Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics Letters 21 (2008) 581–587

Applied Mathematics Letters

www.elsevier.com/locate/aml

Existence of positive periodic solutions for two kinds of neutral functional differential equations☆

Yan Luo*, Weibing Wang, Jianhua Shen

Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China

Received 14 March 2007; received in revised form 17 July 2007; accepted 27 July 2007

Abstract

In this work, we deal with a new existence theory for positive periodic solutions for two kinds of neutral functional differential equations by employing the Krasnoselskii fixed-point theorem. Applying our results to various mathematical models we improve some previous results.

Keywords: Neutral functional differential equation; Existence; Positive periodic solution; Fixed-point theorem

1. Introduction

In this work, we investigate the existence of positive periodic solutions of the following two kinds of neutral functional differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - cx(t - \tau(t))] = -a(t)x(t) + f(t, x(t - \tau(t))) \tag{1.1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - c\int_{-\infty}^{0} Q(r)x(t+r)\mathrm{d}r\right] = -a(t)x(t) + b(t)\int_{-\infty}^{0} Q(r)f(t, x(t+r))\mathrm{d}r, \tag{1.2}$$

where $a(t), b(t) \in C(R, (0, \infty))$, $\tau(t) \in C(R, R)$, $f \in C(R \times R, R)$, and $a(t), b(t), \tau(t), f(t, x)$ are ω-periodic functions, $\omega > 0$ and $|c| < 1$ are constants, $Q(r) \in C((-\infty, 0], [0, \infty))$, $\int_{-\infty}^{0} Q(r)\mathrm{d}r = 1$.

It is well known that the functional differential equations (1.1) and (1.2) include many mathematical ecological models and population models (directly or after some transformation), for example:

☆ This work was supported by NNSF of China (No. 10571050).
* Corresponding author.
E-mail address: luoyan2527@126.com (Y. Luo).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.07.009

(1) Hematopoiesis models [7,10,11]

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - cx(t - \tau(t))] = -a(t)x(t) + b(t)\mathrm{e}^{-\beta(t)x(t-\tau(t))}, \tag{1.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - c\int_{-\infty}^{0} Q(r)x(t+r)\mathrm{d}r\right] = -a(t)x(t) + b(t)\int_{-\infty}^{0} Q(r)\mathrm{e}^{-\beta(t)x(t+r)}\mathrm{d}r, \tag{1.4}$$

(2) Nicholson's blowflies models [2,4,9]

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - cx(t - \tau(t))] = -a(t)x(t) + b(t)x(t - \tau(t))\mathrm{e}^{-\beta(t)x(t-\tau(t))}, \tag{1.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - c\int_{-\infty}^{0} Q(r)x(t+r)\mathrm{d}r\right] = -a(t)x(t) + b(t)\int_{-\infty}^{0} Q(r)x(t+r)\mathrm{e}^{-\beta(t)x(t+r)}\mathrm{d}r, \tag{1.6}$$

(3) models for blood cell production [1,5,6]

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - cx(t - \tau(t))] = -a(t)x(t) + b(t)\frac{x(t-\tau(t))}{1 + x^n(t-\tau(t))}, \quad n > 0, \tag{1.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - c\int_{-\infty}^{0} Q(r)x(t+r)\mathrm{d}r\right] = -a(t)x(t) + b(t)\int_{-\infty}^{0} Q(r)\frac{x(t+r)}{1 + x^n(t+r)}\mathrm{d}r, \quad n > 0. \tag{1.8}$$

Meanwhile, since a growing population is likely to consume more (or less) food than a matured one, depending on individual species, this leads to the neutral functional differential equations. Moreover, it is well known that periodic solutions of differential equations describe the important modality of the systems. So it is important to study the existence of periodic solutions to (1.1) and (1.2).

In the work, we obtain sufficient conditions for the existence of positive periodic solutions for Eqs. (1.1) and (1.2). Our results improve and generalize the corresponding results of [11,12], where the authors discussed the special case $c = 0$ of (1.1). Meanwhile, the authors in [8] discussed Eqs. (1.1) and (1.2) with $c \in [0, 1)$. When $c \in (-1, 0)$ in (1.1) and (1.2), our results are new, and the method of proof of the existence of positive periodic solutions for the Eqs. (1.1) and (1.2) is different from that of [8].

