \(S\) but \((F, A) \cap (G, B)\) cannot be computed.

4. Application of new operations in soft ordered semigroups

We have seen in the previous section that the concept of intersection in soft sets given by Maji et al. suffers many problems. Therefore, in order to remove these problems Ali et al. defined some new operations in soft set theory. In our view, results given by Jun et al. in [23] suffer due to the definition of intersection of soft sets given by Maji et al. in [6].

Note first that if \((F, A)\) and \((G, B)\) are two different soft sets, then it is not necessary for these two soft sets to have the same subset of \(U\) for a particular common parameter say \(c \in A \cap B\), i.e., \(F(c) \neq G(c)\) in general. Hence the intersection of two soft sets as defined in [6] may only be a partial operation. That is to say, not any two soft sets result in a new soft set when we calculate such an intersection operation on them.

On the contrary, if this kind of intersection must be regarded as a binary operation, then we can deduce from the definition that any two soft sets over a common universe must have the same approximation value-set for any common parameter, but this is surely not the case.

In set theory, there is a unique empty set and a whole set but this is not the case in soft sets. In order to make this situation clear we have the following.

Definition 19 ([7]). Let \(U\) be a universal set, \(E\) be the universal set of parameters, and \(A \subseteq E\).
(a) \((F, A)\) is called a relative null soft set (with respect to the parameter set \(A\), denoted by \(\emptyset_A\), if \(F(e) = \emptyset\) for all \(e \in A\).
(b) \((G, E)\) is called a relative whole soft set (with respect to the parameter set \(E\)), denoted by \(\emptyset_E\), if \(G(e) = U\) for all \(e \in A\).

The relative whole soft set with respect to the universal set of parameters \(E\) is called the absolute soft set over \(U\) and simply denoted by \(\emptyset_U\). In a similar way, the relative null soft set with respect to \(E\) is called the null soft set over \(U\) and simply denoted by \(\emptyset_E\).

We shall denote by \(\emptyset_A\) the unique soft set over \(U\) with an empty parameter set, which is called the empty soft set over \(U\). Note that \(\emptyset_B\) and \(\emptyset_A\) are different soft sets over \(U\) and \(\emptyset_B \subseteq (F, A) \subseteq \emptyset_A \subseteq \emptyset_U\) for all soft sets \((F, A)\) over \(U\).

In the sequel, for a soft set \((F, A)\), if \((F, A) \neq \emptyset_A\), then it is called normal. If \((F, A) \neq \emptyset_B\) we shall say \((F, A)\) is non-trivial.

As in this paper, we are dealing only with ordered semigroups, so throughout \(S\) will denote an ordered semigroup and \((S, A)\) a soft set over \(S\) such that \(S(\alpha) = S\) for all \(\alpha \in A\).

A general criteria has been suggested in [5] to define a binary operation on soft sets as follows.

Definition 20 ([5]). Suppose that \(\oplus_E : P(E) \times P(E) \to P(E)\) is a binary operation on \(P(E)\) and \(\oplus_U : P(U) \times P(U) \to P(U)\) is a binary operation on \(P(U)\). Then for any two soft sets \((F, A)\) and \((G, B)\) over \(U(F, A) \oplus (G, B)\) is defined as the soft set \((H, C)\) where \(C = A \oplus E\) and \(H(x) = F(x) \oplus_U G(x)\) for all \(x \in C\).

Definition 21 ([7]). The extended intersection of two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), is the soft set \((H, C)\), where \(C = A \cup B\), and for all \(c \in C\),

\[ H(c) = \begin{cases} F(c), & \text{if } c \in A \setminus B \\ G(c), & \text{if } c \in B \setminus A \\ F(c) \cap G(c), & \text{if } c \in A \cap B. \end{cases} \]

We write \((F, A) \cap_E (G, B) = (H, C)\).
In addition, we may sometimes adopt a different definition of intersection given as follows.

**Definition 22** ([7]). Let \( (F, A) \) and \( (G, B) \) be two soft sets over a common universe \( U \) such that \( A \cap B \neq \emptyset \). The restricted intersection of \( (F, A) \) and \( (G, B) \) is denoted by \( (F, A) \cap_{\mathbb{R}} (G, B) \), and is defined as \( (F, A) \cap_{\mathbb{R}} (G, B) = (H, C) \), where \( C = A \cap B \neq \emptyset \) and for all \( c \in C \), \( H(c) = F(c) \cap G(c) \). If \( A \cap B = \emptyset \), then \( (H, C) = \emptyset_{\emptyset} \).

