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Neumann problem for the Korteweg–de Vries equation

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Abstract

We consider Neumann initial-boundary value problem for the Korteweg–de Vries equation on a half-line

$$\begin{cases} u_t + \lambda u u_x + u_{xxx} = 0, & t > 0,\ x > 0, \\ u(x,0) = u_0(x), & x > 0, \\ u_x(0,t) = 0, & t > 0. \end{cases} \tag{0.1}$$

We prove that if the initial data $u_0 \in \mathbf{H}_1^{0,21/4} \cap \mathbf{H}_2^{1,7/2}$ and the norm $\|u_0\|_{\mathbf{H}_1^{0,21/4}} + \|u_0\|_{\mathbf{H}_2^{1,7/2}} \leqslant \varepsilon$, where $\varepsilon > 0$ is small enough $\mathbf{H}_p^{s,k} = \{f \in \mathbf{L}^2;\ \|f\|_{\mathbf{H}_p^{s,k}} = \|\langle x\rangle^k \langle i\partial_x\rangle^s f\|_{\mathbf{L}^p} < \infty\}$, $\langle x\rangle = \sqrt{1+x^2}$ and $\lambda \int_0^\infty x u_0(x)\,dx = \lambda\theta < 0$. Then there exists a unique solution $u \in \mathbf{C}([0,\infty), \mathbf{H}_2^{1,7/2}) \cap \mathbf{L}^2(0,\infty; \mathbf{H}_2^{2,3})$ of the initial-boundary value problem (0.1). Moreover there exists a constant C such that the solution has the following asymptotics

$$u(x,t) = C\theta(1+\eta\log t)^{-1} t^{-2/3} Ai'\left(\frac{x}{\sqrt[3]{t}}\right) + O\big(\varepsilon^2 t^{-2/3}(1+\eta\log t)^{-6/5}\big)$$

for $t \to \infty$ uniformly with respect to $x > 0$, where $\eta = -9\theta\lambda \int_0^\infty Ai'^2(z)\,dz$ and $Ai(q)$ is the Airy function

$$Ai(q) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-z^3+zq}\,dz = \frac{1}{\pi}\operatorname{Re}\int_0^\infty e^{-i\xi^3+i\xi q}\,d\xi.$$

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1. Introduction

We consider the Neumann initial-boundary value problem on a half-line for the Korteweg–de Vries (KdV) equation

$$\begin{cases} u_t + \lambda u u_x + u_{xxx} = 0, & t > 0,\ x > 0, \\ u(x,0) = u_0(x), & x > 0, \\ u_x(0,t) = 0, & t > 0. \end{cases} \tag{1.1}$$

Korteweg–de Vries equation (1.1) is a simple universal model appearing in many fields of physics and technology as the first approximation in the description of the dispersive nonlinear waves [5,18,20]. In the case of the Cauchy problem on the line KdV equation was solved via the inverse scattering transform method [1] and the well-posedness on the line and the periodic domain was studied extensively by many authors, see, e.g., [3,16,17] and references cited therein. Asymptotic behavior of small solutions was studied in the case of the generalized KdV equations in [4,12–14,19]. However large time asymptotics of small solutions to the Cauchy problem of KdV on the line is not obtained completely still now. From the heuristic point of view the quadratic nonlinearity of the shallow water type uu_x is subcritical for large time: the nonlinear term decays more slowly than the linear part of the equation. The case of the Dirichlet initial-boundary value problem for KdV equation on the positive line was considered recently in paper [10]. It was proved that the solution has an additional time-decay and as a result the nonlinear term of KdV equation behaves as a supercritical in the contrary to the corresponding Cauchy problem. In the case of the Neumann boundary-value problem (1.1) we expect that the time decay rate of the solutions will be more slow so that the nonlinear term in the problem (1.1) behaves as a critical one. Our main goal in the present paper is to obtain the large time asymptotics of solutions to problem (1.1). Our approach is also based on the estimates of the integral equation in the weighted Sobolev spaces

$$\mathbf{H}_p^{s,k} = \left\{ f \in \mathbf{L}^2;\ \|f\|_{\mathbf{H}_p^{s,k}} = \left\| \langle x \rangle^k \langle i\partial_x \rangle^s f \right\|_{\mathbf{L}^p} < \infty \right\}, \quad \langle x \rangle = \sqrt{1+x^2},$$

and weighted $\mathbf{L}^2$ space is used to get smooth solutions, where

$$\|f\|_{\mathbf{L}^p} = \left(\int_0^\infty \left| f(x) \right|^p dx \right)^{1/p}.$$

We also use the methods of our previous papers [7,8,15]. Integral formula is obtained by using the Laplace transform with respect to space variable. The Laplace transform requires the boundary data $u(0,t)$, $u_x(0,t)$, $u_{xx}(0,t)$ and so $u(0,t)$, $u_{xx}(0,t)$ should be determined by the given data. In order to do it we use the method developed in the previous works [9,11]. Our method to derive an integral formula is different from that [2] or [6], but the representation of the integral formula of ours (2.9) stated below is the same as one obtained in [6]. In [6] integral formulas for various boundary value problem of linear equation including Dirichlet and Neumann problems were constructed. However estimates of solutions applied to nonlinear problem were not given. In [2] an integral formula for Dirichlet problem which are different from ones used in [6] was constructed and used to derive the various smoothing properties of solutions.

For simplicity we denote $\|\cdot\|_{\mathbf{L}^\infty} = \|\cdot\|_\infty$, $\|\cdot\|_{\mathbf{L}^2} = \|\cdot\|$, $\|\cdot\|_{\mathbf{H}_2^{s,k}} = \|\cdot\|_{s,k}$ and $\|\cdot\|_{\mathbf{H}_2^{0,k}} = \|\cdot\|_k$.

Now we state our result.

Theorem 1.1. *Suppose that the initial data* $u_0 \in \mathbf{H}_1^{0,21/4} \cap \mathbf{H}_2^{1,7/2}$ *and the norm*

$$\|u_0\|_{\mathbf{H}_1^{0,21/4}} + \|u_0\|_{1,7/2} \leqslant \varepsilon,$$

where $\varepsilon > 0$ *is small enough and*

$$\lambda \int_0^\infty x u_0(x)\,dx = \lambda\theta < 0.$$

Then there exists a unique solution

$$u \in \mathbf{C}\big([0,\infty), \mathbf{H}_2^{1,7/2}\big) \cap \mathbf{L}^2\big(0,T;\mathbf{H}_2^{2,3}\big)$$

of the initial-boundary value problem (1.1) *satisfying the boundary condition such that* $u_x(0,t) = 0$ *for* $t > 0$. *Moreover the solution has the following asymptotics*

$$u(x,t) = 3\theta(1+\eta\log t)^{-1} t^{-\frac{2}{3}} Ai'\left(\frac{x}{\sqrt[3]{t}}\right) + O\big(\varepsilon^2 t^{-\frac{2}{3}}(1+\eta\log t)^{-\frac{6}{5}}\big)$$

for $t \to \infty$ *uniformly with respect to* $x > 0$, *where*

$$\eta = -9\theta\lambda \int_0^\infty Ai'^2(x)\,dx > 0$$

and $Ai(q)$ *is Airy function*

$$Ai(q) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-z^3+zq}\,dz.$$

We note here that solutions of our problem (1.1) decay faster than those of linear problem and $Ai^{(2)}(0) = 0$, then we see that the main term satisfies the Neumann boundary condition for $t > 0$.