The proof of the main results in our work is based on Krasnoselskii's fixed-point theorem. One of the key steps is to find operators T and S satisfying the conditions in the cited fixed-point theorem. To conclude the main results, firstly, we state Krasnoselskii's fixed-point theorem.

Lemma 1.1 (*[13]*). *Let X be a Banach space. Assume K is a bounded closed subset of X. Let*

$$T, S : K \to X$$

satisfy the following conditions:

(i) $Tx + Sy \in K$, $\forall x, y \in K$,
(ii) *S is a contractive operator,*
(iii) *T is a completely continuous operator in K.*

Then $T + S$ has a fixed point in K.

The rest of this work is organized as follows. In the second section, we give and prove our main results. As applications, in the final section, we apply our main results to some models. Besides, new results are obtained.

2. Main results

Let $X = \{x(t) : x(t) \in C(R, R), x(t) = x(t + \omega), t \in R\}$ with the norm $\|x\| = \sup_{t\in[0,\omega]} |x(t)|$; then X is a Banach space with the norm $\|\cdot\|$. Put

$$F(t, x) = \frac{f(t, x)}{a(t)} - cx, \qquad H(t, x) = \frac{b(t)}{a(t)} f(t, x) - cx.$$

Theorem 2.1. *Assume that $c \in [0, 1)$ and that there exist nonnegative constants $m, M : m < M$ such that*

$$(1-c)m \le F(t, x) \le (1-c)M \quad \text{for } \forall t \in [0, \omega], x \in [m, M]$$

and there exists a $t_0 \in [0, \omega]$ such that $F(t_0, x) > (1-c)m$ for any $x \in [m, M]$. Then (1.1) has at least one positive ω-periodic solution $x(t) \in (m, M]$.

Remark 2.1. Theorem 2.1 extends and improves the corresponding results from [12].

Remark 2.2. The perturbed Hill method in [3] allows us to consider equations with constant delays and more general nonlinearities than that in Theorem 2.1, but the application of the Krasnoselskii theorem, as Theorem 2.1 shows, enables us to consider variable delays.

Proof. First, we consider the integral equation

$$x(t) = \int_t^{t+\omega} G(t, s)[f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s))]\mathrm{d}s + cx(t-\tau(t)), \tag{2.1}$$

where

$$G(t, s) := \frac{\exp\left(\int_t^s a(r)\mathrm{d}r\right)}{\exp\left(\int_0^\omega a(r)\mathrm{d}r\right) - 1}. \tag{2.2}$$

We claim that $\varphi(t)$ is an ω-periodic solution of (1.1) if and only if $\varphi(t)$ is an ω-periodic solution of (2.1). In fact, let $\varphi(t)$ be an ω-periodic solution of (1.1); then

$$\frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t) - c\varphi(t-\tau(t))] = -a(t)[\varphi(t) - c\varphi(t-\tau(t))] + f(t, \varphi(t-\tau(t))) - ca(t)\varphi(t-\tau(t)).$$

Let $y(t) = \varphi(t) - c\varphi(t-\tau(t))$; then $y(t)$ is an ω-periodic function and

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -a(t)y(t) + f(t, \varphi(t-\tau(t))) - ca(t)\varphi(t-\tau(t)).$$

Hence,

$$y(t) = \int_t^{t+\omega} G(t, s)[f(s, \varphi(s-\tau(s))) - ca(s)\varphi(s-\tau(s))]\mathrm{d}s,$$

i.e.,

$$\varphi(t) = \int_t^{t+\omega} G(t, s)[f(s, \varphi(s-\tau(s))) - ca(s)\varphi(s-\tau(s))]\mathrm{d}s + c\varphi(t-\tau(t)).$$

This implies that $\varphi(t)$ is an ω-periodic solution of (2.1).

On the other hand, suppose that $\varphi(t)$ is an ω-periodic solution of (2.1); then

$$\varphi(t) - c\varphi(t-\tau(t)) = \int_t^{t+\omega} G(t, s)[f(s, \varphi(s-\tau(s))) - ca(s)\varphi(s-\tau(s))]\mathrm{d}s.$$

Differentiating both sides of the above equality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t) - c\varphi(t-\tau(t))] = -a(t)\varphi(t) + f(t, \varphi(t-\tau(t))).$$

This implies that $\varphi(t)$ is an ω-periodic solution of (1.1).