**Definition 23.** Let \( (F, A) \) and \( (G, B) \) be two soft sets over the same universe \( U \). The restricted difference of \( (F, A) \) and \( (G, B) \) is denoted by \( (F, A) \setminus_{\mathbb{R}} (G, B) \), and is defined as \( (F, A) \setminus_{\mathbb{R}} (G, B) = (H, C) \), where \( C = A \cap B \neq \emptyset \) and for all \( c \in C \), \( H(c) = F(c) \setminus G(c) \). If \( A \cap B = \emptyset \), then \( (H, C) = \emptyset_{\emptyset} \).

**Definition 24.** The restricted complement of a soft set \( (F, A) \) is denoted by \( (F, A)^{\prime} \) and is defined by \( (F, A)^{\prime} = (F^{\prime}, A) \) where \( F^{\prime} : A \rightarrow P(U) \) is a mapping given by \( F^{\prime}(a) = U \setminus F(a) \) for all \( a \in A \).

Clearly, \( (F, A)^{\prime} = \mathcal{U}_{A} \setminus_{\mathbb{R}} (F, A) \) and \( ((F, A)^{\prime})^{\prime} = (F, A) \).

**Definition 25.** The restricted product \( (H, C) \) of two soft sets \( (F, A) \) and \( (G, B) \) over an ordered semigroup \( S \) is defined as the soft set \( (H, C) = (F, A) \tilde{\otimes} (G, B) \) where \( C = A \cap B \neq \emptyset \) and \( H \) is a set valued function from \( C \) to \( P(S) \) defined as \( H(c) = F(c)G(c) \) for all \( c \in C \). If \( A \cap B = \emptyset \), then \( (H, C) = \emptyset_{\emptyset} \).

In the following, we define soft ordered semigroups over an ordered semigroup in another equivalent way as suggested in [23].

**Definition 26.** A normal and non-trivial soft set \( (F, A) \) over an ordered semigroup \( S \) is called a soft ordered semigroup over \( S \) if \( (F, A) \tilde{\otimes} (F, A) \subseteq (F, A) \).

**Lemma 27.** A normal and non-trivial soft set \( (F, A) \) over an ordered semigroup \( S \) is a soft ordered semigroup if and only if \( F(a) \neq \emptyset \) is an ordered subsemigroup of \( S \) for all \( a \in A \).

**Proof.** Straightforward. \( \square \)

Previously, a soft ordered semigroup over an ordered semigroup and soft ordered subsemigroup of a soft ordered semigroup were considered as different soft structures. In the following proposition, we see that there is a certain relation between these two.

**Proposition 28.** A soft set \( (F, A) \) over an ordered semigroup \( S \) is a soft ordered semigroup over \( S \) if and only if \( (F, A) \) is a soft ordered subsemigroup of \( (S, A) \).

**Theorem 29.** Let \( (F, A) \) and \( (G, B) \) be two soft ordered semigroups over \( S \). Then \( (F, A) \cap_{\mathbb{R}} (G, B) \) is also a soft ordered semigroup over \( S \), whenever \( (F, A) \cap_{\mathbb{R}} (G, B) \) is normal and non trivial.

**Proof.** We have \( (F, A) \cap_{\mathbb{R}} (G, B) = (H, C) \) where \( C = A \cap B \neq \emptyset \) and \( H(c) = F(c) \cap G(c) \) for all \( c \in C \). This is clearly either empty or a subsemigroup of \( S \). Consequently, \( (H, C) \) is a soft ordered semigroup over \( S \), whenever normal and non trivial. \( \square \)

**Theorem 30.** Let \( (F, A) \) and \( (G, B) \) be two soft ordered semigroups over \( S \). Then \( (F, A) \cap_{\mathbb{R}} (G, B) \) is also a soft ordered semigroup over \( S \), whenever \( (F, A) \cap_{\mathbb{R}} (G, B) \) is non trivial.

