A New Proof of a Theorem of Harper on the Sperner–Erdős Problem

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Let $P$ be a finite, partially ordered set and $v$ a weight on $P$, i.e., a function $v: P \rightarrow \mathbb{R}_+ \setminus \{0\}$. A subset $F \subseteq P$ is called a $k$-family, if there are not $c_1, \ldots, c_k \in F$ such that $c_1 < \cdots < c_k$. Let $d_\ell(P, v) = \max \{\sum_{x \in F} v(x); F \text{ is } k \text{-family}\}$. It is given a new proof of a theorem of Harper which states that $d_\ell(P, v) = d_\ell(Q, w)$, if there is a flow morphism from $(P, v)$ onto $(Q, w)$.

In all that follows we consider finite, weighted partially ordered sets $(P, v)$, i.e., finite posets $P$ for which there is a function $v: P \rightarrow \mathbb{R}_+ \setminus \{0\}$, where $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$. For any subset $S$ of $P$ we define $v(S) = \sum_{x \in S} v(x)$.

The poset $P$ (and also $(P, v)$) is ranked, if there is a function $r: P \rightarrow \mathbb{N}$ such that $r(x) = 0$, if $x$ is minimal, and $r(x) = r(y) + 1$, if $x > y$ (i.e., $x > y$ and $x \geq z \geq y$ implies $x = z$ or $y = z$). Let the levels $N_i = \{x \in P; r(x) = i\}$ and the rank of $P$ $r(P) = \max_{x \in P} r(x)$. Let briefly $N^x$ be the level containing $x$. The weighted and ranked poset $(P, v)$ is called normal, if

$$\frac{v(A)}{v(N_i)} \leq \frac{v(R(A))}{v(N_{i+1})} \quad \text{for all } A \subseteq N_i \text{ and } i \in \{0, \ldots, r(P) - 1\},$$

where $R(A) = \{x \in N_{i+1}; x \geq a \text{ for some } a \in A\}$. Let $E(P)$ be the edge set of the Hasse graph of the ranked poset $P$. A function $f: E(P) \rightarrow \mathbb{R}_+$ is called a normalized flow, if

$$\sum_{y; y \geq x} f((x \ll y)) = \frac{v(x)}{v(N^x)} \quad \text{for all } x \in P \text{ not maximal},$$

$$\sum_{y; y \ll x} f((y \ll x)) = \frac{v(x)}{v(N^x)} \quad \text{for all } x \in P \text{ not minimal}.$$
Applying the Maxflow–Mincut Theorem of Ford and Fulkerson (see [3] or [1, pp. 131f.]) to the poset induced on \( N_i \) and \( N_{i+1} \) one easily obtains that \((P, \nu)\) is normal iff there is a normalized flow on \( E(P)\).

Let \((P, \nu)\) and \((Q, \omega)\) be weighted posets. A function \(\varphi: P \to Q\) is called a flow morphism, if

(i) \(\varphi\) is surjective,
(ii) \(x < y \implies \varphi(x) < \varphi(y)\),
(iii) \(\omega(x) = \nu(\varphi^{-1}(x))\) for all \(x \in Q\),
(iv) the poset induced on \(\varphi^{-1}(x) \cup \varphi^{-1}(y)\) is normal and has rank 1 for all \(x, y \in Q\) with \(x < y\).

Finally, let \(d(P, \nu) = \max \{\nu(A); A\) is an antichain\}. Using the framework of category theory, Harper [8] proved the following

**Theorem.** If there is a flow morphism \(\varphi\) from \((P, \nu)\) onto \((Q, \omega)\), then \(d(P, \nu) = d(Q, \omega)\).

In this note we will give another proof of this Theorem. Since many results in the Sperner theory can be derived from it, a short proof of this Theorem seems to be of general interest.

Let \(\mathcal{C}(P)\) be the set of all saturated chains from a minimal to a maximal element in \(P\). A function \(s: P \to \mathbb{R}\) is called a Sperner weight of \((P, \nu)\), if

\[
\sum_{x \in C} s(x) \leq 1 \quad \text{for all } C \in \mathcal{C}(P),
\]

\[
s(x) \geq 0 \quad \text{for all } x \in P.
\]

A function \(u: \mathcal{C}(P) \to \mathbb{R}\) is called a covering of \((P, \nu)\) by chains, if

\[
\sum_{\substack{C \in \mathcal{C}(P) \\
C \ni x \in E}} u(C) \geq \nu(x) \quad \text{for all } x \in P,
\]

\[
u(x) \geq 0 \quad \text{for all } C \in \mathcal{C}(P).
\]

**Lemma 1.**

\[
d(P, \nu) = \max_{s \text{ Sperner weight}} \sum_{x \in P} \nu(x) s(x) = \min_{u \text{ covering}} \sum_{C \in \mathcal{C}(P)} u(C).
\]