For the convenience of the reader we now state of our strategy of the proof. In Section 2 we construct the integral formula of the solution of the linear problem corresponding to (1.1):

$$u(x,t) = \Phi(x,t)u_0 = \int_0^\infty F(x,y,t)u_0(y)\,dy,$$

where

$$F(x,y,t)=\frac{1}{2\pi i}\left(\int\limits_{-i\infty}^{i\infty} e^{-p^3t+p(x-y)}\,dp-\int\limits_{-i\infty}^{i\infty} e^{\xi t+\phi_3(\xi)x}\phi_3^{-1}(\xi)\sum_{j=1}^{2}e^{-\phi_j(\xi)y}\phi_j(\xi)\phi_j'(\xi)\,d\xi\right)$$

and $\phi_j(\xi)$ are the roots of equation $p^3=-\xi$ such that

$$p=\phi_1(\xi)=\big(\xi\exp(i\pi)\big)^{\frac{1}{3}},\qquad p=\phi_2(\xi)=\big(\xi\exp(-i\pi)\big)^{\frac{1}{3}}.$$

$$p=\phi_3(\xi)=\big(\xi\exp(3i\pi)\big)^{\frac{1}{3}}.$$

Section 3 is devoted to the study of the asymptotic behavior of solutions to the linear problem by using the integral formula constructed in Section 2. We will show that for $t>1$, $0<\delta<1$

$$\Phi(t)u_0=3\theta t^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{t}}\right)+O\big(t^{-\frac{2+\delta}{3}}\big),$$

where $\theta=\int_0^\infty xu_0\,dx$. Therefore nonlinear term in the equation $u_t+\lambda uu_x+u_{xxx}=0$ has the same decay rate as linear terms. Section 5 is devoted to prove a global result and to establish asymptotic formulas of solutions. Here as in paper [7] we make a change of variable $u=e^{-\phi(t)}v$. Then for the new function v we get the following problem

$$\begin{cases} v_t-\phi_t v+\lambda e^{-\phi(t)}vv_x+v_{xxx}=0, & t>0,\ x>0,\\ v(x,0)=e^{\phi(0)}u_0(x), & x>0,\\ v_x(0,t)=0, & t>0. \end{cases}$$

In order to obtain an additional time decay rate we assume that real-valued function $\phi(t)$ satisfies the following condition

$$\int\limits_0^\infty x\big(-\phi_t v+\lambda e^{-\phi(t)}vv_x\big)\,dx=0,\qquad \phi(0)=0,$$

which implies that

$$e^{\phi(t)}=g(t)=1+\lambda\theta^{-1}\int\limits_0^t d\tau\int\limits_0^\infty xvv_x\,dx=1-\lambda\theta^{-1}\int\limits_0^t\big\|v(\tau)\big\|^2\,d\tau.$$

We look for the solution v in the neighborhood of the first approximation

$$\Phi(t)u_0\approx 3\theta t^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{t}}\right)=\theta G.$$

To do it, we put $r = v - \Phi(t)u_0$ then we get the following integral formula

$$\begin{cases} r = \int_0^t g(\tau)^{-1}\Phi(x, t-\tau)\big(\lambda G\theta^2\|G\|^2 + GG_x\big)\,d\tau + R_1, \\ g = 1 - \lambda\theta \int_0^t \|G\|^2\,d\tau + R_2, \end{cases}$$

where R_1, R_2 are considered as remainder terms in our function space defined later. By Lemmas 3.5, 3.6 in Section 3 we obtain

$$g(t) = \frac{C}{1+\eta\log(1+t)} \quad \text{and} \quad \|r\|_\infty \leqslant Ct^{-\frac{2}{3}}g(t)^{-\frac{1}{5}},$$

where under condition $\lambda\theta < 0$

$$\eta = -9\theta\lambda \int_0^\infty Ai'^2(z)\,dz > 0.$$

Hence from representation $u = v/g = (r + \Phi)/g$ we get result of theorem. We prove a local existence result of solutions in separate Section 4 by using energy method. However it seems that the energy method is not sufficient to get a global result and a sharp asymptotics of solutions. Therefore we show various estimates of solutions of linear problem with Neumann boundary condition in Section 3.

2. Linear problem

We consider the linear initial-boundary value problem corresponding to (1.1)

$$\begin{cases} u_t + u_{xxx} = f(x,t), & t > 0,\ x > 0, \\ u(x,0) = u_0(x), & x > 0, \\ u_x(0,t) = h(t), & t > 0. \end{cases} \tag{2.1}$$

Taking the Laplace transformation of the problem (2.1) with respect to the space variable x we have

$$\hat{u}_t(p,t) + p^3\hat{u}(p,t) = f_1(p,t),$$

where

$$f_1(p,t) = u(0,t)p^2 + u_{xx}(0,t) + \hat{f}(p,t) + ph(t),$$
$$\hat{u}(p,t) = \int_0^\infty e^{-px}u(x,t)\,dx, \tag{2.2}$$

and so we have the following representation for the Laplace transform of the solution

$$\hat{u}(p,t) = e^{-p^3 t}\hat{u}_0(p) + \int_0^t e^{-p^3(t-\tau)}f_1(p,\tau)\,d\tau. \tag{2.3}$$

In order to get the integral formula associated with (2.1), we find $u(0,t)$ and $u_{xx}(0,t)$ by using given data. The condition

$$\left|\hat{u}(p,t)\right| \leqslant M\left(1+|p|\right)^{\beta}, \quad \text{for all } \operatorname{Re} p \geqslant 0, \tag{2.4}$$

with some M, $\beta > 0$ is necessary and sufficient for the existence of the inverse Laplace transformation. It is easy to see that condition (2.4) is fulfilled in domains $\operatorname{Re} p^3 > 0$ of the right half-complex plane $\operatorname{Re} p \geqslant 0$. In domains $\operatorname{Re} p^3 < 0$ of the right half-complex plane $\operatorname{Re} p \geqslant 0$, we rewrite formula (2.3) as

$$\hat{u}(p,t) = e^{-p^3 t}\left(\hat{u}_0(p) + \int_0^\infty e^{p^3\tau} f_1(p,\tau)\,d\tau\right) - \int_t^\infty e^{p^3(t-\tau)} f_1(p,\tau)\,d\tau.$$

It is clear that the last integral

$$\int_t^\infty e^{p^3(t-\tau)} f_1(p,\tau)\,d\tau$$

satisfies condition (2.4) for all $\operatorname{Re} p \geqslant 0$, $\operatorname{Re} p^3 < 0$. Therefore in order to satisfy condition (2.4) we have to assume that

$$\hat{u}_0(p) + \int_0^\infty e^{p^3\tau} f_1(p,\tau)\,d\tau = 0 \tag{2.5}$$

for all $\operatorname{Re} p \geqslant 0$, $\operatorname{Re} p^3 < 0$. We use (2.5) to find the boundary functions $u(0,t)$ and $u_{xx}(0,t)$ involved in (2.2). Making the change of variable $-p^3 = \xi$ we transform domains $\operatorname{Re} p^3 < 0$ of the right half-complex plane $\operatorname{Re} p \geqslant 0$ to the half-complex plane $\operatorname{Re}\xi > 0$. The equation $-p^3 = \xi$ has three roots $\phi_1(\xi)$, $\phi_2(\xi)$ and $\phi_3(\xi)$ such that

$$p = \phi_1(\xi) = \left(\xi \exp(i\pi)\right)^{\frac{1}{3}}, \qquad p = \phi_2(\xi) = \left(\xi \exp(-i\pi)\right)^{\frac{1}{3}}, \qquad p = \phi_3(\xi) = \left(\xi \exp(3i\pi)\right)^{\frac{1}{3}}$$

which transform the half-complex plane $\operatorname{Re}\xi > 0$ to domains, where

$$\operatorname{Re}\phi_1(\xi) > 0, \qquad \operatorname{Re}\phi_2(\xi) > 0 \quad \text{and} \quad \operatorname{Re}\phi_3(\xi) < 0.$$

