Well-graded spaces of valued sets

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Abstract

Well-graded spaces of valued sets and relations are introduced and their properties are investigated. In particular, it is shown that the space of valued partial orders on a finite set is well-graded. This is a generalization of a well-known result of Bogart (J. Math. Soc. 3 (1973) 49). Motivation for these studies comes from media theory (Falmagne, J. Math. Psych. 41 (2) (1997) 129; Discrete Appl. Math., submitted) where well-graded families of usual sets play an important role. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The concept of a well-graded family of subsets of a finite set was introduced by Doignon and Falmagne in [4,9] in connection with their studies in the area of ‘stochastic evolution of preference structures’ [7,9]. (It should be noted that Bogart [3] used this concept in the special case of partial orders as early as in 1973.) The following definition uses the standard (Hamming) distance \(d(A,B) = |A \Delta B|\) between subsets \(A\) and \(B\) of a finite set \(X\) (\(A \Delta B\) stands for the symmetric difference between sets \(A\) and \(B\)).

**Definition 1.1.** A nonempty family \(\mathcal{M}\) of subsets of a finite set \(X\) is well-graded if, for any two distinct subsets \(P\) and \(Q\) in \(\mathcal{M}\), there exists a sequence of subsets \(P = R_0, \ldots, R_k = Q\) in \(\mathcal{M}\) such that \(d(P,Q) = k\) and \(d(R_i, R_{i+1}) = 1\) for \(i = 0, \ldots, k - 1\).

Kuz’min and Ovchinnikov [13] and Ovchinnikov [15] use an equivalent ‘completeness property’ in their ‘geometric approach’ to group choice. In a more general setting, Ovchinnikov [14] introduces the completeness property in connection with ‘convexity in subsets of lattices’. (Equivalence of the completeness and well-gradedness properties follows from our Theorem 5.1.)

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The concept of well-gradedness is also quite useful in other areas including, for instance, the theory of knowledge spaces [5]. One particularly important application is in Falmagne’s media theory [8,10]. Well-graded families of sets are common examples of media. Moreover, Ovchinnikov and Dukhovny [16] show that any medium can be represented as a well-graded family of subsets of a finite set.

In this paper we present a theory of well-graded spaces of valued sets and relations. A valued set is a function from a given set $X$ into a given linearly ordered set $L$. (Technical definitions will be given in the next section. In the particular case when $L = [0,1]$, valued sets are called fuzzy sets [19]. A comprehensive study of valued relations and their application is found, for instance, in [11].) Motivation for extending the concept of well-gradedness to the case of valued sets comes from the media theory. Our preliminary results [6] show that even the simple medium of all valued sets on a given set $X$ gives rise to quite interesting stochastic properties.

We call families of valued sets ‘spaces’ and use concepts of ‘betweenness’, ‘interval’, and ‘line segment’ to build a theory of well-graded spaces of valued sets. This ‘geometric’ approach has its roots in the works of Kemeny and Snell [12], Barbut and Monjardet [1] and Bogart [3] and was used in our studies in the area of group choice [15].

The paper is organized as follows.

After introducing some basic notations and conventions in Section 2, we develop a general theory of spaces of valued sets in Section 3.

The definition of a well-graded space is given in Section 4 together with some examples. It is also shown in this section that, in the case of usual (classical, crisp, non-valued) sets, a well-graded family of sets in the sense of Definition 1.1 is a special case of a well-graded space.

In Section 5, we consider a special case of spaces of finite valued sets. We show that in this case well-graded spaces can be modeled as well-graded families of usual sets.

Unlike in the case of well-graded families of usual sets, we do not use the distance function as a primitive notion in our constructions. Rather we develop a theory based on the concepts of betweenness and line segment. The metric structure on well-graded spaces appears naturally as a consequence of rather weak assumptions about the distance function. We introduce this metric structure in Section 6.

One particular goal of this paper is to show that the space of all valued partial orders on a finite set is well-graded. We establish this result in Section 8 as a consequence of a more general result obtained in Section 7.

2. Preliminaries

We denote $X$ a given finite set and $L$ a given linearly ordered set with universal bounds $O$ and $I$. In other words, we assume that $L$ has the least element $O$ and the greatest element $I$. As usual, symbols $\lor$ and $\land$ denote sup and inf operations,
respectively, and \( L^X \) denotes the lattice of all functions from \( X \) to \( L \). Operations \( \lor \) and \( \land \) on \( L^X \) are defined pointwise by
\[
(A \lor B)(x) = A(x) \lor B(x), \quad (A \land B)(x) = A(x) \land B(x)
\]
for all \( x \in X \) and \( A, B \in L^X \).

We use letters \( x, y, z, \ldots \) to represent elements of \( X \) and letters \( \alpha, \beta, \gamma, \ldots \) to represent elements of \( L \). Upper-case letters \( A, B, C, \ldots \) denote elements of the lattice \( L^X \) and letters \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \) denote subsets of \( L^X \).

We assume that there is a positive valuation \( v \) on \( L \), i.e. a function \( v : L \to \mathbb{R} \) such that
\[
\alpha < \beta \iff v(\alpha) < v(\beta)
\]
for all \( \alpha, \beta \in L \). It is well-known (Birkhoff–Milgram’s theorem [17]) that such a function \( v \) exists if and only if \( L \) has a countable order-dense subset. Function \( v \) defines a distance function \( d_L(\alpha, \beta) = |v(\alpha) - v(\beta)| \) on \( L \). We shall use \( d_L(\alpha, \beta) = |\alpha - \beta| \) when \( L \) is a subset of the set of real numbers (including the cases when \( L = n = \{0, 1, \ldots, n-1\} \) and \( L = [0,1] \)).

By using the distance function \( d_L(\alpha, \beta) \), we define the distance function \( d_H(P, Q) \) on \( L^X \) by
\[
 d_H(P, Q) = \sum_{x \in X} d_L(P(x), Q(x)).
\]
This distance function is the Hamming distance in the case of \( L = 2 \).