**Proof.** The second equality follows immediately by the Duality Theorem in linear programming. If \(P = \{x_1, \ldots, x_p\}\) and \(s_i = s(x_i)\), then each Sperner weight can be regarded as an element of a polyhedron in the \(\mathbb{R}^p\).
defined by (1). By a theorem of Fulkerson [4] (see [1, p. 138]) all vertices of this polyhedron have integers, i.e., 0, 1 coordinates (the relation graph of a poset $P$, i.e., the undirected graph with vertex set $P$ and edges connecting related elements of $P$, is known to be perfect; the elements of $C(P)$ can be regarded as cliques in the relation graph). Thus each vertex $(s_1, ..., s_P)$ of the polyhedron corresponds to an antichain $A = \{x \in P; s(x) = 1\}$. Since the optimal solution of a linear program is attained on a vertex, the first equality of the Lemma follows.

**Lemma 2.** $(P, \nu)$ is normal iff $P$ is ranked and there exists a function $g: C(P) \to \mathbb{R}_+$ such that

$$
\sum_{C \in C(P)} g(C) = \frac{\nu(x)}{\nu(N_x)} \quad \text{and} \quad \sum_{C \in C(P)} g(C) = 1.
$$

**Proof.** If $P$ is ranked and there exists such a function $g$, we obtain a normalized flow $f$ on $E(P)$ by

$$
f((x \ll y)) = \sum_{C \in C(P)} g(C).
$$

Conversely, let $f$ be a normalized flow on $E(P)$. We may construct the searched function $g$ by the following algorithm. Let $f_i: E(P) \to \mathbb{R}_+$ and $g_i: C(P) \to \mathbb{R}_+$, $i = 0, 1, ...$. Further let $f_0 = f$ and $g_0 = 0$. If $f_i$ and $g_i$ are given and if $f_i \not= 0$, then let $C' = (c'_0 < \cdots < c'_{k-1})$ be such an element of $K(P)$ for which

$$
m = \min_{j = 0, ..., k-1} f_i((c'_j < c'_{j+1})) > 0.
$$

Now let

$$
f_{i+1}((x \ll y)) = f_i((x \ll y)) - m, \quad \text{if} \quad (x \ll y) = (c'_j < c'_{j+1})
$$

for some $j \in \{0, ..., k-1\}$,

$$
f_{i+1}((x \ll y)) = f_i((x \ll y)), \quad \text{otherwise},
$$

$$
g_{i+1}(C) = g_i(C) + m, \quad \text{if} \quad C = C',
$$

$$
g_{i+1}(C) = g_i(C), \quad \text{otherwise}.
$$

Obviously, there is an $n_0$ such that $f_{n_0} = 0$. Then $g = g_{n_0}$ is the sought after function.

Q.E.D.
Proof of the theorem. Let $u$ be a covering of $(Q, w)$ by chains such that

$$d(Q, w) = \sum_{C' \in \mathcal{C}(Q)} u(C')$$

(Lemma 1).

By definition of a flow morphism the poset induced on $\varphi^{-1}(C')$ is normal for all $C' \in \mathcal{C}(Q)$. Let $g_{C'}: \mathcal{C}(P) \to \mathbb{R}_+$ be its corresponding function of Lemma 2. Now we define $t: \mathcal{C}(P) \to \mathbb{R}_+$ by

$$t(C) = g_{\varphi(C)}(C) \cdot u(\varphi(C)).$$

(Obviously, $C \in \mathcal{C}(P)$ implies $\varphi(C) \in \mathcal{C}(Q)$.) Using Lemma 2 it is easy to check that $t$ is a covering. Further

$$\sum_{C \in \mathcal{C}(P)} t(C) = \sum_{C' \in \mathcal{C}(Q)} u(C') \sum_{C \in \mathcal{C}(P)} g_{C'}(C) = \sum_{C' \in \mathcal{C}(Q)} u(C') = d(Q, w).$$

By Lemma 1 we have

$$d(P, v) \leq \sum_{C \in \mathcal{C}(P)} t(C) = d(Q, w).$$

At last, $d(P, v) \geq d(Q, w)$ since $\varphi^{-1}(A)$ is an antichain in $P$ if $A$ is an antichain in $Q$, and $v(\varphi^{-1}(A)) = w(A)$. Q.E.D.

A subset $F$ of $P$ is called a $k$-family, if there are not $c_0, \ldots, c_k \in F$ such that $c_0 < \cdots < c_k$. Let $d_k(P, v) = \max\{v(F); F \text{ is a } k\text{-family}\}$.

Corollary 1. If there is a flow morphism $\varphi$ from $(P, v)$ onto $(Q, w)$, then $d_k(P, v) = d_k(Q, w)$ for all $k = 1, 2, \ldots$.