Condition (2.5) can be written as a system of equations in the half-complex plane $\operatorname{Re}\xi > 0$

$$\begin{aligned} 0 = {} & \int_0^\infty e^{-\phi_l(\xi)x} u_0(x)\,dx + \int_0^\infty e^{-\xi\tau} u(0,\tau)\,d\tau\,\phi_l(\xi)^2 + \int_0^\infty e^{-\xi\tau}\partial_{xx}u(0,\tau)\,d\tau \\ & + \int_0^\infty\int_0^\infty e^{-(\phi_l(\xi)x+\xi\tau)} f(x,\tau)\,dx\,d\tau + \left(\int_0^\infty e^{-\xi\tau} h(\tau)\,d\tau\right)\phi_l(\xi) \end{aligned}$$

for $l=1,2$. From which it follows that

$$\begin{pmatrix} \phi_1^2(\xi) & 1 \\ \phi_2^2(\xi) & 1 \end{pmatrix} \begin{pmatrix} \int_0^\infty e^{-\xi t} u(0,t)\,dt \\ \int_0^\infty e^{-\xi t} u_{xx}(0,t)\,dt \end{pmatrix} = -\begin{pmatrix} g(\phi_1) \\ g(\phi_2) \end{pmatrix}, \tag{2.6}$$

where

$$g(\phi_l) = \hat{u}_0(\phi_l) + \hat{\hat{f}}\bigl(\phi_l(\xi),\xi\bigr) + \phi_l \hat{h}(\xi)$$

and

$$\hat{u}_0\bigl(\phi_l(\xi)\bigr) = \int_0^\infty e^{-\phi_l(\xi)y} u_0(y)\,dy,$$

$$\hat{\hat{f}}\bigl(\phi_l(\xi),\xi\bigr) = \int_0^\infty \int_0^\infty e^{-(\phi_l(\xi)y+\xi t)} f(y,t)\,dy\,dt,$$

$$\hat{h}(\xi) = \int_0^\infty e^{-\xi t} h(t)\,dt.$$

Solving (2.6) we find

$$\begin{pmatrix} \int_0^\infty e^{-\xi t} u(0,t)\,dt \\ \int_0^\infty e^{-\xi t} u_{xx}(0,t)\,dt \end{pmatrix} = \frac{1}{\phi_2^2(\xi) - \phi_1^2(\xi)} \begin{pmatrix} g(\phi_1) - g(\phi_2) \\ \phi_1^2 g(\phi_2) - \phi_2^2 g(\phi_1) \end{pmatrix}.$$

By the inverse Laplace transform with respect to time variable we get

$$\begin{pmatrix} u(0,t) \\ u_{xx}(0,t) \end{pmatrix} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\xi t}\,d\xi \frac{1}{\phi_2^2(\xi) - \phi_1^2(\xi)} \begin{pmatrix} g(\phi_1(\xi)) - g(\phi_2(\xi) \\ \phi_1^2(\xi) g(\phi_2(\xi)) - \phi_2^2(\xi) g(\phi_1(\xi)) \end{pmatrix}. \tag{2.7}$$

By (2.3),

$$u(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-p^3 t + px} \hat{u}_0(p)\,dp + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} \int_0^t e^{-p^3(t-\tau)} \bigl(\hat{f}(p,\tau) + h(\tau)p\bigr)\,d\tau\,dp$$
$$+ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} e^{-p^3 t} \int_0^t e^{p^3 \tau} \bigl(u(0,\tau)p^2 + u_{xx}(0,\tau)\bigr)\,d\tau\,dp. \tag{2.8}$$

We consider the last term of the right-hand side of (2.8). We put

$$H(p,\xi) = -\frac{1}{\phi_1^2(\xi) - \phi_2^2(\xi)} p^2 \bigl(g\bigl(\phi_1(\xi)\bigr) - g\bigl(\phi_2(\xi)\bigr) + \phi_1^2(\xi) g\bigl(\phi_2(\xi)\bigr) - \phi_2^2(\xi) g\bigl(\phi_1(\xi)\bigr)\bigr).$$

Using (2.7) we rewrite

$$I = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} e^{-p^3 t} \int_0^t e^{p^3 \tau} \big(u(0,\tau) p^2 + u_{xx}(0,\tau) \big) \, d\tau \, dp$$
$$= -\frac{1}{4\pi^2} \int_{-i\infty}^{i\infty} e^{px} \int_{-i\infty}^{i\infty} \frac{e^{\xi t} - e^{-p^3 t}}{p^3 + \xi} H(p,\xi) \, d\xi \, dp.$$

Since by Cauchy Theorem for all Re $p = 0$

$$\int_{-i\infty}^{i\infty} \frac{1}{p^3 + \xi} H(p,\xi) \, d\xi = 0$$

we find that

$$\int_{-i\infty}^{i\infty} e^{px} \int_{-i\infty}^{i\infty} \frac{e^{-p^3 t}}{p^3 + \xi} H(p,\xi) \, d\xi \, dp = 0.$$

Also by the facts that

$$\phi_1'(\xi) = -\frac{1}{(\phi_1(\xi) - \phi_2(\xi))(\phi_1(\xi) - \phi_3(\xi))},$$
$$\phi_2'(\xi) = -\frac{1}{(\phi_2(\xi) - \phi_1(\xi))(\phi_2(\xi) - \phi_3(\xi))}, \quad \sum_{j=1}^{3} \phi_j(\xi) = 0$$

we get

$$\int_{-i\infty}^{i\infty} \frac{e^{px}}{p^3 + \xi} H(p,\xi) \, dp = \int_{-i\infty}^{i\infty} \frac{e^{px}}{\prod_{j=1}^{3} (p - \phi_j(\xi))} H(p,\xi) \, dp$$
$$= -2\pi i e^{\phi_3(\xi)x} \frac{g(\phi_1)(\phi_3^2 - \phi_2^2) + g(\phi_2)(\phi_1^2 - \phi_3^2)}{(\phi_1^2 - \phi_2^2)(\phi_3 - \phi_1)(\phi_3 - \phi_2)}$$
$$= -2\pi i e^{\phi_3(\xi)x} \left(\frac{g(\phi_1)(\phi_3 + \phi_2)}{(\phi_1^2 - \phi_2^2)(\phi_3 - \phi_1)} - \frac{g(\phi_2)(\phi_1 + \phi_3)}{(\phi_1^2 - \phi_2^2)(\phi_3 - \phi_2)} \right)$$
$$= -2\pi i e^{\phi_3(\xi)x} \phi_3^{-1} \big(g(\phi_1)\phi_1\phi_1' + g(\phi_2)\phi_2\phi_2' \big).$$

Therefore we obtain

$$I=-\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}e^{px}\int_{-i\infty}^{i\infty}\frac{e^{\xi t}}{p^3+\xi}H(p,\xi)\,d\xi\,dp=-\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}e^{\xi t}\int_{-i\infty}^{i\infty}\frac{e^{px}}{p^3+\xi}H(p,\xi)\,d\xi\,dp$$

$$=-\frac{1}{2\pi i}\int_0^{\infty}\sum_{j=1}^{2}\int_{-i\infty}^{i\infty}e^{\xi t}e^{\phi_3(\xi)x}\phi_3^{-1}(\xi)\phi_j(\xi)\phi_j'(\xi)e^{-y\phi_j(\xi)}$$

$$\times\left(u_0(y)+\int_0^{\infty}e^{-\xi\tau}f(y,\tau)\,d\tau\right)d\xi\,dy$$

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\xi t}e^{\phi_3(\xi)x}\phi_3^{-1}(\xi)\sum_{j=1}^{2}\phi_j^2(\xi)\phi_j(\xi)\int_0^{\infty}e^{-\xi\tau}h(\tau)\,d\tau\,d\xi$$

$$=-\frac{1}{2\pi i}\int_0^{\infty}\left(\sum_{j=1}^{2}\int_{-i\infty}^{i\infty}e^{\xi t}e^{\phi_3(\xi)x}\phi_3^{-1}(\xi)\phi_j(\xi)\phi_j'(\xi)e^{-y\phi_j(\xi)}\,d\xi\right)u_0(y)\,dy$$

$$-\frac{1}{2\pi i}\int_0^{t}d\tau\int_0^{\infty}\left(\sum_{j=1}^{2}\int_{-i\infty}^{i\infty}e^{\xi(t-\tau)}e^{\phi_3(\xi)x}\phi_3^{-1}(\xi)\phi_j(\xi)\phi_j'(\xi)e^{-y\phi_j(\xi)}\,d\xi\right)f(y,\tau)\,dy$$

$$-\frac{1}{\pi i}\int_0^{t}\int_{-i\infty}^{i\infty}e^{\xi(t-\tau)}e^{\phi_3(\xi)x}\phi_3(\xi)\phi_3'(\xi)\,d\xi\,h(\tau)\,d\tau.$$