**Definition 2.1.** A valued set \( A \) on \( X \) is a function \( A : X \to L \).

The set of all valued sets on \( X \) is the lattice \( L^X \). Note that \( L^X \) is a distributive but not necessarily complete lattice. Typical examples of valued sets include the usual sets \( (L = 2 = \{0,1\}) \) and fuzzy sets \( (L = [0,1]) \). We shall also use as an example the case when \( L = n = \{0,1,\ldots,n-1\} \).

We adhere to the usual definition of the union and intersection of valued sets [11]. Given two valued sets \( A \) and \( B \), we say that a valued set \( C \) is the union (resp. intersection) of \( A \) and \( B \), if \( C = A \lor B \) (resp. \( C = A \land B \)), and that \( A \) is a subset of \( B \) if \( A \leq B \) in \( L^X \).

In the paper, we shall not distinguish between usual subsets of \( X \) and their characteristic (indicator) functions. In other words, we shall identify (and use the same symbol for both) a subset \( A \subseteq X \) with the valued set \( A \) defined by
\[
 A(x) = \begin{cases} 
 I & \text{if } x \in A, \\
 O & \text{otherwise.}
 \end{cases}
\]

An \( \alpha \)-level set \( A_\alpha \) of a valued set \( A \in L^X \) is a subset of \( X \) defined by
\[
 A_\alpha = \{x \in X : A(x) \geq \alpha\},
\]
where \( \alpha \in L \setminus \{O\} \), the set of positive elements in \( L \) (cf. [19]). The corresponding characteristic function is the valued set given by

\[
A_\alpha(x) = \begin{cases} 
1 & \text{if } A(x) \geq \alpha, \\
O & \text{otherwise}.
\end{cases}
\]

Clearly, the family \( \{A_\alpha\}_{\alpha \in L \setminus \{O\}} \) is a nested family of subsets of \( X \):

\[ \alpha \leq \beta \Rightarrow A_\alpha \supseteq A_\beta. \]

The next proposition follows immediately from our definitions (cf. [20]).

**Proposition 2.1.** For any \( A \in L^X \), \( \bigvee_{\alpha \in L \setminus \{O\}} \alpha \wedge A_\alpha(x) \) exists and

\[
A(x) = \bigvee_{\alpha \in L \setminus \{O\}} \alpha \wedge A_\alpha(x)
\]

for all \( x \in X \).

Since \( X \) is a finite set, (1) can be written in the form

\[
A(x) = \bigvee_{i=1}^k \alpha_i \wedge A^{(i)}(x),
\]

where \( \alpha_1 < \alpha_2 < \cdots < \alpha_k \) are all distinct positive values (\( \alpha_i > O \)) of \( A \) and \( A^{(i)} = A_{\alpha_i} \) for \( 1 \leq i \leq k \).

Conversely, one can use (2) to construct valued sets on \( X \). Namely, let \( \alpha_1 < \alpha_2 < \cdots < \alpha_k \) be positive elements of \( L \) and let \( A^{(1)} \supseteq A^{(2)} \supseteq \cdots \supseteq A^{(k)} \) be subsets of \( X \). Then (2) defines a valued set \( A \) on \( X \). The proofs of the following two propositions are straightforward and omitted.

**Proposition 2.2.** Let \( A \) be a valued set defined by

\[
A(x) = \bigvee_{i=1}^k \alpha_i \wedge A^{(i)}(x)
\]

for all \( x \in X \), where \( O = \alpha_0 < \alpha_1 < \cdots < \alpha_k \) and \( A^{(1)} \supseteq \cdots \supseteq A^{(k)} \). Then level sets \( A_\alpha \) of \( A \) are given by

\[
A_\alpha = \begin{cases} 
A^{(j)} & \text{if } \alpha_{j-1} < \alpha \leq \alpha_j, \\
\emptyset & \text{if } \alpha > \alpha_k.
\end{cases}
\]

**Proposition 2.3.** For any three valued sets \( A, B, C \in L^X \):

(i) \( A \leq B \) if and only if \( A_\alpha \subseteq B_\alpha \) for all \( \alpha \in L \setminus \{O\} \).
(ii) \( A \wedge B)_\alpha = A_\alpha \cap B_\alpha \), for all \( \alpha \in L \setminus \{O\} \).
(iii) \( A \vee B)_\alpha = A_\alpha \cup B_\alpha \), for all \( \alpha \in L \setminus \{O\} \).
(iv) \( A \wedge B \leq C \leq A \vee B \) if and only if \( A_\alpha \cap B_\alpha \subseteq C_\alpha \subseteq A_\alpha \cup B_\alpha \), for all \( \alpha \in L \setminus \{O\} \).
3. Spaces of valued sets

**Definition 3.1.** A space of valued sets is an arbitrary nonempty subset $\mathcal{M}$ of $L^X$. Elements of $\mathcal{M}$ are called points in $\mathcal{M}$.

In what follows, we introduce some ‘geometric’ concepts in a space $\mathcal{M}$ of valued sets. Our approach is motivated by the studies found, in particular, in [1,3,12,14,15].

**Definition 3.2.** Let $P$ and $Q$ be two points in $\mathcal{M}$. An interval with end points $P$ and $Q$ is a subset $[P;Q]$ of $\mathcal{M}$ defined by

$$[P;Q] = \{ R \in \mathcal{M}; P \wedge Q \leq R \vee P \cup Q \}.$$ 

A point $R$ lies between $P$ and $Q$ if and only if

$$R \in [P;Q].$$

Thus, by definition, an interval $[P;Q]$ in $\mathcal{M}$ is the set of all points in $\mathcal{M}$ that lie between the end points $P$ and $Q$. It is possible that $[P;Q]=[S;T]$ for two distinct sets of end points. A simple example is given by $\mathcal{M}=2^X$ where $X=\{a,b\}$. Clearly, $[\emptyset,X]=[\{a\},\{b\}]$. On the other hand, $[P;Q]=[Q;P]$ in any space $\mathcal{M}$.

The following two lemmas establish some important technical properties of intervals.

