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# THE EFFECT OF TRUNCATION AND ROUND-OFF ON COMPUTER SIMULATED CHAOTIC TRAJECTORIES

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**Abstract**—In this short report I demonstrate that due to truncation and round-off, computer generated trajectories of chaotic dynamical systems eventually will become periodic.

Many systems of differential equations have been found to be chaotic. When a system is chaotic a trajectory in its state space (depicting the evolution of the system from some initial condition) never repeats or crosses itself. If that were to happen the system would return to a state it was at some time in its past and then it would have to follow the same path. As a result an aperiodic but deterministic evolution is obtained. The trajectory, however, may at some time in the future come very, very close to a state of the past. If those two states differ at the $n$th decimal point then a computer that truncates or rounds-off at the $(n-1)$th decimal point will or may make those two points equivalent.

The above can be illustrated by considering the logistic equation $x_{n+1} = 4x_n(1 - x_n)$ [1,2]. In this form the logistic equation is chaotic, i.e., aperiodic. Let us now assume that we desire to generate a long time series and that we are dealing with a computer which in its calculations carries only two decimal points. Let us then start with the initial condition $x_0 = 0.121134\ldots$ The computer will read this value as $x_0 = 0.12$ and will produce a value of $x_1 = 0.42$ (in fact the exact value would be $x_1 = 0.4224$ but it is truncated to 0.42). If we continue this iteration process we find that $x_2 = 0.97$, $x_3 = 0.11$, $x_4 = 0.39$, $x_5 = 0.95$, $x_6 = 0.19$, $x_7 = 0.61$, $x_8 = 0.95$, $x_9 = 0.19$, $x_{10} = 0.61$, $x_{11} = 0.95$, and so on. The evolution has become periodic of period three after eight time steps. We can repeat this experiment many times (each time starting with a different initial condition) and find the mean number of time steps before the evolution becomes periodic. We then can proceed with other truncation scenarios and obtain a graph like Figure 1 which shows $\log N$, the logarithm of the number of time steps before the chaotic evolution becomes periodic, as a function of $n$, the number of digits after the decimal points that are considered in the calculations.

As can be seen from Figure 1 the evolution sooner or later will become periodic. Approximately one can consider that $N$ varies as $N \propto 10^{\frac{n}{2}}$. Thus with a computer which truncates at the 16th decimal point a chaotic evolution will become periodic after $10^8$ time steps.

Similar results are obtained when we are considering round-off instead of truncation. Starting again with the initial condition of $x_0 = 0.121134\ldots$ and assuming that our computer rounds-off at the second decimal point, we obtain that $x_1 = 0.42$, $x_2 = 0.97$, $x_3 = 0.12$, $x_4 = 0.42$, $x_5 = 0.97$, and so on. The evolution has become again periodic of period three after (in this case) four time steps. In general, it is found that round-off "induced" periodicity is achieved somewhat faster than truncation "induced" periodicity. The reason for this is that apparently rounding-off causes very close states to become equivalent more often than truncation does. Take for example two states represented by the values 0.1212436... and 0.1211733... Rounding-off at the fourth decimal point will make both these values equal to 0.1212 while truncation will make them equal to 0.1212 and 0.1211, respectively.

The results reported here provide some confidence in computer generated chaotic trajectories for low dimensional systems, since in such cases we do not usually require more than $10^8$ data

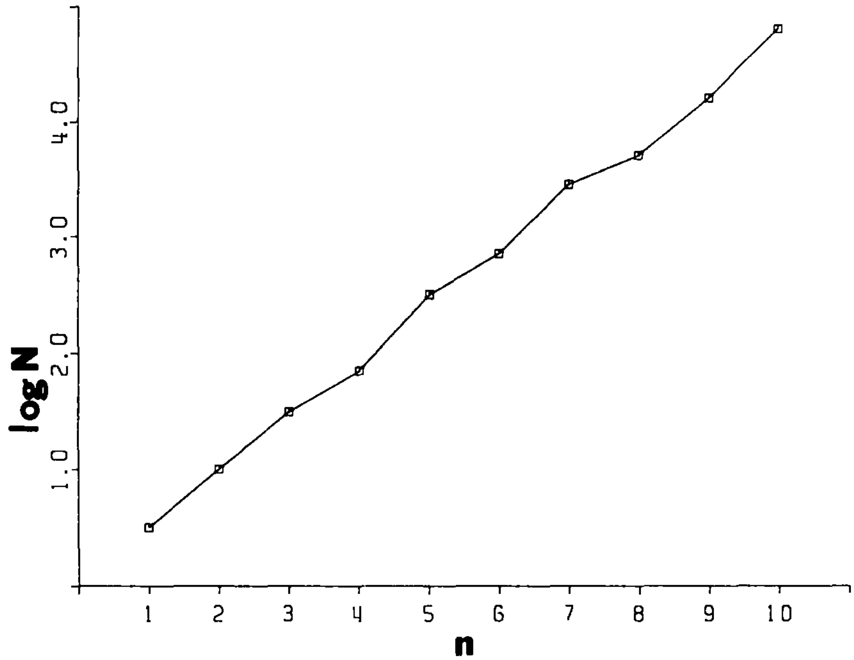


Figure 1. Truncation will cause any chaotic sequence to become periodic after some time steps. In this figure the logarithm of the average number of time steps ($\log N$) before a chaotic sequence from the logistic equation becomes periodic is plotted against the number of digits after the decimal point that are carried in the calculations ($n$).

points. At the same time the results point out the ultimate limitations in computer simulations of chaotic evolutions, which might become important when we begin to deal with high dimensional systems. I view the results reported here as complimentary to the results reported by Yorke [3] on the shadowing property. According to his results when computer calculations of chaotic trajectories are accurate to a certain number of digits, the generated trajectories will be correlated to the true trajectories (a property coined shadowing) for some time only. Beyond that time the computer trajectories do not represent true trajectories. Here we show that what causes shadowing to fail will also cause a chaotic trajectory to become periodic.

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