Hence (2.7) and (2.8) give the following integral representation of the solution $u(x,t)$ to the problem (2.1) with $h(t)\equiv 0$

$$u(x,t)=\int_0^{\infty}u_0(y)F(x,y,t)\,dy+\int_0^{t}d\tau\int_0^{\infty}f(x,y,\tau)F(x,y,t-\tau)\,dy, \tag{2.9}$$

where

$$F(x,y,t)=\frac{1}{2\pi i}\left(\int_{-i\infty}^{i\infty}e^{-p^3t+p(x-y)}\,dp-\int_{-i\infty}^{i\infty}e^{\xi t+\phi_3(\xi)x}\phi_3^{-1}(\xi)\sum_{j=1}^{2}e^{-\phi_j(\xi)y}\phi_j(\xi)\phi_j'(\xi)\,d\xi\right).$$

3. Preliminaries

In this section we prepare useful lemma which is needed to obtain our results. As in Section 2 we denote by

$$\phi_1(\xi)=\big(\xi\exp(i\pi)\big)^{\frac{1}{3}},\qquad \phi_2(\xi)=\big(\xi\exp(-i\pi)\big)^{\frac{1}{3}},\qquad \phi_3(\xi)=\big(\xi\exp(3i\pi)\big)^{\frac{1}{3}}, \tag{3.1}$$

the roots of the equation $p^3 = -\xi$, such that $\operatorname{Re}\phi_l(\xi) > 0$, $l = 1, 2$, $\operatorname{Re}\phi_3(\xi) < 0$, for all $\operatorname{Re}\xi > 0$. We introduce the following functions

$$\mathcal{G}_k(x, y, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-p^3 t + px} \left(e^{-py} - \sum_{j=0}^{k} (-1)^j \frac{p^j y^j}{j!} \right) dp \tag{3.2}$$

and

$$\mathcal{F}_k(x, y, t) = -\frac{1}{2\pi i} \sum_{j=1}^{2} \int_{-i\infty}^{i\infty} e^{\xi t + \phi_3(\xi) x} \phi_3^{-1}(\xi) \left(e^{-\phi_j(\xi) y} - \sum_{l=0}^{k} (-1)^l \frac{\phi_j^l(\xi) y^l}{l!} \right) \phi_j \phi_j'(\xi)\, d\xi. \tag{3.3}$$

Lemma 3.1. *We have for $m \geqslant 0$, $t > 0$*

$$\begin{aligned} &\left\| x^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}_0(x, y, t) \right\| \\ &\quad \leqslant C \begin{cases} t^{-\frac{2n+1-m}{6}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle + t^{-\frac{n+1}{3}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle^{\frac{m+3}{2}} + t^{-\frac{3n+2}{6} + \frac{1}{12}} \langle t \rangle^{-\frac{1}{4}} \langle y \rangle^{\frac{m+2+n}{2}}, \\ t^{-\frac{2n+1-m}{6}} + t^{-\frac{n+1}{3}} \langle y \rangle^{\frac{m+1}{2}} + t^{-\frac{3n+2}{6} + \frac{1}{12}} \langle y \rangle^{\frac{m+2+n}{2} - \frac{3}{4}}, \end{cases} \end{aligned}$$

for $n \geqslant 1$ and

$$\left\| x^{\frac{m}{2}} \mathcal{G}_0(x, y, t) \right\| \leqslant C \begin{cases} t^{-\frac{1-m}{6}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle + t^{-\frac{1}{3}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle^{\frac{m+3}{2}} + t^{-\frac{1}{2}} \langle t \rangle^{-\frac{1}{4}} \langle y \rangle^{\frac{m+2}{2} + \frac{3}{4}}, \\ t^{-\frac{1-m}{6}} + t^{-\frac{1}{3}} \langle y \rangle^{\frac{m+1}{2}} + t^{-\frac{1}{2}} \langle y \rangle^{\frac{m+2}{2}}. \end{cases}$$

Proof. We let $p^3 t = z^3$, then we have

$$\begin{aligned} \partial_x^{(n)} \mathcal{G}_0(x, y, t) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-p^3 t} e^{px} \left(e^{-py} - 1 \right) p^n\, dp \\ &= \frac{1}{2\pi i} t^{-\frac{n+1}{3}} \int_{-i\infty}^{i\infty} e^{-z^3} z^n e^{z\tilde{x}} \left(e^{-z\tilde{y}} - 1 \right) dz = t^{-\frac{n+1}{3}} \left(Ai^{(n)}(\tilde{x} - \tilde{y}) - Ai^{(n)}(\tilde{x}) \right), \end{aligned}$$

where $Ai(q)$ is Airy function

$$Ai(q) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-z^3 + zq}\, dz.$$

It is known that we have

$$\left| Ai^{(n)}\left(-|q| \right) \right| \leqslant C \left(1 + |q| \right)^{\frac{2n-1}{4}}$$

and for any $\beta > 0$

$$\left|Ai^{(n)}\left(|q|\right)\right| \leqslant C\left(1+|q|\right)^{-\beta},$$

see, e.g., [1]. So we get for $\xi \in (\tilde{x}-\tilde{y}, \tilde{x})$, $\tilde{x} = x/\sqrt[3]{t}$ and $\tilde{y} = y/\sqrt[3]{t}$, $\mu \in [0,1]$, $x < 2y$

$$\begin{aligned}\left|\partial_x^{(n)}\mathcal{G}_0(x,y,t)\right| &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}\left|Ai^{(n+1)}(\xi)\right|^{\mu}\left(\left|Ai^{(n)}(\tilde{x}-\tilde{y})\right|^{1-\mu}+\left|Ai^{(n)}(\tilde{x})\right|^{1-\mu}\right)\\ &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}\left(1+|\tilde{y}-\tilde{x}|\right)^{\frac{(2n+1)\mu}{4}}\left(\left(1+|\tilde{y}-\tilde{x}|\right)^{\frac{(2n-1)}{4}(1-\mu)}+(1+\tilde{x})^{-\beta}\right).\end{aligned}$$

Therefore we get for $y > \sqrt[3]{t}$ and $(2n-1)/2+\mu \geqslant 0$

$$\begin{aligned}\left(\int_0^{2y} x^m\left|\partial_x^{(n)}\mathcal{G}_0(x,y,t)\right|^2 dx\right)^{\frac{1}{2}} &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}\left(\int_0^{2y} x^m\left(1+|\tilde{y}-\tilde{x}|\right)^{\frac{(2n-1)}{2}+\mu} dx\right)^{\frac{1}{2}}\\ &\leqslant Ct^{-\frac{n+1+\mu}{3}-\frac{1}{3}(\frac{(2n-1)}{4}+\frac{1}{2}\mu)} y^{\frac{(2n-1)}{4}+\frac{3}{2}\mu}\left(\int_0^{2y} x^m\, dx\right)^{\frac{1}{2}}\\ &\leqslant Ct^{-\frac{n+1+\mu}{3}-\frac{1}{3}(\frac{(2n-1)}{4}+\frac{1}{2}\mu)} y^{\frac{(2n-1)}{4}+\frac{3}{2}\mu+\frac{m+1}{2}}\end{aligned}$$

and for $y < \sqrt[3]{t}$

$$\left(\int_0^{2y} x^m\left|\mathcal{G}_0(x,y,t)\right|^2 dx\right)^{\frac{1}{2}} \leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}\left(\int_0^{2y} x^m\, dx\right)^{\frac{1}{2}} \leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu+\frac{m+1}{2}}.$$

Since $Ai^{(n)}(|\xi|) \leqslant C(1+|\xi|)^{-10}$ we get for $x > 2y$

$$\begin{aligned}\left|\partial_x^{(n)}\mathcal{G}_0(x,y,t)\right| &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}\left|Ai^{(n+1)}(\xi)\right|^{\mu}\left(\left|Ai^{(n)}(\tilde{x}-\tilde{y})\right|^{1-\mu}+\left|Ai^{(n)}(\tilde{x})\right|^{1-\mu}\right)\\ &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}(1+\tilde{x})^{-10}.\end{aligned}$$

So letting $x = \sqrt[3]{t}z$ we obtain

$$\begin{aligned}\left(\int_y^{\infty} x^m\left|\partial_x^{(n)}\mathcal{G}_0(x,y,t)\right|^2 dx\right)^{\frac{1}{2}} &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu}\left(\int_y^{\infty} x^m(1+\tilde{x})^{-20} dx\right)^{\frac{1}{2}}\\ &\leqslant Ct^{-\frac{n+1+\mu}{3}} y^{\mu} t^{-\frac{m}{6}+\frac{1}{6}}\left(\int_0^{\infty} z^{2m}(2+z)^{-20} dz\right)^{\frac{1}{2}}\\ &\leqslant Ct^{-\frac{2n+1+2\mu-m}{6}} y^{\mu}.\end{aligned}$$