**Lemma 3.1.** For any $P,Q,R,S,T \in \mathcal{M}$,

$$S,T \in [P,Q] \text{ if and only if } [S,T] \subseteq [P,Q].$$

**Proof.** Clearly

$$P \wedge Q \leq S \leq P \vee Q, \quad P \wedge Q \leq T \leq P \vee Q, \quad \text{and} \quad S \wedge T \leq R \leq S \vee T$$

imply

$$P \wedge Q \leq S \wedge T \leq R \leq S \vee T \leq P \vee Q. \quad \square$$

**Lemma 3.2.** For any $P,Q,S,T \in \mathcal{M}$,

$$S \in [P,T], \quad R \in [S,T] \text{ if and only if } R \in [P,T], \quad S \in [P,R].$$

**Proof.** Suppose $S \in [P,T]$ and $R \in [S,T]$. By Lemma 3.1, $R \in [P,T]$.

Since

$$P \wedge T \leq S \leq P \vee T \quad \text{and} \quad S \wedge T \leq R \leq S \vee T$$

we have

$$P \wedge R \leq P \wedge (S \cup T) = (P \wedge S) \vee (P \wedge T) \leq (P \wedge S) \vee S = S$$
and
\[ P \lor R \leq P \lor (S \land T) = (P \lor S) \land (P \lor T) \leq (P \lor S) \land S = S. \]

Thus, \( S \in [P, R] \). We proved that
\[ S \in [P, T], \quad R \in [S, T] \quad \text{implies} \quad R \in [P, T], \quad S \in [P, R]. \]

The converse statement is obtained from the previous one by substituting \( P \) for \( T \), \( S \) for \( R \), and vice versa.

The concept of a ‘base-point order’ plays an important role in our constructions. In a more general form it is introduced in the framework of general theory of convex structures [18, Chapter 1, Section 5].

Definition 3.3. Let \( R \) be a point in \( \mathcal{M} \). A base-point order at \( R \) is a binary relation \( \preceq_R \) on \( \mathcal{M} \) defined by
\[ P \preceq_R Q \quad \text{if and only if} \quad P \in [R, Q]. \]

Theorem 3.1. The base-point order at \( R \) is a reflexive partial ordering.

Proof. Let \( \preceq_R \) be a base-point order on \( \mathcal{M} \). Clearly, \( \preceq_R \) is a reflexive relation.
Suppose \( P \preceq_R Q \) and \( Q \preceq_R P \). Then
\[ R \land Q \preceq P \preceq R \lor Q \quad \text{and} \quad R \land P \preceq Q \preceq R \lor P \]

imply
\[ P = P \land (R \lor Q) = (P \land R) \lor (P \land Q) \leq Q \lor (P \land Q) = Q. \]

By symmetry, \( Q \preceq P \). Hence, \( P = Q \) implying that \( \preceq_R \) is an antisymmetric relation.

Suppose now that \( P \preceq_R Q \) and \( Q \preceq_R S \) for some \( P, Q, S \in \mathcal{M} \). Then \( P \in [R, Q] \) and \( Q \in [R, S] \). By Lemma 3.1 \([P, Q] \subseteq [R, Q] \subseteq [R, S]\). Hence, \( P \in [R, S] \) implying \( P \preceq_R S \).
Therefore \( \preceq_R \) is transitive.

Any interval \([P, Q]\) in \( \mathcal{M} \) is a partially ordered set with respect to \( \preceq_P \). Clearly, \( P \) and \( Q \) are, respectively, the minimal and the maximal elements in \([P, Q]\) relative to \( \preceq_P \). It follows immediately from Lemma 3.2 that
\[ S \preceq_P T \quad \text{if and only if} \quad T \preceq_Q S \]
for all \( S, T \in [P, Q] \). Thus \( \preceq_Q \) is the converse of \( \preceq_P \) on \([P, Q]\).

The following three lemmas establish some properties of base-point orders on \( \mathcal{M} \).

The proofs are immediate and omitted.

Lemma 3.3. Suppose \( S \preceq_P T \) in \( \mathcal{M} \). Then \( R \) lies between \( S \) and \( T \) if and only if \( S \preceq_P R \preceq_P T \).
Lemma 3.4. Let \( P = R^{(0)} \preceq_P R^{(1)} \preceq_P \cdots \preceq_P R^{(k)} = Q \) be a chain of points in \( \mathcal{M} \). Then

(i) If \( S \in [R^{(i)}, R^{(i+1)}] \), \( T \in [R^{(j)}, R^{(j+1)}] \), and \( i < j \), then \( S \preceq_P T \).

(ii) For \( i \neq j \), the intervals \([R^{(i)}, R^{(i+1)}]\) and \([R^{(j)}, R^{(j+1)}]\) are disjoint except the case when \(|j - i| = 1\). In this case the intersection of these two intervals is a singleton which is an end point of each.

Lemma 3.5. Let \( P = R^{(0)} \preceq_P R^{(1)} \preceq_P \cdots \preceq_P R^{(k)} = Q \) be a chain of points in \( \mathcal{M} \). Then, for any two points \( S, T \in [R^{(i)}, R^{(i+1)}] \),

\[ S \preceq_{R^{(i)}} T \quad \text{if and only if} \quad S \preceq_P T. \]

Assuming Hausdorff’s Maximal Principle [2, Chapter VIII, Section 7] (which is a variant of the Axiom of Choice), we conclude that \([P, Q]\) contains a maximal chain with respect to \( \preceq_P \). This justifies the following definition.

Definition 3.4. Let \( P \) and \( Q \) be two points in \( \mathcal{M} \) and \( \preceq_P \) be the base-point order at \( P \). A line segment \( \mathcal{L}[P, Q] \) from \( P \) to \( Q \) is a maximal chain in \([P, Q]\) relative to \( \preceq_P \).

Clearly, any line segment from \( P \) to \( Q \) is also a line segment from \( Q \) to \( P \) (relative to \( \preceq_Q \)) and, generally speaking, there are more than one line segment from \( P \) to \( Q \) in a given space \( \mathcal{M} \).