**Proof.** As we know that \( (F, A) \cap_{\mathbb{R}} (G, B) = (H, C) \) where \( C = A \cup B \) and

\[
H(c) = \begin{cases} 
F(c) & \text{if } c \in A \setminus B \\
G(c) & \text{if } c \in B \setminus A \\
F(c) \cap G(c) & \text{if } c \in A \cap B.
\end{cases}
\]

If \( c \in A \setminus B \) then \( H(c) = F(c) \) if \( c \in B \setminus A \) then \( H(c) = G(c) \) in both the cases \( H(c) \neq \emptyset \) is an ordered subsemigroup of \( S \). Now if \( c \in A \cap B \) then \( H(c) = F(c) \cap G(c) \) if \( H(c) \neq \emptyset \) then it is an ordered subsemigroup of \( S \), because both \( F(c) \) and \( G(c) \) are ordered subsemigroups of \( S \). Hence \( (H, C) \) is a soft ordered subsemigroup over \( S \) whenever non trivial. \( \square \)

**Theorem 31.** Let \( (F, A) \) and \( (G, B) \) be two soft ordered semigroups over \( S \). Then \( (F, A) \tilde{\otimes} (G, B) \) is also a soft ordered semigroup over \( S \), whenever \( (F, A) \tilde{\otimes} (G, B) \) is non trivial.

**Proof.** Since \( (F, A) \tilde{\otimes} (G, B) = (H, A \times B) \), and \( H(a, b) = F(a) \cap G(b) \). As \( F(a) \) and \( G(b) \) are ordered subsemigroup of \( S \), therefore either \( F(a) \cap G(b) = \emptyset \) or \( F(a) \cap G(b) \) is an ordered subsemigroup of \( S \). Consequently, \( (H, A \times B) \) is a soft ordered semigroup of \( S \), whenever non trivial. \( \square \)

**Theorem 32.** Let \( (F, A) \) be a soft ordered semigroup over \( S \) and let \( (G_1, B_1) \) and \( (G_2, B_2) \) be soft ordered subsemigroups of \( (F, A) \) then \( (G_1, B_1) \cap_{\mathbb{R}} (G_2, B_2) \tilde{\leq} (F, A) \), whenever \( (G_1, B_1) \cap_{\mathbb{R}} (G_2, B_2) \) is normal and non trivial.
Proof. We have \((G_1, B_1) \cap \mathcal{R}(G_2, B_2) = (H, C)\) where \(C = B_1 \cap B_2 \neq \emptyset\) and \(H(c) = G_1(c) \cap G_2(c)\) for all \(c \in C\). This is clearly either empty or an ordered subsemigroup of \(F(c)\). Consequently, \((H, C)\) is a soft ordered subsemigroup of \((F, A)\), whenever normal and non-trivial. \(\Box\)

**Theorem 33.** Let \((F, A)\) be a soft ordered semigroup over \(S\) and let \((G_1, B_1)\) and \((G_2, B_2)\) be soft ordered subsemigroups of \((F, A)\) then \((G_1, B_1) \cap \mathcal{R}(G_2, B_2) \subseteq (F, A)\), whenever \((F, A) \cap \mathcal{R}(G, B)\) is non-trivial.

**Proof.** As we know that \((G_1, B_1) \cap \mathcal{R}(G_2, B_2) = (H, C)\) where \(C = A \cup B\) and

\[
H(c) = \begin{cases} 
G_1(c) & \text{if } c \in A \setminus B \\
G_2(c) & \text{if } c \in B \setminus A \\
G_1(c) \cap G_2(c) & \text{if } c \in A \cap B.
\end{cases}
\]

If \(c \in A \setminus B\) then \(H(c) = G_1(c)\) if \(c \in B \setminus A\) then \(H(c) = G_2(c)\) in both the cases \(H(c) \neq \emptyset\) is an ordered subsemigroup of \(G(c)\). Now if \(c \in A \cap B\) then \(H(c) = G_1(c) \cap G_2(c)\) if \(H(c) \neq \emptyset\) then it is an ordered subsemigroup of \(S\), because both \(G_1(c)\) and \(G_2(c)\) are ordered subsemigroups of \(G(c)\). \(\Box\)

**Definition 34.** Let \(S\) be an ordered semigroup. A soft set \((\leq, A)\) over \(S \times S\) is called a soft partial order over \(S\) if \(\leq (a) \neq \emptyset\) (may be denoted by \(\leq_a\)) is a partial order on \(S\) for all \(a \in A\).