Therefore we have for $t > 0$ and $\mu_j \in [0, 1]$, $j = 1, 2, 3$, $(2n-1)/2 + \mu_1 \geqslant 0$

$$\left\| x^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}_0(x, y, t) \right\| \leqslant C t^{-\frac{n+1+\mu_1}{3} - \frac{1}{3}(\frac{(2n-1)}{4} + \frac{1}{2}\mu_1)} y^{\frac{(2n-1)}{4} + \frac{3}{2}\mu_1 + \frac{m+1}{2}} + C t^{-\frac{n+1+\mu_2}{3}} y^{\mu_2 + \frac{m+1}{2}} + C t^{-\frac{2n+1+2\mu_3-m}{6}} y^{\mu_3}.$$

Thus if we take $\mu_2, \mu_3 = 0, 1$ and $\mu_1 = 1/2, 1$ for $n = 0$ and $\mu_1 = 0, 1/2$ for $n = 1, 2$, then we have the desired estimates. Lemma 3.1 is proved. □

Lemma 3.2. *We have* $m \geqslant 0$, $t > 0$

$$\left\| x^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}_1(x, y, t) \right\| \leqslant \begin{cases} C t^{-\frac{n+1}{3}} \langle y \rangle^{\frac{m+1}{2}} + C t^{-\frac{2n+1-m}{6}} + C t^{-\frac{3n+2}{6} + \frac{1}{12}} \langle y \rangle^{\frac{m+1+n}{2} - \frac{1}{4}}, \\ C t^{-\frac{n+1}{3}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle^{\frac{m+3}{2}} + C t^{-\frac{2n+1-m}{6}} \langle t \rangle^{-\frac{2}{3}} \langle y \rangle^{2} + C t^{-\frac{3n+2}{6} + \frac{1}{12}} \langle t \rangle^{-\frac{1}{4}} \langle y \rangle^{\frac{m+2+n}{2}}, \end{cases}$$

for $n = 1, 2$ *and*

$$\left\| x^{\frac{m}{2}} \mathcal{G}_1(\cdot, y, t) \right\| \leqslant C t^{-\frac{1}{3}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle^{\frac{m+3}{2}} + C t^{-\frac{1-m}{6}} \langle t \rangle^{-\frac{2}{3}} \langle y \rangle^{2} + C t^{-\frac{1}{2}} \langle t \rangle^{-\frac{1}{3}} \langle y \rangle^{\frac{m+3}{2}}.$$

Proof. Changing the variable of integration $p^3 t = z^3$ we have

$$\begin{aligned} \partial_x^{(n)} \mathcal{G}_1(x, y, t) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-p^3 t} e^{px} \left(e^{-py} - \sum_{k=0}^{1} (-1)^k (py)^k \right) p^n \, dp \\ &= \frac{1}{2\pi i} t^{-\frac{n+1}{3}} \int_{-i\infty}^{i\infty} e^{-z^3} z^n e^{z\tilde{x}} \left(e^{-z\tilde{y}} - \sum_{k=0}^{1} (-1)^k (z\tilde{y})^k \right) dz \\ &= t^{-\frac{n+1}{3}} \left(Ai^{(n)}(\tilde{x} - \tilde{y}) - \sum_{j=0}^{1} (-1)^j \frac{\tilde{y}^j}{j!} Ai^{(n+j)}(\tilde{x}) \right). \end{aligned}$$

We again use the asymptotic properties of the Airy function such that

$$\left| Ai^{(n)}\big(-|q|\big) \right| \leqslant C \big(1 + |q|\big)^{\frac{2n-1}{4}}$$

and for any $\beta > 0$

$$\left| Ai^{(n)}\big(|q|\big) \right| \leqslant C \big(1 + |q|\big)^{-\beta}.$$

Then we get for $\xi \in (\tilde{x}-\tilde{y},\tilde{x})$, $\tilde{x} = x/\sqrt[3]{t}$ and $\tilde{y} = y/\sqrt[3]{t}$, $\mu \in [0,1]$, $x < 2y$

$$\begin{aligned}
\left|\partial_x^{(n)}\mathcal{G}_1(x,y,t)\right| &\leqslant Ct^{-\frac{n+1+2\mu}{3}} y^{2\mu}\left|Ai^{(n+2)}(\xi)\right|^{\mu} \\
&\quad\times \left(\left|Ai^{(n)}(\tilde{x}-\tilde{y})\right|^{1-\mu} + \left|Ai^{(n)}(\tilde{x})\right|^{1-\mu} + \left|Ai^{(n+1)}(\tilde{x})\right|^{1-\mu}\tilde{y}^{1-\mu}\right) \\
&\leqslant t^{-\frac{n+1+2\mu}{3}} y^{2\mu}\left(1+|\tilde{y}-\tilde{x}|\right)^{\frac{(2n+3)\mu}{4}} \\
&\quad\times \left(\left(1+|\tilde{y}-\tilde{x}|\right)^{\frac{2n-1}{4}(1-\mu)} + (1+\tilde{x})^{-\beta} + (1+\tilde{x})^{-\beta}\tilde{y}^{1-\mu}\right).
\end{aligned}$$

Therefore we get for $y > \sqrt[3]{t}$ and $(2n-1)/2 + 2\mu \geqslant 0$

$$\begin{aligned}
\left(\int_0^{2y} x^m \left|\partial_x^{(n)}\mathcal{G}_1(x,y,t)\right|^2 dx\right)^{\frac{1}{2}} &\leqslant Ct^{-\frac{n+1+2\mu}{3}} y^{2\mu}\left(\int_0^{2y} x^m \left(1+|\tilde{y}-\tilde{x}|\right)^{\frac{(2n-1)}{2}+2\mu} dx\right)^{\frac{1}{2}} \\
&\leqslant Ct^{-\frac{n+1+2\mu}{3}-\frac{1}{3}(\frac{(2n-1)}{4}+\mu)} y^{\frac{(2n-1)}{4}+3\mu}\left(\int_0^{2y} x^m\, dx\right)^{\frac{1}{2}} \\
&\leqslant Ct^{-\frac{n+1+2\mu}{3}-\frac{1}{3}(\frac{(2n-1)}{4}+\mu)} \langle y\rangle^{\frac{(2n-1)}{4}+3\mu+\frac{m+1}{2}}.
\end{aligned}$$

In the case $y < \sqrt[3]{t}$ we get

$$\left(\int_0^{2y} x^m \left|\mathcal{G}_1(x,y,t)\right|^2 dx\right)^{\frac{1}{2}} \leqslant Ct^{-\frac{1+2\mu}{3}} y^{2\mu}\left(\int_0^{2y} x^m\, dx\right)^{\frac{1}{2}} \leqslant Ct^{-\frac{1+2\mu}{3}}\langle y\rangle^{2\mu+\frac{1+m}{2}}.$$

Since $|Ai^{(n)}(|\xi|)| \leqslant C(1+|\xi|)^{-10}$ we have for $x > 2y$

$$\begin{aligned}
\left|\partial_x^{(n)}\mathcal{G}_1(x,y,t)\right| &\leqslant Ct^{-\frac{n+1}{3}} \tilde{y}^{2\mu}\left|Ai^{(n+2)}(\xi)\right|^{\mu} \\
&\quad\times \left(\left|Ai^{(n)}(\tilde{x}-\tilde{y})\right|^{1-\mu} + \left|Ai^{(n)}(\tilde{x})\right|^{1-\mu} + \left|Ai^{(n+1)}(\tilde{x})\right|^{1-\mu}\tilde{y}^{1-\mu}\right) \\
&\leqslant Ct^{-\frac{n+1+2\mu}{3}} y^{2\mu}(1+\tilde{y})^{1-\mu}(1+\tilde{x})^{-10} \leqslant Ct^{-\frac{n+1+2\mu}{3}} y^{2\mu}(1+\tilde{x})^{-9}
\end{aligned}$$

and which gives us with $x = \sqrt[3]{t}z$

$$\begin{aligned}
\left(\int_{2y}^{\infty} x^m \left|\partial_x^{(n)}\mathcal{G}_1(x,y,t)\right|^2 dx\right)^{\frac{1}{2}} &\leqslant Ct^{-\frac{n+1+2\mu}{3}} y^{2\mu}\left(\int_{2y}^{\infty} x^m (2+\tilde{x})^{-18}\, dx\right)^{\frac{1}{2}} \\
&\leqslant Ct^{-\frac{n+1+2\mu}{3}} y^{2\mu} t^{-\frac{m}{6}+\frac{1}{6}}\left(\int_0^{\infty} z^{2m}(2+z)^{-18}\, dz\right)^{\frac{1}{2}} \\
&\leqslant Ct^{-\frac{2n+1-m+4\mu}{6}} y^{2\mu} \leqslant Ct^{-\frac{2n+1-m+4\mu}{6}}\langle y\rangle^{2\mu}.
\end{aligned}$$