Theorem 3.2. (i) Let \( P = R^{(0)} \preceq_P R^{(1)} \preceq_P \cdots \preceq_P R^{(k)} = Q \) be a chain of points in \( \mathcal{M} \) and, for \( 0 \leq i < k \), let \( \mathcal{L}[R^{(i)}, R^{(i+1)}] \) be a line segment. Then the union

\[ \bigcup_{i=0}^{k-1} \mathcal{L}[R^{(i)}, R^{(i+1)}] \]

of these line segments is a line segment from \( P \) to \( Q \).

(ii) Conversely, let \( \mathcal{L}[P, Q] \) be a line segment in \( \mathcal{M} \) and let

\[ P = R^{(0)} \preceq_P R^{(1)} \preceq_P \cdots \preceq_P R^{(k)} = Q \]

be a chain of points in \( \mathcal{L}[P, Q] \). Then each set \( \mathcal{L}[P, Q] \cap [R^{(i)}, R^{(i+1)}] \) is a line segment from \( R^{(i)} \) to \( R^{(i+1)} \).

Proof. (i) By Lemma 3.5, each \( \mathcal{L}[R^{(i)}, R^{(i+1)}] \) is a chain in \([R^{(i)}, R^{(i+1)}]\) with respect to \( \preceq_P \). By Lemma 3.4, the union of all these chains is a chain from \( P \) to \( Q \). Clearly, this is a maximal chain.

(ii) By Lemma 3.5, each \( \mathcal{L}[P, Q] \cap [R^{(i)}, R^{(i+1)}] \) is a chain in \([R^{(i)}, R^{(i+1)}]\) with respect to \( \preceq_{R^{(i)}} \). Clearly, this chain is a maximal chain in \([R^{(i)}, R^{(i+1)}]\). □
The following theorem asserts that there is a numerical representation of \( \mathcal{L}[P, Q] \).

**Theorem 3.3.** Let \( \mathcal{L}[P, Q] \) be a line segment in \( \mathcal{M} \). There is a one-to-one function \( \phi: \mathcal{L}[P, Q] \to [0, 1] \) such that \( \phi(P) = 0, \phi(Q) = 1 \) and

\[
S \preceq_P T \text{ if and only if } \phi(S) \leq \phi(T).
\]

Moreover, \( R \) lies between \( S \) and \( T \) if and only if the number \( \phi(R) \) lies between the numbers \( \phi(S) \) and \( \phi(T) \) on the number line.

**Proof.** We define

\[
\phi(R) = \frac{d_H(P, R)}{d_H(P, Q)} = \frac{\sum_{x \in X} d_L(P(x), R(x))}{\sum_{x \in X} d_L(P(x), Q(x))}.
\]

Clearly, \( \phi(P) = 0 \), and \( \phi(Q) = 1 \).

Suppose that \( S \preceq_P T \), i.e.

\[
P(x) \land T(x) \leq S(x) \leq P(x) \lor T(x) \quad \text{for all } x \in X.
\]

It follows that \( d_L(P(x), S(x)) \leq d_L(P(x), T(x)) \). Thus, \( \phi(S) \leq \phi(T) \). Note that, in this case, \( \phi(S) < \phi(T) \) if \( S \neq T \). Suppose that \( \phi(S) < \phi(T) \). If \( S \preceq_P T \), then \( T \preceq_P S \) and \( S \neq T \), since \( \mathcal{L}[P, Q] \) is a chain with respect to \( \preceq_P \). Hence, \( \phi(T) < \phi(S) \), a contradiction.

Suppose now that \( R \) lies between \( S \) and \( T \). We may assume that \( S \preceq_P T \). By Lemma 3.3, \( S \preceq_P R \preceq_P T \). By the first part of this proof, it is equivalent to \( \phi(S) \leq \phi(R) \leq \phi(T) \). \( \square \)

The function \( \phi \) gives a natural one-dimensional parameterization of the line segment \( \mathcal{L}[P, Q] \).

### 4. Well-graded spaces of valued sets

Usually, there may be many line segments connecting two points \( P \) and \( Q \) in a given space \( \mathcal{M} \). All these line segments are, of course, subsets of the interval \( [P, Q] \). In the following definition we introduce a special class of *simple* line segments.

**Definition 4.1.** A line segment \( \mathcal{L}[P, Q] \) in a given space \( \mathcal{M} \) is a *simple line segment* if it coincides with the interval \( [P, Q] \) in \( L^X \).

The following theorem, in some sense, justifies this definition.

**Theorem 4.1.** Let \( \mathcal{L}[P, Q] \) be a simple line segment. There is an element \( a \in X \) such that \( P(x) = Q(x) \) for all \( x \neq a \).
Proof. Suppose that there are elements $a, b \in X$ such that $P(a) \neq Q(a)$ and $P(b) \neq Q(b)$. Let us define

$$S(x) = \begin{cases} P(x) & \text{if } x \neq a, \\ Q(a) & \text{if } x = a, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} P(x) & \text{if } x \neq b, \\ Q(b) & \text{if } x = b. \end{cases}$$

Clearly, $S, T \in \mathcal{L}[P, Q]$ but neither $S \leq_P T$ nor $T \leq_P S$, a contradiction. \qed

Corollary 4.1. Let $L = 2$. If $P \neq Q$ and $\mathcal{L}[P, Q]$ is a simple line segment, then the symmetric difference $P \Delta Q$ is a singleton. Equivalently, $d_H(P, Q) = 1$.

Note that $P$ and $Q$ are treated as usual sets in the above statement.

Now we introduce the main concept of the paper.

Definition 4.2. A space of valued sets $\mathcal{M}$ is well-graded if and only if for any two distinct points $P, Q \in \mathcal{M}$ there is a line segment $L[P, Q]$ and a sequence of points $P = R(0) \leq_P R(1) \leq_P \cdots \leq_P R(k) = Q$ in $\mathcal{L}[P, Q]$ such that each set $\mathcal{L}[P, Q] \cap [R(i), R(i+1)]$ is a simple line segment in $\mathcal{M}$.

By Theorem 3.2, each $\mathcal{L}[P, Q] \cap [R(i), R(i+1)]$ is a line segment from $R(i)$ to $R(i+1)$.

Our definition requires that this line segment coincides with the interval $[R(i), R(i+1)]$ in $L^X$. Let us consider some examples.