We like to mention here that if \((F, A)\) is a soft set over an ordered semigroup \(S\) then \(F(a) \neq \emptyset\) inherits a partial order for all \(a \in A\). Hence there is an associated soft partial order with each soft set over an ordered semigroup.

**Definition 35.** Let \((F, A)\) be a soft ordered semigroup over an ordered semigroup \(S\). A soft subset \((G, I)\) of \((F, A)\) is called a soft \(l\)-ideal (\(r\)-ideal) of \((F, A)\) if

1. \((F, A)\a\mathcal{O}(G, I) \subseteq (G, I)\) \((G, I)\a\mathcal{O}(F, A) \subseteq (G, I)\).
2. There is an associated soft partial order with \((G, I)\) denoted by \((\leq, I)\) such that for all \(a \in G(i), b \in F(i)\) \((b \leq_i a \Rightarrow b \in G(i))\) for all \(i \in I\).

**Theorem 36.** Let \((F, A)\) be a soft ordered semigroup over an ordered semigroup \(S\). Then a normal and non-trivial soft subset \((G, I)\) of \((F, A)\) is a soft \(l\)-ideal of \((F, A)\) if and only if \(G(i) \neq \emptyset\) is an \(l\)-ideal of \(F(i)\) for all \(i \in I\).

**Proof.** Suppose that \((G, I)\) is a soft subset of the soft ordered semigroup \((F, A)\) over \(S\) such that it satisfies both conditions of the **Definition 35** then

\[(F, A)\a\mathcal{O}(G, I) \subseteq (G, I).\]

Now \((F, A)\a\mathcal{O}(G, I) = (H, C)\) where \(C = A \cap I = I\) because \(I \subseteq A\) and \(H(i) = F(i)G(i)\). Since \((F, A)\a\mathcal{O}(G, I) \subseteq (G, I)\) therefore for every \(G(i) \neq \emptyset\), we have \(F(i)G(i) \subseteq G(i)\). Also for all \(a \in G(i), b \in F(i)(b \leq_i a \Rightarrow b \in G(i))\) for all \(i \in I\). That is \(G(i) \leq F(i)\) for all \(i \in I\).

Conversely, let each \(G(i) \neq \emptyset\) is an \(l\)-ideal of \(F(i)\) for all \(i \in I\). Then by definition \(F(i)G(i) \subseteq G(i)\) for all \(i \in I\). This implies

\[(F, A)\a\mathcal{O}(G, I) \subseteq (G, I).\]

Also there is an associated soft partial order \((\leq, I)\) with \((G, I)\) such that for all \(a \in G(i), b \in F(i)\) \((b \leq_i a \Rightarrow b \in G(i))\) for all \(i \in I\). Hence \((G, I)\) is a soft \(l\)-ideal of \((F, A)\). \(\Box\)

The above Theorem can also be proved for soft \(r\)-ideals of \((F, A)\).

**Theorem 37.** Let \((F, A)\) be a soft ordered semigroup over \(S\). For any soft sets \((G_1, I_1)\) and \((G_2, I_2)\) over \(S\) we have

1. If \((G_1, I_1)\a\mathcal{O}(F, A)\) and \((G_2, I_2)\a\mathcal{O}(F, A)\), then \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\a\mathcal{O}(F, A)\), whenever \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\) is normal and non-trivial.
2. If \((G_1, I_1)\a\mathcal{O}(F, A)\) and \((G_2, I_2)\a\mathcal{O}(F, A)\), then \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\a\mathcal{O}(F, A)\), whenever \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\) is normal and non-trivial.

**Proof.** (1) Since \((G_1, I_1) \cap \mathcal{R}(G_2, I_2) = (H, C)\) where \(C = I_1 \cap I_2 \neq \emptyset\) and \(H(c) = G_1(c) \cap G_2(c)\), is either empty or an ideal of \(S\). Consequently \((H, C)\) is a soft ideal over \(S\), whenever normal and non-trivial.

(2) Proof is similar to (1). \(\Box\)

**Theorem 38.** Let \((F, A)\) be a soft ordered semigroup over \(S\). For any soft sets \((G_1, I_1)\) and \((G_2, I_2)\) over \(S\) we have

1. If \((G_1, I_1)\a\mathcal{O}(F, A)\) and \((G_2, I_2)\a\mathcal{O}(F, A)\), then \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\a\mathcal{O}(F, A)\), whenever \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\) is non-trivial.
2. If \((G_1, I_1)\a\mathcal{O}(F, A)\) and \((G_2, I_2)\a\mathcal{O}(F, A)\), then \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\a\mathcal{O}(F, A)\), whenever \((G_1, I_1) \cap \mathcal{R}(G_2, I_2)\) is non-trivial.