Therefore we obtain for $t > 0$,

$$\left\| x^{\frac{m}{2}} \partial_x^{(n)} \mathcal{G}_1(x, y, t) \right\| \leqslant C t^{-\frac{n+1+2\mu_1}{3} - \frac{1}{3}(\frac{(2n-1)}{4} + \mu_1)} \langle y \rangle^{\frac{(2n-1)}{4} + 3\mu_1 + \frac{m+1}{2}} + C t^{-\frac{n+1+2\mu_2}{3}} \langle y \rangle^{2\mu_2 + \frac{1+m}{2}} + C t^{-\frac{2n+1-m+4\mu_3}{6}} \langle y \rangle^{2\mu_3},$$

where $\mu_j \in [0, 1]$, $j = 1, 2, 3$, $(2n-1)/2 + 2\mu_1 \geqslant 0$. If we take $\mu_2 = 0, 1/2$; $\mu_3 = 0, 1$ and $\mu_1 = 0, 1/4$ for $n = 1, 2$, then we have the result for $n = 1, 2$ and if we take $\mu_1 = 1/4, 5/12$ for $n = 0$, then we have the last estimate of the lemma. Lemma 3.2 is proved. □

Lemma 3.3. *We have for $t > 1$, $n, m \geqslant 0$, $k = 0, 1$, $\delta \in [0, 1]$*

$$\left\| x^{\frac{m}{2}} \partial_x^{(n)} \mathcal{F}_k(x, y, t) \right\| \leqslant C t^{-\frac{1-m+2n+2k+2\delta}{6}} \langle y \rangle^{k+\delta}.$$

Proof. Using $\phi_j'(\xi) = -\frac{1}{3\phi_j^2}$ we get for Laplace transform of function $\mathcal{F}_k(x, y, t)$

$$\mathcal{L}\left\{ \partial_x^{(n)} \mathcal{F}_k(x, y, t) \right\} = \widehat{\partial_x^{(n)} \mathcal{F}_k}(p, y, t) = -\frac{1}{6\pi i} \sum_{j=1}^{2} \int_{-i\infty}^{i\infty} e^{\xi t} \frac{\phi_3^{n-1}(\xi)}{(p - \phi_3(\xi))} \left(e^{-\phi_j(\xi) y} - \sum_{l=0}^{k} (-1)^l \frac{\phi_j^l(\xi) y^l}{l!} \right) \phi_j^{-1}(\xi)\, d\xi.$$

Using the Plancherel Theorem we get for $t > 1$, $m \geqslant 0$ is an integer

$$\begin{aligned} &\left\| x^m \partial_x^{(n)} \mathcal{F}_k(x, y, t) \right\| \\ &\leqslant \left\| \partial_p^{(m)} \widehat{\partial_x^{(n)} \mathcal{F}_k}(p, y, t) \right\| \\ &\leqslant C \left\| \sum_{j=1}^{2} \int_{-i\infty}^{i\infty} e^{\xi t} \frac{\phi_3^{n-1}(\xi)}{(p - \phi_3(\xi))^{m+1}} \left(e^{-\phi_j(\xi) y} - \sum_{l=0}^{k} (-1)^l \frac{\phi_j^l(\xi) y^l}{l!} \right) \phi_j^{-1}(\xi)\, d\xi \right\|. \end{aligned}$$

Since $e^{\xi t} = \frac{1}{1+\xi t} \partial_\xi \xi e^{\xi t}$ integrating by part and using estimate for $\delta \in [0, 1]$

$$\sum_{j=1}^{2} \left| e^{-\phi_j(\xi) y} - \sum_{l=0}^{k} (-1)^l \frac{\phi_j^l(\xi) y^l}{l!} \right| \leqslant C \sum_{j=1}^{2} \left| \phi_j(\xi) y \right|^{k+\delta} \leqslant C y^{k+\delta} |\xi|^{\frac{k+\delta}{3}}$$

we get for $t > 1$

$$\sum_{j=1}^{2} \int_{-i\infty}^{i\infty} e^{\xi t} \frac{\phi_3^{n-1}(\xi)}{(p - \phi_3(\xi))^{m+1}} \left(e^{-\phi_j(\xi) y} - \sum_{l=0}^{k} (-1)^l \frac{\phi_j^l(\xi) y^l}{l!} \right) \phi_j^{-1}(\xi)\, d\xi = \int_{-i\infty}^{i\infty} \frac{e^{\xi t}}{1 + \xi t} \left(\xi M_\xi(\xi, p) + \frac{M(\xi, p)\xi}{1 + \xi t} \right) d\xi,$$

where for $k = 0, 1$

$$M(\xi,p) = \phi_3^{n-1}(\xi)\frac{1}{(p-\phi_3(\xi))^{m+1}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-1}(\xi)$$
$$= \frac{y^{k+\delta}}{(p-\phi_3(\xi))^{m+1}}O\big(|\xi|^{\frac{n-2+k+\delta}{3}}\big),$$
$$\xi M_\xi(\xi,p) = \xi\phi_3^{n-4}(\xi)\frac{1}{(p-\phi_3(\xi))^{m+1}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-1}(\xi)$$
$$-\frac{\xi y\phi_3^{n-1}(\xi)}{3}\frac{1}{(p-\phi_3(\xi))^{m+1}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k-1}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-3}(\xi)$$
$$+\frac{\xi\phi_3^{n-1}(\xi)}{3}\frac{1}{(p-\phi_3(\xi))^{m+1}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-4}(\xi)$$
$$+\xi\frac{\phi_3^{n-3}(\xi)}{(p-\phi_3(\xi))^{m+2}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-1}(\xi)$$
$$= \frac{y^{k+\delta}}{(p-\phi_3(\xi))^{m+1}}O\big(|\xi|^{\frac{n-2+k+\delta}{3}}\big).$$

Note that when we integrate by parts substitution in $\xi = 0$ vanish for $\operatorname{Re} p = 0$ since integrand is estimated as

$$\left|e^{\xi t}\frac{1}{1+\xi t}\xi\frac{\phi_3^{n-1}(\xi)}{(p-\phi_3(\xi))^{m+1}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-1}(\xi)\right| \leqslant C|\xi|^{\frac{n+1}{3}}.$$

So we get

$$\big\|x^m\partial_x^{(n)}\mathcal{F}_k(\cdot,y,t)\big\|$$
$$= \left(\int_{-i\infty}^{i\infty}|dp|\left|\int_{-i\infty}^{i\infty}e^{\xi t}\frac{\phi_3^{n-1}(\xi)}{(p-\phi_3(\xi))^{m+1}}\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y} - \sum_{l=0}^{k}(-1)^l\frac{\phi_j^l(\xi)y^l}{l!}\right)\phi_j^{-1}(\xi)\,d\xi\right|^2\right)^{\frac{1}{2}}$$
$$\times\left(\int_{-i\infty}^{i\infty}|dp|\int_{-i\infty}^{i\infty}\frac{1}{1+\xi t}\left(\xi M_\xi(\xi,p)+\frac{M(\xi,p)\xi t}{1+\xi t}\right)d\xi\right.$$
$$\left.\times\overline{\int_{-i\infty}^{i\infty}\frac{1}{1+\xi_1 t}\left(\xi_1 M_{\xi_1}(\xi_1,p)+\frac{M(\xi_1,p)\xi_1 t}{1+\xi_1 t}\right)d\xi_1}\right)^{\frac{1}{2}}$$

$$= y^{1+k}\Bigg(\int_{-i\infty}^{i\infty} \frac{1}{1+\xi t}\bigg(O\big(|\xi|^{\frac{n-2+k+\delta}{3}}\big) + \frac{O(|\xi|^{\frac{n-2+k+\delta}{3}})\xi t}{1+\xi t}\bigg)\, d\xi$$

$$\overline{\times \int_{-i\infty}^{i\infty} \frac{1}{1+\xi_1 t}\bigg(O\big(|\xi_1|^{\frac{n-2+k+\delta}{3}}\big) + \frac{O(|\xi_1|^{\frac{n-2+k+\delta}{3}})\xi_1 t}{1+\xi_1 t}\bigg)\, d\xi_1}$$

$$\times \int_{-i\infty}^{i\infty} \frac{1}{(p-\phi_3(\xi))^{m+1}\overline{(p-\phi_3(\xi_1))^{m+1}}}\, |dp| \Bigg)^{\frac{1}{2}}.$$