Example 4.1. The space $L^X$ is well-graded. Let $P$ and $Q$ be two distinct points in $L^X$ and let $\{x_1, \ldots, x_k\}$ be the set of all elements of $X$ such that $P(x_i) \neq Q(x_i)$. We define $R(0) = P$ and, inductively

$$R(i)(x) = \begin{cases} R(i-1)(x) & \text{if } x \neq x_i, \\ Q(x) & \text{if } x = x_i \end{cases} \quad \text{for } i = 1, \ldots, k.$$ 

Clearly, $R(k) = Q$. It is easy to verify that the sequence $R(0), R(1), \ldots, R(k)$ satisfies the conditions of Definition 4.2.

Example 4.2. Let $X$ be a two element set and $L = [0, 1]$. Then $L^X$ is the unit square $\{(x, \beta) : 0 \leq x \leq 1, \ 0 \leq \beta \leq 1\}$. Consider the space

$$\mathcal{M} = \{(x, \beta) \in L^X : x + \beta = 1\}.$$ 

Clearly, there are no simple line segments in $\mathcal{M}$. Thus, $\mathcal{M}$ is not well-graded. On the other hand, it can be shown that the space

$$\mathcal{M}' = \{(x, \beta) \in L^X : 0.9 \leq x + \beta \leq 1.1\}$$ 

is a well-graded space.

Example 4.3. Let $\mathcal{M}$ be the space of all weak orderings on the set $A = \{a, b, c\}$. Thus, $\mathcal{M} \subset 2^{A \times A}$. Let $P = (a \sim b \sim c)$ and $Q = (a \sim b \prec c)$. Clearly, $[P, Q] = \{P, Q\}$ in $\mathcal{M}$ (there is no weak ordering that lies between $P$ and $Q$ and distinct from $P$ and $Q$).
Thus, \( \mathcal{L}[P, Q] = \{P, Q\} \). Clearly, this line segment is not simple. Hence \( \mathcal{M} \) is not well-graded.

Any space \( \mathcal{M} \subseteq 2^X \) is just a family of subsets of \( X \). The next theorem shows that our definition of a well-graded space coincides with the definition of a well-graded family of sets (Definition 1.1) in the case when \( L = 2 \). We need the following lemma which is a special case of Exercise 2 on p. 234 in [2] (see also [1]).

**Lemma 4.1.** For any \( P, Q, R \in 2^X \), \( R \) lies between \( P \) and \( Q \), i.e.,

\[
P \cap Q \subseteq R \subseteq P \cup Q
\]

if and only if

\[
d_H(P, Q) = d_H(P, R) + d_H(R, Q).
\]

**Theorem 4.2.** A space \( \mathcal{M} \subseteq 2^X \) is well-graded if and only if the family \( \mathcal{M} \) of subsets of \( X \) is well-graded in the sense of Definition 1.1.

**Proof.** (i) Suppose \( \mathcal{M} \) is a well-graded space. Let \( P \) and \( Q \) be two distinct points in \( \mathcal{M} \) and

\[
\mathcal{L}[P, Q] = \bigcup_{i=0}^{k-1} \mathcal{L}[R^{(i)}, R^{(i+1)}]
\]

be a line segment from \( P \) to \( Q \), where \( (R^{(i)}) \) is a sequence of distinct points in \( \mathcal{L}[P, Q] \) required by Definition 4.2. By Corollary 4.1, \( d_H(R^{(i)}, R^{(i+1)}) = 1 \). By Lemmas 3.3 and 4.1, \( d_H(P, Q) = k \).

(ii) Suppose \( \mathcal{M} \) is a well-graded family of subsets of \( X \). Let

\[
P = R^{(0)}, R^{(1)}, \ldots, R^{(k)} = Q
\]

be a sequence of elements of \( \mathcal{M} \) such that \( d_H(R^{(i)}, R^{(i+1)}) = 1 \) for \( 0 \leq i < k \) and \( d_H(P, Q) = k \). By Lemma 4.1, \( R^{(k-1)} \in [P, Q] \), i.e., \( R^{(k-1)} \preceq P R^{(k)} = Q \). By repeating this argument, we show that

\[
P = R^{(0)} \preceq P R^{(1)} \preceq P \cdots \preceq P R^{(k-1)} \preceq P R^{(k)} = Q.
\]

Clearly, each \( [R^{(i)}, R^{(i+1)}] \) is a simple line segment. This proves that \( \mathcal{M} \) is a well-graded space. \( \square \)

### 5. The case of finite \( L \)

In this section we consider spaces of valued sets assuming that \( L \) is a finite ordinal \( L = n = \{0, 1, \ldots, n-1\} \). In this case the lattice \( L^X \) is a finite lattice with \( n^m \) elements, where \( m = |X| \). We shall demonstrate that in this case well-graded spaces can be treated as well-graded families of sets in the sense of Definition 1.1.
The next definition is motivated by the proof of Theorem 4.2.

**Definition 5.1.** Two points $P$ and $Q$ in a given space $\mathcal{M}$ are *adjacent* if and only if $[P, Q] = \{P, Q\}$ in $\mathcal{M}$.

The adjacency relation depends on the choice of $\mathcal{M}$—two points that are adjacent in one space are not necessarily adjacent in another space.

Clearly, two points $P$ and $Q$ are adjacent in $\mathcal{L}$ if and only if there is $a \in X$ such that $P(x) = Q(x)$ for all $x \neq a$ and either $P(x) = Q(x) + 1$ or $Q(x) = P(x) + 1$.

**Lemma 5.1.** If $\mathcal{L}[P, Q]$ is a line segment in $\mathcal{M}$, then there are points $R(i)$, $0 \leq i \leq k$, in $\mathcal{L}[P, Q]$ such that $\mathcal{L}[P, Q] = \{R(0), \ldots, R(k)\}$ and points $R(i)$ and $R(i+1)$ are adjacent in $\mathcal{M}$ for $0 \leq i < k$.