**Proof.** Straightforward. \(\Box\)

We can define soft \(l\)-idealistic (\(r\)-idealistic) soft ordered semigroups over \(S\) in another equivalent way as given in [23].
Definition 39. A normal and non trivial soft set \((F, A)\) over a semigroup \(S\) is called an \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\), if

1. \((S, A)\) is a \((F, A)\) over a semigroup \(S\) is called an \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\), if
2. \((F, A)\) over a semigroup \(S\) is called an \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\), if

We may also call the above defined \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\) soft ordered \(l\)-ideal \((r\)-ideal) over the soft ordered semigroup \(S\).

Proposition 40. A normal and non trivial soft set \((F, A)\) over an ordered semigroup \(S\), is an \(l\)-idealistic \((r\)-idealistic) soft ordered semigroup over \(S\) if and only if \(F(a)\) is an ordered \(l\)-ideal \((r\)-ideal) of \(S\), for all \(a \in A\).

Proof. Straightforward. \(\square\)

Soft idealistic ordered semigroup and soft ideal of a soft ordered semigroup have been treated separately [23]. In the following proposition, we see the relationship between these two soft structures.

Proposition 41. A soft set \((F, A)\) over an ordered semigroup is soft idealistic if and only if \((F, A)\) is a soft ideal over \((S, A)\).

Theorem 42. Let \((F, A)\) and \((G, B)\) be \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\). Then \((F, A)\) and \((G, B)\) be \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\), whenever \((F, A)\) and \((G, B)\) is normal and non trivial.

Proof. We have \((F, A)\) and \((G, B)\) be \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\). Then \((F, A)\) and \((G, B)\) is normal and non trivial.

Theorem 43. Let \((F, A)\) and \((G, B)\) be \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\). Then \((F, A)\) and \((G, B)\) is normal and non trivial.

Proof. We have \((F, A)\) and \((G, B)\) be \(l\)-idealistic \((r\)-idealistic) soft ordered semigroups over \(S\). Then \((F, A)\) and \((G, B)\) is normal and non trivial.

5. Soft filters of a soft ordered semigroup

In this section, the concept of soft filters of a soft ordered semigroup is initiated. Main result of this paper is presented in this section.

Definition 45. Let \((F, A)\) be a soft ordered semigroup over an ordered semigroup \(S\). Then \((F, A)\) is called a soft filter over \(S\) if \(F(\alpha)\) is normal and non trivial.

Definition 46. Let \((F, A)\) be a soft ordered semigroup over an ordered semigroup \(S\). A soft subset \((G, B)\) of \((F, A)\) is called a soft filter of \((F, A)\) if \(G(\beta)\) is normal and non trivial.

Proposition 47. A soft set \((F, A)\) over an ordered semigroup is a soft filter over \(S\) if and only if \((F, A)\) is a soft filter of \((S, A)\).

Proof. Straightforward. \(\square\)

Theorem 48. If \((G, A)\) is a soft subset of a soft ordered semigroup over an ordered semigroup \(S\). Then the following are equivalent.

1. \((G, A)\) is a soft filter of \((F, A)\),
2. \((F, A)\) is a soft prime ideal of \((G, A)\).