Since for $\operatorname{Re}\xi = 0$

$$\left| \int_{-i\infty}^{i\infty} \frac{1}{(p-\phi_3(\xi))^{m+1}\overline{(p-\phi_3(\xi_1))^{m+1}}}\, |dp| \right|$$

$$\leqslant C\left| \int_{-i\infty}^{i\infty} \frac{1}{(|p+\xi^{\frac{1}{3}}| + |\xi|^{\frac{1}{3}})^{m+1}(|p+\xi_1^{\frac{1}{3}}| + |\xi_1|^{\frac{1}{3}})^{m+1}}\, |dp| \right|$$

$$\leqslant C\left(\left| \int_{-i\infty}^{i\infty} \frac{1}{(|p+\xi^{\frac{1}{3}}|^2 + |\xi|^{\frac{2}{3}})^{m+1}}\, |dp| \right|\right)^{\frac{1}{2}} \left(\left| \int_{-i\infty}^{i\infty} \frac{1}{(|p+\xi_1^{\frac{1}{3}}|^2 + |\xi_1|^{\frac{2}{3}})^{m+1}}\, |dp| \right|\right)^{\frac{1}{2}}$$

$$\leqslant C|\xi|^{-\frac{m+1}{3}}|\xi_1|^{-\frac{m+1}{3}}$$

we find that

$$\left\| x^m \partial_x^{(n)} \mathcal{F}_k(\cdot, y, t) \right\| \leqslant C y^{k+\delta} \int_{-i\infty}^{i\infty} \frac{1}{1+|\xi| t} |\xi|^{\frac{n-2+k+\delta}{3} - \frac{m+1}{6}}\, |d\xi|.$$

And therefore we have for $m \geqslant 0$

$$\left\| x^m \partial_x^{(n)} \mathcal{F}_k(\cdot, y, t) \right\| \leqslant C y^{k+\delta} t^{-\frac{1-2m+2n+2k+2\delta}{6}}. \tag{3.4}$$

For any fractional $m \geqslant 0$ estimate (3.4) is obtained by interpolation. Lemma 3.3 is proved. □

We introduce the following operator

$$\Phi(x,t)\psi(\tau) = \int_0^\infty F(x,y,t)\psi(y,\tau)\, dy, \tag{3.5}$$

where the function $F(x,y,t)$ is the Green function of linear problem (2.1)

$$F(x,y,t)=\frac{1}{2\pi i}\left(\int_{-i\infty}^{i\infty} e^{-p^3t+p(x-y)}\,dp-\int_{-i\infty}^{i\infty} e^{\xi t+\phi_3(\xi)x}\phi_3^{-1}(\xi)\sum_{j=1}^{2}e^{-\phi_j(\xi)y}\phi_j(\xi)\phi_j'(\xi)\,d\xi\right).$$

Lemma 3.4. *The following estimates are valid for $m\geqslant 0$, $t>0$*

$$\left\|x^{\frac{m}{2}}\partial_x^{(n)}\Phi(t)f\right\|\leqslant C\begin{cases} t^{-\frac{4+2n}{6}}\|f\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}+t^{-\frac{3+2n-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}+t^{-\frac{3+3n}{6}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2+n}{2}}}, \\ t^{-\frac{n+1}{3}}\|f\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}}+t^{-\frac{1+2n-m}{6}}\|f\|_{\mathbf{L}^1} \\ \quad +t^{-\frac{3n+2}{6}+\frac{1}{12}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2+n}{2}-\frac{3}{4}}}+t^{-\frac{3+2n-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}\end{cases}$$

if $n=1,2$ and

$$\left\|x^{\frac{m}{2}}\Phi(t)f\right\|\leqslant C\begin{cases} t^{-\frac{2}{3}}\|f\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}+t^{-\frac{3-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}+t^{-\frac{3}{4}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}}, \\ t^{-\frac{1}{3}}\|f\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}}+t^{-\frac{1-m}{6}}\|f\|_{\mathbf{L}^1} \\ \quad +t^{-\frac{1}{2}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2}{2}}}+t^{-\frac{3-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}\end{cases}$$

provided that the right-hand sides are finite.

Proof. Using

$$\sum_{j=1}^{3}\phi_j'(\xi)\phi_j(\xi)=0,$$

we obtain the following representation for Green function $F(x,y,t)$

$$\begin{aligned}F(x,y,t)&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-p^3t+px}\left(e^{-py}-1\right)dp\\ &\quad-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t+\phi_3(\xi)x}\phi_3^{-1}(\xi)\sum_{j=1}^{2}\left(e^{-\phi_j(\xi)y}-1\right)\phi_j(\xi)\phi_j'(\xi)\,d\xi\\ &=\mathcal{G}_0(x,y,t)+\mathcal{F}_0(x,y,t).\end{aligned}$$

The functions $\mathcal{G}_0(x,y,t)$, $\mathcal{F}_0(x,y,t)$ was defined in (3.3) and (3.2) with $k=0$. We obtain the following representation

$$\Phi(x,t)f(x)=\int_0^\infty F(x,y,t)f(y)\,dy=\int_0^\infty \mathcal{G}_0(x,y,t)f(y)\,dy+\int_0^\infty \mathcal{F}_0(x,y,t)f(y)\,dy.$$

From Lemma 3.1 we have for $t>0$

$$\begin{aligned}&\left\|x^{\frac{m}{2}}\partial_x^{(n)}\mathcal{G}_0(x,y,t)\right\|\\ &\quad\leqslant C\begin{cases}t^{-\frac{2n+1-m}{6}}\langle t\rangle^{-\frac{1}{3}}\langle y\rangle+t^{-\frac{n+1}{3}}\langle t\rangle^{-\frac{1}{3}}\langle y\rangle^{\frac{m+3}{2}}+t^{-\frac{3n+2}{6}+\frac{1}{12}}\langle t\rangle^{-\frac{1}{4}}\langle y\rangle^{\frac{m+2+n}{2}},\\ t^{-\frac{2n+1-m}{6}}+t^{-\frac{n+1}{3}}\langle y\rangle^{\frac{m+1}{2}}+t^{-\frac{3n+2}{6}+\frac{1}{12}}\langle y\rangle^{\frac{m+2+n}{2}-\frac{3}{4}}\end{cases}\end{aligned}$$

if $n=1,2$ and

$$\left\|x^{\frac{m}{2}}\mathcal{G}_0(x,y,t)\right\|\leqslant C\begin{cases}t^{-\frac{1-m}{6}}\langle t\rangle^{-\frac{1}{3}}\langle y\rangle+t^{-\frac{1}{3}}\langle t\rangle^{-\frac{1}{3}}\langle y\rangle^{\frac{m+3}{2}}+t^{-\frac{1}{2}}\langle t\rangle^{-\frac{1}{4}}\langle y\rangle^{\frac{m+2}{2}+\frac{3}{4}},\\ t^{-\frac{1-m}{6}}+t^{-\frac{1}{3}}\langle y\rangle^{\frac{m+1}{2}}+t^{-\frac{1}{2}}\langle y\rangle^{\frac{m+2}{2}}.\end{cases}$$