Let $K = \{x \in X : m \le x(t) \le M\}$. Obviously, K is a bounded closed convex set. Define operators T and S on K by

$$(Tx)(t) = \int_t^{t+\omega} G(t, s)[f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s))]\mathrm{d}s, \tag{2.3}$$

$$(Sx)(t) = cx(t-\tau(t)). \tag{2.4}$$

For any $x \in K$, we have

$$
\begin{aligned}
(Tx)(t+\omega) &= \int_{t+\omega}^{t+2\omega} G(t+\omega, s)[f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s))]\mathrm{d}s \\
&= \int_{t}^{t+\omega} G(t+\omega, u+\omega)[f(u, x(u-\tau(u))) - ca(u)x(u-\tau(u))]\mathrm{d}u \\
&= \int_{t}^{t+\omega} G(t, u)[f(u, x(u-\tau(u))) - ca(u)x(u-\tau(u))]\mathrm{d}u \\
&= (Tx)(t)
\end{aligned}
$$

and

$$
(Sx)(t+\omega) = cx(t+\omega-\tau(t+\omega)) = cx(t-\tau(t)) = (Sx)(t).
$$

So, we have

$$
T(K) \subset X, \qquad S(K) \subset X. \tag{2.5}
$$

Next we will show that $Tx + Sy \in K$ for all $x, y \in K$. In fact, $\forall x, y \in K$,

$$
\begin{aligned}
(Tx)(t) + (Sy)(t) &= \int_{t}^{t+\omega} G(t, s)[f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s))]\mathrm{d}s + cy(t-\tau(t)) \\
&\le \int_{t}^{t+\omega} G(t, s)[(cx(s-\tau(s)) + (1-c)M)a(s) - ca(s)x(s-\tau(s))]\mathrm{d}s + cM \\
&= (1-c)M \int_{t}^{t+\omega} G(t, s)a(s)\mathrm{d}s + cM \\
&= M.
\end{aligned} \tag{2.6}
$$

On the other hand,

$$
\begin{aligned}
(Tx)(t) + (Sy)(t) &\ge \int_{t}^{t+\omega} G(t, s)[(cx(s-\tau(s)) + (1-c)m)a(s) - ca(s)x(s-\tau(s))]\mathrm{d}s + cm \\
&= (1-c)m \int_{t}^{t+\omega} G(t, s)a(s)\mathrm{d}s + cm \\
&= m.
\end{aligned} \tag{2.7}
$$

So, from (2.6) and (2.7), we get that $Tx + Sy \in K$ for all $x, y \in K$.

Clearly, S is a contractive operator on X. We will show that T is a completely continuous operator on K. It is not difficult to see that $T(K)$ is a uniformly bounded set and T is continuous on K, so it suffices to show that $T(K)$ is equi-continuous by the Ascoli–Arzela theorem. For any $x \in K$, by (2.3), we have

$$
\begin{aligned}
|(Tx)'(t)| &\le |a(t)||x(t)| + |f(t, x(t-\tau(t)))| \\
&\le \|a\|M + [(1-c)M + cM]\|a\| \\
&= 2M\|a\|.
\end{aligned} \tag{2.8}
$$

This implies that $T(K)$ is equi-continuous. So T is a completely continuous operator on K. Therefore, $T + S$ has a fixed point $x \in K$ by Lemma 1.1. Furthermore, $m \le x(t) \le M$, which means $x(t)$ is a nonnegative ω-periodic solution of (1.1).

Next, we prove that $x(t)$ is a positive ω-periodic solution of (1.1). We just need to prove that for all $t \in [0, \omega]$, $x(t) > m$. Otherwise, there exists $t^* \in [0, \omega]$ satisfying $x(t^*) = m$. From (2.1), we have

$$
\begin{aligned}
m &= \int_{t^*}^{t^*+\omega} G(t^*, s)[f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s))]\mathrm{d}s + cx(t^*-\tau(t^*)) \\
&\ge \int_{t^*}^{t^*+\omega} G(t^*, s)a(s)\left[\frac{f(s, x(s-\tau(s)))}{a(s)} - cx(s-\tau(s))\right]\mathrm{d}s + cm.
\end{aligned}
$$

From $\int_t^{t+\omega} G(t,s)a(s)\mathrm{d}s = 1$, it follows that

$$\int_{t^*}^{t^*+\omega} G(t^*,s)a(s)[F(s,x)-(1-c)m]\mathrm{d}s \le 0. \tag{2.9}$$