Proof. (1) \(\Rightarrow\) (2) Let \((F, A)\) be a soft filter of \((F, A)\), then \(H(\alpha) = F(\alpha) \setminus G(\alpha)\) for all \(\alpha \in A\). We have to show that \(H(\alpha)\) is a prime ideal of \(F(\alpha)\) for all \(\alpha \in A\). If possible let \(x \in F(\alpha) \setminus G(\alpha)\) and \(y \in F(\alpha)\) such that \(xy \notin F(\alpha) \setminus G(\alpha)\), then \(xy \in G(\alpha)\). Since \(G(\alpha)\) is a filter, \(x \in G(\alpha)\). This is a contradiction; hence \(xy \notin F(\alpha) \setminus G(\alpha)\). Similarly, it can be shown that \(yx \notin F(\alpha) \setminus G(\alpha)\). That is to say \(H(\alpha) = F(\alpha) \setminus G(\alpha)\) is an ideal of \(F(\alpha)\) for all \(\alpha \in A\). Therefore \((G, A)\) is a soft ideal of \((F, A)\). Now if possible let there exist some \(x, y \in F(\alpha)\) such that \(xy \notin H(\alpha)\) and \(y \notin H(\alpha)\) this implies \(xy \notin F(\alpha) \setminus G(\alpha)\) and \(y \notin F(\alpha) \setminus G(\alpha)\), then \(x \in \alpha \in A\) and \(y \in \alpha \in A\). Since \(G(\alpha)\) is a filter, \(xy \in G(\alpha)\). This is a
contradiction, therefore either \(x \in F(\alpha) \setminus G(\alpha)\) or \(y \in F(\alpha) \setminus G(\alpha)\). This shows that \(H(\alpha) \neq \emptyset\) is a prime ideal of \(F(\alpha)\) for all \(\alpha \in A\). Hence \((H, A)\) is a soft prime ideal of \((F, A)\).

(2) \(\Rightarrow\) (1) If \((F, A) \sim_{\mathcal{R}} (G, A) = \emptyset_A\), then \(F(\alpha) \setminus G(\alpha) = \emptyset\) for all \(\alpha \in A\), this implies \(F(\alpha) = G(\alpha)\). Since each \(F(\alpha)\) is an ordered subsemigroup of \(F(A)\), \(F(\alpha)\) is a filter of \(F(\alpha)\). Now assume that \((F, A) \sim_{\mathcal{R}} (G, A) = (H, A)\) is a soft prime ideal of \((F, A)\). Suppose there exist some \(x, y \in G(\alpha) \neq \emptyset\) such that \(xy \notin G(\alpha)\), this implies \(xy \in F(\alpha) \setminus G(\alpha)\). Since \(F(\alpha) \setminus G(\alpha)\) is a prime ideal of \(F(\alpha)\), \(x \in F(\alpha) \setminus G(\alpha)\) or \(y \in F(\alpha) \setminus G(\alpha)\). This implies \(x \notin G(\alpha)\) or \(y \notin G(\alpha)\), which is a contradiction. Therefore for all \(x, y \in G(\alpha) \neq \emptyset\) we have \(xy \in G(\alpha)\). This shows that \(G(\alpha)\) is a subsemigroup of \(F(\alpha)\) for all \(\alpha \in A\). That is \((G, A)\) is a soft subsemigroup of \((F, A)\).

Now if possible let \(xy \in G(\alpha)\), where \(x, y \in F(\alpha)\) and \(x \notin G(\alpha)\) and \(y \notin G(\alpha)\). This implies \(x \in F(\alpha) \setminus G(\alpha)\) and \(y \in F(\alpha) \setminus G(\alpha)\). As \(F(\alpha) \setminus G(\alpha)\) is an ideal, \(xy \in F(\alpha) \setminus G(\alpha)\), which is a contradiction. Therefore \(x \in G(\alpha)\) and \(y \in G(\alpha)\). Now let \(x \in G(\alpha)\) and \(y \in F(\alpha)\) such that \(x \leq y\). If possible let \(y \notin G(\alpha)\), this implies \(y \in F(\alpha) \setminus G(\alpha)\). Since \(F(\alpha) \setminus G(\alpha)\) is an ideal of \(F(\alpha)\), \(x \in F(\alpha) \setminus G(\alpha)\). This is a contradiction, so \(y \in G(\alpha)\). That is \((G, A) \neq \emptyset\) is a filter of \(F(\alpha)\) for all \(\alpha \in A\). Hence \((G, A)\) is a filter of \((F, A)\).

**Corollary 49.** Let \((F, A)\) be a normal and non trivial soft set over an ordered semigroup \(S\). Then \((F, A)\) is a soft filter over \(S\) if and only if \((F, A) \uparrow = \emptyset_A\) or \((F, A) \downarrow\) is a soft prime ideal of \((S, A)\).

### 6. Conclusion

Due to various applications of ordered structures, especially ordered semigroups, they are one among the important topics of research in algebra. Wide ranging applications of ordered semigroups in computer science, coding theory and automata make them an important area of study. Soft ordered semigroups are a relatively new area of investigation. Soft ideals of a soft ordered semigroup are studied here. The concept of soft filters of soft ordered semigroups is defined and some of their properties are studied.

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### References