**Proof.** Let $\{R(0), \ldots, R(k)\}$ be the set of all points in $\mathcal{L}[P, Q]$ enumerated according to $\preceq_P$. Since $\mathcal{L}[P, Q]$ is a maximal chain in $[P, Q]$, we have $[R(i), R(i+1)] = \{R(i), R(i+1)\}$ for all $0 \leq i < k$ (cf. Lemma 3.3).

**Theorem 5.1.** A space $\mathcal{M}$ is well-graded if and only if any two points $P$ and $Q$ that are adjacent in $\mathcal{M}$ are also adjacent in the space $\mathcal{L}^X$.

**Proof.** Suppose $\mathcal{M}$ is a well-graded space and let $P$ and $Q$ be two adjacent points in this space. Since $\mathcal{M}$ is well-graded and $[P, Q] = \{P, Q\}$ in $\mathcal{M}$, the only line segment $\mathcal{L}[P, Q] = \{P, Q\}$ from $P$ to $Q$ in $\mathcal{M}$ is simple, i.e., $\{P, Q\} = [P, Q]$ in $\mathcal{L}^X$. Thus, $P$ and $Q$ are adjacent in $\mathcal{L}^X$.

To prove sufficiency, consider a line segment $\mathcal{L}[P, Q]$ in $\mathcal{M}$. By the preceding lemma, $\mathcal{L}[P, Q] = \{R(0), \ldots, R(k)\}$ where points $R(i)$ and $R(i+1)$ are adjacent in $\mathcal{M}$ and $P = R(0) \preceq_P R(1) \preceq_P \ldots \preceq_P R(k) = Q$.

Since these points are also adjacent in $\mathcal{L}^X$, each line segment $\mathcal{L}[R(i), R(i+1)]$ is simple. Thus, $\mathcal{M}$ is well-graded.

Theorem 5.1 shows that the notion of well-gradedness can be defined in terms of the adjacency relation. This approach is used in [13–15], in the case $L = 2$, where a ‘complete’ space $M$ of subsets is defined as a family of subsets of $X$ such that any two adjacent points in $M$ are also adjacent in $2^X$. Thus Theorem 5.1 establishes equivalence of the concepts of well-gradedness and completeness in the case of spaces of usual sets.

The following arguments demonstrate that well-graded spaces of valued sets can be treated as well-graded families of sets in the case of finite $L$. 


Consider a mapping \( F : L^X \rightarrow 2^{X \times L} \) defined by
\[
F(R) = \{(x, z) \in X \times L : R(x) \geq z\}.
\]

Clearly, \( F \) is a lattice monomorphism, i.e., it preserves unions, intersections, and therefore the betweenness relation. It is easy to verify that two points \( P \) and \( Q \) are adjacent in \( L^X \) if and only if \( F(P) \) and \( F(Q) \) are adjacent in \( 2^{X \times L} \). Thus we have the following theorem.

**Theorem 5.2.** A space \( \mathcal{M} \subseteq L^X \) is well-graded if and only if \( F(\mathcal{M}) \) is a well-graded space in \( 2^{X \times L} \).

### 6. The distance function

The aim of this section is to show that any well-graded space is a metric space with respect to a natural uniquely defined distance function.

We begin with some motivations. Suppose \(|X| = 1\). Then \( L^X \cong L \) and we already have the distance function \( d_L \) on \( L \). In general, consider a simple line segment \( \mathcal{L}[P, Q] \). By Theorem 4.1, there is \( a \in X \) such that \( P(x) = Q(x) \) for all \( x \neq a \). We shall call this element \( a \) the base of the simple line segment \( \mathcal{L}[P, Q] \). It is natural to assume that the ‘length’ of a simple line segment \( \mathcal{L}[P, Q] \) with the base \( a \) is \( d_L(P(a), Q(a)) \).

Consider now a point \( R \) in a line segment \( \mathcal{L}[P, Q] \). This point ‘divides’ the linear segment \( \mathcal{L}[P, Q] \) into two line segments \( \mathcal{L}[P, R] \) and \( \mathcal{L}[R, Q] \). It is natural to assume that the length of \( \mathcal{L}[P, Q] \) is equal to the sum of lengths of its ‘parts’ \( \mathcal{L}[P, R] \) and \( \mathcal{L}[R, Q] \). Any point \( R \) that lies between \( P \) and \( Q \) can be included in a line segment from \( P \) to \( Q \). To be consistent, we must assume that the distance from \( P \) to \( Q \) is the sum of the distance from \( P \) to \( R \) and the distance from \( R \) to \( Q \).

The following theorem shows that these assumptions define a unique distance function on a well-graded space.

**Theorem 6.1.** Let \( \mathcal{M} \) be a well-graded space. There exists a unique function \( d(P, Q) \) on \( \mathcal{M} \) that satisfies conditions:

(i) If \( R \in [P, Q] \), then \( d(P, Q) = d(P, R) + d(P, Q) \) for all \( P, Q, R \in \mathcal{M} \).
(ii) If \( \mathcal{L}[P, Q] \) is a simple line segment in \( \mathcal{M} \) with base \( a \), then \( d(P, Q) = d_L(P(a), Q(a)) \).

This function is the generalized Hamming distance on \( \mathcal{M} \):
\[
d(P, Q) = d_H(P, Q) = \sum_{x \in X} d_L(P(x), Q(x)).
\]

**Proof.** Clearly, \( d_H(P, Q) \) satisfies conditions (i) and (ii) of the theorem.

To prove uniqueness, let us consider a line segment \( \mathcal{L}[P, Q] \). Since \( \mathcal{M} \) is well-graded, there is a sequence of points \( P = R^{(0)} \preceq_P R^{(1)} \preceq_P \cdots \preceq_P R^{(k)} = Q \) in \( \mathcal{L}[P, Q] \) such
that each \( L[R^i, R^{i+1}] = L[P, Q] \cap [R^i, R^{i+1}] \) is a simple line segment. Let \( a_i \) be the base of \( L[R^i, R^{i+1}] \). By conditions (i) and (ii), we have

\[
d(P, Q) = \sum_{i=0}^{k-1} d(R^i, R^{i+1}) = \sum_{i=0}^{k-1} d_L(R^i(a_i), R^{i+1}(a_i))
\]

\[
= \sum_{i=0}^{k-1} \sum_{x \in X} d_L(R^i(x), R^{i+1}(x)) = \sum_{x \in X} \sum_{i=0}^{k-1} d_L(R^i(x), R^{i+1}(x))
\]

\[
= \sum_{x \in X} d_L(P(x), Q(x)) = d_H(P, Q).
\]

Thus \( d_H(P, Q) \) is the unique function satisfying the conditions of the theorem.