Noting that $F(s,x) \ge (1-c)m$ and $F(t_0,x) > (1-c)m$, $t_0 \in [0,\omega]$, we obtain

$$\int_{t^*}^{t^*+\omega} G(t^*,s)a(s)[F(s,x)-(1-c)m]\mathrm{d}s > 0.$$

This is a contradiction. So, $\forall t \in [0,\omega]$, $0 \le m < x(t) < M$, which means that $x(t)$ is a positive ω-periodic solution of (1.1). The proof is complete. □

Theorem 2.2. *Assume that $c \in [0,1)$ and there exist nonnegative constants $m, M : m < M$ such that*

$$(1-c)m \le H(t,x) \le (1-c)M \quad \text{for } \forall t \in [0,\omega], x \in [m,M]$$

and there exists a $t_0 \in [0,\omega]$ such that $H(t_0,x) > (1-c)m$ for any $x \in [m,M]$. Then (1.2) has at least one positive ω-periodic solution $x(t) \in (m,M]$.

Proof. First of all, like in the proof of Theorem 2.1, we can point out that finding an ω-periodic solution of (1.2) is equivalent to finding an ω-periodic solution of the integral equation

$$\begin{aligned} x(t) = \int_t^{t+\omega} G(t,s)\Bigg[b(s)\int_{-\infty}^0 Q(r)f(s,x(s+r))\mathrm{d}r \\ - ca(s)\int_{-\infty}^0 Q(r)x(s+r)\mathrm{d}r\Bigg]\mathrm{d}s + c\int_{-\infty}^0 Q(r)x(t+r)\mathrm{d}r. \end{aligned} \tag{2.10}$$

Define operators T and S on K by

$$(Tx)(t) = \int_t^{t+\omega} G(t,s)\left[b(s)\int_{-\infty}^0 Q(r)f(s,x(s+r))\mathrm{d}r - ca(s)\int_{-\infty}^0 Q(r)x(s+r)\mathrm{d}r\right]\mathrm{d}s$$

and

$$(Sx)(t) = c\int_{-\infty}^0 Q(r)x(t+r)\mathrm{d}r$$

where K is defined in Theorem 2.1.

The rest of the proof is similar to that of Theorem 2.1. The proof is complete. □

Theorem 2.3. *Assume that $c \in (-1,0)$ and there exist nonnegative constants $m, M : m < M$ such that*

$$m - cM \le F(t,x) \le M - cm \quad \text{for } \forall t \in [0,\omega], x \in [m,M]$$

and there exists a $t_0 \in [0,\omega]$ such that $F(t_0,x) > m - cM$ for any $x \in [m,M]$. Then (1.1) has at least one positive ω-periodic solution $x(t) \in (m,M]$.

Theorem 2.4. *Assume that $c \in (-1,0)$ and there exist nonnegative constants $m, M : m < M$ such that*

$$m - cM \le H(t,x) \le M - cm \quad \text{for } \forall t \in [0,\omega], x \in [m,M]$$

and there exists a $t_0 \in [0,\omega]$ such that $H(t_0,x) > m - cM$ for any $x \in [m,M]$. Then (1.2) has at least one positive ω-periodic solution $x(t) \in (m,M]$.

The proofs of Theorems 2.3 and 2.4 are similar to the proofs of Theorems 2.1 and 2.2, respectively; we omit them here.

3. Some applications

In this section, we apply the results obtained in the previous section to the study of Eqs. (1.3)–(1.8). In view of Theorems 2.1 and 2.2, we obtain the following results.

Corollary 3.1. *Assume that*

(i) $a(t), b(t) \in C(R, (0, \infty))$, $\tau(t) \in C(R, R)$, *and* $a(t)$, $b(t)$, $\tau(t)$, $\beta(t)$ *are* ω*-periodic functions,* $\omega > 0$ *and* $c \in [0, 1)$ *are two constants.*

(ii) $(1-c)m < \frac{b(t)e^{-\beta(t)x}}{a(t)} - cx \leq (1-c)M$ *for all* $(t, x) \in [0, \omega] \times [m, M]$.

Then (1.3) *has at least one positive* ω*-periodic solution* $x(t) \in (m, M]$.

Corollary 3.2. *Assume* (i) *in Corollary* 3.1 *holds and*

(ii) $(1-c)m < \frac{b(t)x\mathrm{e}^{-\beta(t)x}}{a(t)} - cx \leq (1-c)M$ *for all* $(t, x) \in [0, \omega] \times [m, M]$.