Proof. Following an idea of Saks [9] we associate to $(P, v)$ the poset $(P_k, v_k)$ as follows. Let $P_k = P \times \{1, \ldots, k\}$;

$$(x, i) \leq_P (y, j)\text{ iff } x \leq y \text{ and } i \leq j; \quad v_k(x, i) = v(x).$$

We will show that $d(P_k, v_k) = d_k(P, v)$. For that let $A$ be an antichain in $P_k$ with $v_k(A) = d_k(P, v)$. Let $F = \{x \in P; (x, i) \in A \text{ for some } i\}$. Obviously, $F$ is a $k$-family in $P$ with $v(F) = v_k(A)$ (note that $(x, i), (x, j) \in A$ implies $i = j$). Hence, $d_k(P) \geq d(P_k, v_k)$. Conversely, let $F$ be a $k$-family in $P$ with $v(F) = d_k(P, v)$. It is known that $F$ can be partitioned into $k$ antichains $F = F_1 \cup \cdots \cup F_k$ such that $a \in F_i, b \in F_j, k \geq i > j \geq 1$ imply $a \not\geq b$ (see
[6, 2]). Now define $A = \bigcup_{i=1}^{k} (F_i, i)$, where $(F_i, i)$ denotes the set $\{(x, i); x \in F_i\}$. Obviously, $A$ is an antichain in $(P_k, v_k)$ and $v(F) = v_k(A)$. Hence, $d_k(P) \leq d(P_k, v_k)$. Analogously, we have $d(Q_k, w_k) = d_k(Q, w)$. Finally, it is easy to see that the function $\psi$ defined by $\psi((x, i)) = (\varphi(x), i)$ is a flow morphism from $(P_k, v_k)$ onto $(Q_k, w_k)$. Q.E.D.

The notion of a flow morphism can be weakened. A function $\varphi: P \to Q$ is called a weak flow morphism from $(P, v)$ onto $(Q, w)$ if it satisfies the conditions (i), (iii), (iv), and

$$(ii') \ x < y \implies \varphi(x) < \varphi(y).$$

**Corollary 2.** If there is a weak flow morphism $\varphi$ from $(P, v)$ onto $(Q, w)$, then $d_k(P, v) = d_k(Q, w)$ for all $k = 1, 2, \ldots$.

**Proof.** Consider the poset $(P^*, v)$ which has the same elements and weight as $(P, v)$, but where $x <^* y$ iff $x < y$ and $\varphi(x) < Q \varphi(y)$. Obviously, $\varphi: P = P^* \to Q$ is a flow morphism from $(P^*, v)$ onto $(Q, w)$. By Corollary 1 we have $d_k(P^*, v) = d_k(Q, w)$. Evidently, $d_k(P, v) \leq d_k(P^*, v)$. On the other hand, $d_k(P, v) \geq d_k(Q, w)$, because $\varphi^{-1}(F)$ is a $k$-family in $P$ if $F$ is a $k$-family in $Q$ and it holds $v(\varphi^{-1}(F)) = w(F)$. Q.E.D.

One application of Corollary 2 is the following. Let $(P, v)$ be a weighted poset. A bijective function $\psi: P \to P$ with $p < p'$ iff $\psi(p) < \psi(p')$ for all $p, p' \in P$ is called an automorphism of the poset $P$. Let $g$ be a group of automorphisms of $P$ and let $O_1, \ldots, O_t$ be the orbits in $P$ under $g$. To $(P, v)$ and $g$ we can associate the weighted automorphism order $(P_g, w)$ as follows: $P_g = \{O_1, \ldots, O_t\}$, $O_i < O_j$ iff $p < p'$ for some $p \in O_i$ and $p' \in O_j$, and $w(O_i) = v(O_i)$. We suppose that the weight $v$ is constant on the orbits in $P$ or $O_i < O_j$ implies $p < p'$ for all $p \in O_i$ and $p' \in O_j$ ($i, j = 1, \ldots, t$). Then the function $\varphi_g$ defined by $\varphi_g(p) = O_i$ iff $p \in O_i$, is a weak flow morphism from $(P, v)$ onto $(P_g, w)$. By Corollary 2 we have $d(P, v) = d(P_g, w)$. Furthermore, if $F$ is a $k$-family in $P_g$ with $w(F) = d(P_g, w)$, then $\varphi^{-1}(F)$ is a $k$-family in $P$ with $v(\varphi^{-1}(F)) = d(P, v)$ which consists of complete orbits under $g$. The existence of such families was proved in a different way by Kleitman, Edelberg, and Lubell [7]. The function $\varphi$ is not necessarily a flow morphism. A counterexample is given in Fig. 1, where we take $v \equiv 1$ and the group $g$ of all automorphisms of $P$. Here condition (ii) is violated.

![Figure 1](image-url)
REFERENCES