In the case when \( L \) is \( n \) or \([0, 1]\) and the distance \( d_L \) is given by \( |\alpha - \beta| \), we have

\[
d_H(P, Q) = \sum_{x \in X} |P(x) - Q(x)|.
\]

It should be also noted that in the case of a finite \( L \), condition (ii) could be replaced by the following one:

(iii) \( d(P, Q) = 1 \) for any two adjacent points \( P, Q \in \mathcal{M} \).

7. Spaces closed under intersection

In this section, we use Proposition 2.2 to construct a particular space of valued sets from a given family of usual sets.

Let \( \mathcal{F} \) be a nonempty family of subsets of \( X \) satisfying the following three conditions:

(i) \( \mathcal{F} \) is a well-graded family,
(ii) \( \mathcal{F} \) is closed under intersection,
(iii) \( \emptyset \in \mathcal{F} \).

We denote \( L(\mathcal{F}) \) the space of all valued sets \( A \) in the form

\[
A(x) = \bigvee_{i=1}^{k} a_i \land A^{(i)}(x)
\]

for all \( x \in X \), where \( O < a_1 < a_2 < \cdots < a_k \) is a finite increasing sequence of elements of \( L \) and \( A^{(1)} \supseteq A^{(2)} \supseteq \cdots \supseteq A^{(k)} \) is a nested family of subsets in \( \mathcal{F} \). Note that \( A^{(i)}(x) \) in (5) is the characteristic function of \( A^{(i)} \).

Our first goal is to show that the space \( L(\mathcal{F}) \) satisfies the same conditions (i)–(iii) as the family \( \mathcal{F} \). Clearly, \( \emptyset \in L(\mathcal{F}) \). Next, we have the following theorem.

**Theorem 7.1.** The set \( L(\mathcal{F}) \) is closed under finite intersection.
Proof. Let \( P \) and \( Q \) be two elements of \( L(\mathcal{F}) \). By Proposition 2.2, \( P, Q, \in \mathcal{F} \) for any \( x \in L \setminus \{O\} \). By Proposition 2.3 (ii) and conditions (ii) and (iii) on \( \mathcal{F} \), \((P \land Q)_x = P_x \cap Q_x \in \mathcal{F}\). We have
\[
(P \land Q)(x) = \bigvee_{x \in L \setminus \{O\}} x \land (P \land Q)_x(x).
\]
Since \( X \) is a finite set, the above resolution can be written in the form (5). Thus, \( P \land Q \in L(\mathcal{F}) \). □

Note, that condition (iii) is essential in the proof of this theorem.

It remains to prove that \( L(\mathcal{F}) \) is a well-graded space. First, we prove the following lemma.

Lemma 7.1. Let \( P \) and \( Q \) be two distinct points in \( L(\mathcal{F}) \). There is a point \( R \in [P, Q] \) such that \([P, R]\) is a simple line segment.

Proof. Let \( a_1 < \cdots < a_k \) be all distinct positive values of the functions \( P \) and \( Q \) and let \( a_0 = O \). Then
\[
P(x) = \bigvee_{i=1}^{k} a_i \land P^{(i)}(x) \quad \text{and} \quad Q(x) = \bigvee_{i=1}^{k} a_i \land Q^{(i)}(x)
\]
for all \( x \in X \), where \( P^{(i)} = P_{a_i} \) and \( Q^{(i)} = Q_{a_i} \). Let \( j \) be the smallest index for which \( P^{(j)} \neq Q^{(j)} \). Since \( \mathcal{F} \) is a well-graded family, there is \( S \in \mathcal{F} \) that lies between \( P \) and \( Q \) and is adjacent to \( P \). We define \( R^{(j)} = S \). Then there is an element \( a \in X \) such that either \( R^{(j)} = P^{(j)} \cup \{a\} \) or \( R^{(j)} = P^{(j)} \setminus \{a\} \). We treat these two cases separately.

(i) \( R^{(j)} = P^{(j)} \cup \{a\} \), where \( a \notin P^{(j)} \) and \( a \in Q^{(j)} \). We define \( R^{(i)} = P^{(i)} \) for \( i \neq j \). The family \( \{R^{(i)}\}_{1 \leq i \leq k} \) is a nested family of subsets of \( X \). Indeed, since \( P^{(i)} = Q^{(j)} \) and \( a \in Q^{(j)} \subseteq Q^{(i)} \) for \( i < j \), we have \( a \in P^{(i)} \) for \( i < j \). Thus, \( R^{(i)} = R^{(j)} \) for \( i < j \). Clearly, \( R^{(i)} = P^{(i)} \subseteq P^{(j)} \subseteq P^{(j)} \cup \{a\} \) for \( i > j \).

Since \( \{R^{(i)}\}_{1 \leq i \leq k} \) is a nested family of subsets, the valued set \( R \) defined by
\[
R(x) = \bigvee_{i=1}^{k} a_i \land R^{(i)}(x)
\]
for all \( x \in X \), belongs to \( L(\mathcal{F}) \). Moreover, by Proposition 2.3 (iv), \( R \in [P, Q] \), since \( R^{(i)} \) lies between \( P^{(i)} \) and \( Q^{(i)} \) for all \( i, 1 \leq i \leq k \). Clearly, \( R(x) = P(x) \) for \( x \neq a \). Since \( a \in R^{(i)} \) for \( i \leq j \) and \( a \notin R^{(i)} \) for \( i > j \), \( R(a) = a_j \). Since \( a \in P^{(i)} \) for \( i < j \) and \( a \notin P^{(i)} \) for \( i > j \), \( P(a) = a_{j-1} \). Clearly, \([P, R]\) is a line segment. It remains to show that it is a simple line segment. Let \( S \) be a valued set such that \( S \in [P, R] \). Then \( S(x) = P(x) = R(x) \) for \( x \neq a \) and \( S(a) = \beta \) for some \( \beta \in [a_{j-1}, a_j] \). Clearly, each \( S_x \) is either \( P_x \) or \( R_x \). Therefore, \( S \in L(\mathcal{F}) \).