Then (1.5) *has at least one positive* ω*-periodic solution* $x(t) \in (m, M]$.

Corollary 3.3. *Assume that*

(i) $a(t), b(t) \in C(R, (0, \infty))$, $\tau(t) \in C(R, R)$, *and* $a(t)$, $b(t)$, $\tau(t)$ *are* ω*-periodic functions,* $\omega > 0$ *and* $c \in [0, 1)$ *are two constants.*

(ii) $(1-c)m < \frac{b(t)x}{a(t)(1+x^n)} - cx \leq (1-c)M$ *for all* $(t, x) \in [0, \omega] \times [m, M]$.

Then (1.7) *has at least one positive* ω*-periodic solution* $x(t) \in (m, M]$.

Corollary 3.4. *Assume* (ii) *in Corollary* 3.1 *holds and*

(i) $a(t), b(t) \in C(R, (0, \infty))$, $a(t)$, $b(t)$, $\beta(t)$ *are* ω*-periodic functions,* $\omega > 0$ *and* $c \in [0, 1)$ *are two constants. Moreover,* $Q(r) \in C((-\infty, 0], [0, \infty))$, $\int_{-\infty}^{0} Q(r)\mathrm{d}r = 1$.

Then (1.4) *has at least one positive* ω*-periodic solution* $x(t) \in (m, M]$.

Corollary 3.5. *Assume* (i) *in Corollary* 3.4 *and* (ii) *in Corollary* 3.2 *hold; then* (1.6) *has at least one positive* ω*-periodic solution* $x(t) \in (m, M]$.

Corollary 3.6. *Assume* (ii) *in Corollary* 3.3 *holds and*

(i) $a(t), b(t) \in C(R, (0, \infty))$, $a(t)$, $b(t)$ *are* ω*-periodic functions,* $\omega > 0$ *and* $c \in [0, 1)$ *are two constants. Moreover,* $Q(r) \in C((-\infty, 0], [0, \infty))$, $\int_{-\infty}^{0} Q(r)\mathrm{d}r = 1$.

Then (1.8) *has at least one positive* ω*-periodic solution* $x(t) \in (m, M]$.

When $c \in (-1, 0)$, we have similar results and we omit them here.

Acknowledgements

The authors are grateful to the referees for their valuable comments which have led to improvement of the presentation.

References

[1] K. Gopalsamy, Stability and Oscillation in Delay Differential Equations of Population Dynamics, Kluwer Academic Press, Boston, 1992.
[2] W.S.C. Gurney, S.P. Blythe, R.M. Nisbet, Nicholson's blowflies revisited, Nature 287 (1980) 17–20.
[3] M.I. Gil, Existence and stability of periodic solutions of semilinear neutral type systems, Dyn. Discrete Contin. Syst. 7 (2001) 809–820.
[4] W. Joseph, H. So, J. Yu, Global attractivity and uniform persistence in Nicholson's blowflies, Differential Equations Dynam. Systems 1 (1994) 11–18.
[5] D. Jiang, J. Wei, Existence of positive periodic solutions for Volterra integro-differential equations, Acta Math. Sci. 21B4 (2002) 553–560.
[6] Y. Li, Existence and global attractivity of a positive periodic solution of a class of delay differential equation, Sci. China 41A3 (1998) 273–284.
[7] J. Luo, J. Yu, Global asymptotic stability of nonautonomous mathematical ecological equations with distributed deviating arguments, Acta Math. Sinica 41 (1998) 1273–1282 (in Chinese).
[8] Z. Li, X. Wang, Existence of positive periodic solutions for neutral functional differential equations, Electron. J. Differential Equations 34 (2006) 1–8.

[9] P. Weng, Existence and global attractivity of periodic solution of integrodifferential equation in population dynamics, Acta Appl. Math. 4 (1996) 427–434.
[10] P. Weng, M. Liang, The existence and behavior of periodic solution of Hematopoiesis model, Math. Appl. 4 (1995) 434–439.
[11] A. Wan, D. Jiang, Existence of positive periodic solutions for functional differential equations, Kyushu J. Math. 1 (2002) 193–202.
[12] A. Wan, D. Jiang, X. Xu, A new existence theory for positive periodic solutions to functional differential equations, Comput. Math. Appl. 47 (2004) 1257–1262.
[13] B. You, Ordinary Differential Equation Complementary Curriculum, People Education Press, Beijing, 1982.