(ii) \( R^{(j)} = P^{(j)} \setminus \{a\} \), where \( a \in P^{(j)} \) and \( a \notin Q^{(j)} \). We define \( R^{(i)} = P^{(i)} \) for \( i < j \) and \( R^{(i)} = P^{(i)} \setminus \{a\} \) for \( i > j \). Clearly, \( R^{(i)} \in \mathcal{F} \) if \( i \leq j \). Since \( P^{(i)} \subseteq P^{(j)} \) for \( i > j \) and \( \mathcal{F} \) is closed under intersections, \( R^{(i)} = P^{(i)} \setminus \{a\} = R^{(j)} \cap P^{(i)} \in \mathcal{F} \). Thus all \( R^{(i)} \)’s are in \( \mathcal{F} \).
Clearly, the family \( \{ R(i) \}_{1 \leq i \leq k} \) is a nested family of subsets of \( X \). Let us show that \( R(i) \) lies between \( P(i) \) and \( Q(i) \) for all \( i \). This is obviously true for \( i \leq j \). Suppose that \( i > j \). Since \( a \not\in Q(j) \supseteq Q(i) \), we have

\[
P(i) \cap Q(i) \subseteq P(i) \setminus \{ a \} \subseteq P(i) \cup Q(i).
\]

Hence, \( R(i) = P(i) \setminus \{ a \} \in [P(i), Q(i)] \).

As in the previous case, we conclude that the valued set \( R \) defined by

\[
R(x) = \bigvee_{i=1}^{k} \varphi_i \wedge R(i)(x),
\]

for all \( x \in X \), belongs to \( L(\mathcal{F}) \) and lies between \( P \) and \( Q \). Clearly, \( R(x) = P(x) \) for all \( x \neq a \). Since \( a \in R(i) \) for \( i < j \) and \( a \not\in R(i) \) for \( i \geq j \), \( R(a) = \varphi_{j-1} \). On the other hand, since \( a \in P(i) \) for \( i \leq j \), \( P(a) \supseteq \varphi_{j} \). By repeating the argument from the previous case, we can show that \([P, R]\) is a simple line segment. \( \square \)

**Theorem 7.2.** The space \( L(\mathcal{F}) \) is well-graded.

**Proof.** According to Definition 4.2, we need to show that for any two distinct points \( P \) and \( Q \) in \( L(\mathcal{F}) \) there is a line segment \( L[P, Q] \) and a sequence of points

\[
P = R(0) \leq_P R(1) \leq_P \cdots \leq_P R(k) = Q
\]

in \( L[P, Q] \) such that each set \( L[P, Q] \cap [R(i), R(i+1)] \) is a simple line segment in \( L(\mathcal{F}) \).

According to Lemma 7.1, there is a point \( R(1) \) such that \( R(1) \in [P, Q] \) and \([P, R(1)]\) is a simple line segment. Applying the same lemma to the interval \([R(1), Q]\) we obtain a point \( R(2) \) such that \( R(2) \in [R(1), Q] \) and \([R(1), R(2)]\) is a simple line segment. By repeating this procedure, we obtain a sequence of points \( P = R(0), R(1), R(2), \ldots \) in \( L(\mathcal{F}) \) satisfying the following two conditions:

(i) \( R(i) \in [R(i-1), Q] \),

(ii) \([R(i-1), R(i)]\) is a simple line segment

for all \( i \). By Lemma 3.2, condition (i) implies \( R(i-1) \in [P, R(i)] \) or, equivalently, \( R(i-1) \leq_P R(i) \) for all \( i \). Thus we have

\[
P = R(0) \leq_P R(1) \leq_P \cdots \leq_P R(i) \leq_P \cdots.
\]

By means of the construction in the proof of Lemma 7.1, all valued sets \( R_i \) assume values in the finite set of all distinct values of the valued sets \( P \) and \( Q \). Since the set \( X \) is finite, we conclude that \( (R(i)) \) is a finite sequence

\[
P = R(0) \leq_P R(1) \leq_P \cdots \leq_P R(k) = Q
\]

of points in \( L(\mathcal{F}) \). It suffices to apply Theorem 3.2 to complete the proof. \( \square \)
8. The space of valued partial orders

By definition, a valued binary relation on $X$ is a valued set on the product $X \times X$ [20,11]. Given any family of (usual) binary relations on $X$, one can use (5) to construct a space of valued binary relations. In this section, we apply this construction to the space $\mathcal{P}^0$ of all strict partial orders on $X$. We denote $\mathcal{V} \mathcal{P}^0$ the space $L(\mathcal{P}^0)$ (see Section 7) and call elements of this space valued partial orders. A straightforward calculation (cf. [20]) shows that a valued partial order $P$ on $X$ satisfies the following properties:

(i) Antisymmetry: $P(x,y) > O \Rightarrow P(y,x) = O$ for all $x,y \in X$.

(ii) Transitivity: $P(x,y) \wedge P(y,z) \leq P(x,z)$ for all $x,y,z \in X$.

Moreover, each $\alpha$-level set

$$P_\alpha = \{(x,y) \in X \times X : P(x,y) \geq \alpha\}$$

of a valued binary relation $P$ satisfying properties (i) and (ii) is a strict partial order (cf. [20]). Note that usually valued partial orders are defined as valued binary relations satisfying the antisymmetry and transitivity properties [20].

Clearly, the space $\mathcal{P}^0$ is closed under intersections and $\emptyset \in \mathcal{P}^0$. It is also a well-graded space (cf. [3,4]). By applying Theorem 7.2, we obtain the following result.

Theorem 8.1. The space $\mathcal{V} \mathcal{P}^0$ of all valued partial orders on $X$ is well-graded.

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