From Lemma 3.3 we have for $t>1$, $n\geqslant 0$, $k=0$, $\delta=1$

$$\left\|x^{\frac{m}{2}}\partial_x^{(n)}\mathcal{F}_0(x,y,t)\right\|\leqslant Ct^{-\frac{3+2n-m}{6}}\langle y\rangle.$$

Therefore

$$\left\|x^{\frac{m}{2}}\partial_x^{(n)}\Phi(t)f\right\|\leqslant C\begin{cases}t^{-\frac{4+2n}{6}}\|f\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}+t^{-\frac{3+2n-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}+t^{-\frac{3+3n}{6}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2+n}{2}}},\\ t^{-\frac{n+1}{3}}\|f\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}}+t^{-\frac{1+2n-m}{6}}\|f\|_{\mathbf{L}^1}\\ \qquad+t^{-\frac{3n+2}{6}+\frac{1}{12}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2+n}{2}-\frac{3}{4}}}+t^{-\frac{3+2n-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}\end{cases}$$

for $n=1,2$ and

$$\left\|x^{\frac{m}{2}}\Phi(t)f\right\|\leqslant C\begin{cases}t^{-\frac{2}{3}}\|f\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}+t^{-\frac{3-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}+t^{-\frac{3}{4}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}},\\ t^{-\frac{1}{3}}\|f\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}}+t^{-\frac{1-m}{6}}\|f\|_{\mathbf{L}^1}+t^{-\frac{1}{2}}\|f\|_{\mathbf{H}_1^{0,\frac{m+2}{2}}}+t^{-\frac{3-m}{6}}\|f\|_{\mathbf{H}_1^{0,1}}.\end{cases}$$

Lemma 3.4 is proved. □

Using

$$\sum_{j=1}^{3}\phi_j(\xi)=0,\qquad \sum_{j=1}^{3}\phi_j^2(\xi)=0,$$

which imply

$$\sum_{j=1}^{3}\phi_j'(\xi)=0,\qquad \sum_{j=1}^{3}\phi_j'(\xi)\phi_j(\xi)=0\quad\text{and}\quad \sum_{j=1}^{3}\phi_j^2(\xi)\phi_j'(\xi)=-1,$$

we obtain the following representation for the function $F(x,y,t)$

$$F(x,y,t)=\frac{3y}{2\pi i}\int_{-i\infty}^{i\infty}e^{-p^3t+px}\,dp+G(x,y,t)=3yt^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{t}}\right)+G(x,y,t),\quad (3.6)$$

where $G(x,y,t)=\mathcal{G}_1(x,y,t)+\mathcal{F}_1(x,y,t)$. The functions $\mathcal{G}_1(x,y,t)$, $\mathcal{F}_1(x,y,t)$ were defined in (3.2) and (3.3) with $k=1$.

Lemma 3.5. *Let the function $f(x,t)$ such that $\int_0^\infty xf(x,t)\,dx=0$ and satisfy the estimate*

$$\left\|x^{\frac{l}{2}}f(t)\right\|_{\mathbf{L}^1}\leqslant C\varepsilon^\beta(1+t)^{-\frac{8-l}{6}}g(t)^\alpha,\quad 0\leqslant l\leqslant 5,$$

where $\alpha=\frac{4}{5}$, $0<\beta$. We also assume that the function $g(t)$ satisfies the inequalities for $\eta>0$

$$\frac{1}{2}\big(1+\eta\log(1+t)\big)<g(t)<2\big(1+\eta\log(1+t)\big).$$

Then the following estimates are valid for $n=0,1$, $t>0$

$$\left\|x^{\frac{m}{2}}\int_0^t g^{-1}(\tau)\partial_x^{(n)}\Phi(t-\tau)f(\tau)\,d\tau\right\|\leqslant C\varepsilon^\beta g^{-1+\alpha}(t)t^{-\frac{3-m+2n}{6}},$$

where $0\leqslant m\leqslant 2$.

Proof. Since

$$\int_0^\infty xf(x,t)\,dx=0$$

using (3.6) we obtain the following representation

$$\Phi(x,t-\tau)f(\tau)=\int_0^\infty F(x,y,t-\tau)f(y,\tau)\,dy=\int_0^\infty G(x,y,t-\tau)f(y,\tau)\,dy.$$

From Lemma 3.2 we have for $t>0$, $m\geqslant 0$

$$\left\|x^{\frac{m}{2}}\partial_x\mathcal{G}_1(x,y,t)\right\|\leqslant C\begin{cases}t^{-\frac{2}{3}}\langle y\rangle^{\frac{m+1}{2}}+t^{-\frac{3-m}{6}}+Ct^{-\frac{5}{6}+\frac{1}{12}}\langle y\rangle^{\frac{m+2}{2}-\frac{1}{4}},\\ t^{-\frac{2}{3}}\langle t\rangle^{-\frac{1}{3}}\langle y\rangle^{\frac{m+3}{2}}+t^{-\frac{3-m}{6}}\langle t\rangle^{-\frac{2}{3}}\langle y\rangle^{2}+t^{-\frac{5}{6}+\frac{1}{12}}\langle t\rangle^{-\frac{1}{4}}\langle y\rangle^{\frac{m+3}{2}}\end{cases}\quad (3.7)$$

and

$$\left\| x^{\frac{m}{2}} \mathcal{G}_1(x, y, t) \right\| \leqslant C t^{-\frac{2}{3}} \langle y \rangle^{\frac{m+3}{2}} + C t^{-\frac{5-m}{6}} \langle y \rangle^2. \tag{3.8}$$

From Lemma 3.3 we have $k = 1$, $\delta \in [0, 1]$

$$\left\| x^{\frac{m}{2}} \partial_x^{(n)} \mathcal{F}_1(x, y, t) \right\| \leqslant C t^{-\frac{3-m+2n+2\delta}{6}} \langle y \rangle^{1+\delta}. \tag{3.9}$$

Therefore

$$\begin{aligned} \left\| x^{\frac{m}{2}} \Phi(t-\tau) f(\tau) \right\| &= \left\| \int_0^\infty x^{\frac{m}{2}} G(x, y, t-\tau) f(y, \tau)\, dy \right\| \\ &\leqslant C(t-\tau)^{-\frac{2}{3}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0, \frac{m+3}{2}}} + C(t-\tau)^{-\frac{5-m}{6}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0,2}}. \end{aligned} \tag{3.10}$$

Furthermore we have

$$\begin{aligned} \left\| x^{\frac{m}{2}} \partial_x \Phi(t-\tau) f(\tau) \right\| &= \left\| \int_0^\infty x^{\frac{m}{2}} \partial_x G(x, y, t-\tau) f(y, \tau)\, dy \right\| \\ &\leqslant C(t-\tau)^{-1} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0, \frac{m+3}{2}}} + C(t-\tau)^{-\frac{7-m}{6}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0,2}}, \\ \left\| x^{\frac{m}{2}} \partial_x \Phi(t-\tau) f(\tau) \right\| &= \left\| \int_0^\infty x^{\frac{m}{2}} \partial_x G(x, y, t-\tau) f(y, \tau)\, dy \right\| \\ &\leqslant C(t-\tau)^{-\frac{2}{3}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0, \frac{m+1}{2}}} + C(t-\tau)^{-\frac{3-m}{6}} \left\| f(\tau) \right\|_{\mathbf{L}^1} \\ &\quad + C(t-\tau)^{-\frac{3}{4}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0, \frac{m}{2}+\frac{3}{4}}} + C(t-\tau)^{-\frac{5-m}{6}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0,1}}. \end{aligned} \tag{3.11}$$

Also from the condition of the lemma for the function $g(t)$ we have for $t > 1$

$$t^{-\alpha} < \frac{1}{g(t)}, \quad \alpha > 0, \quad \text{and} \quad \sup_{\tau \in [\sqrt{t}, t]} \frac{1}{g(\tau)} < \frac{C}{g(t)}.$$

Hence we get for $t > 4$

$$\begin{aligned} &\left\| \int_0^t x^{\frac{m}{2}} g^{-1}(\tau) \partial_x \Phi(t-\tau) f(\tau)\, d\tau \right\| \\ &\quad \leqslant C \int_0^{\sqrt{t}} t^{-1} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0, \frac{m+3}{2}}} + t^{-\frac{7-m}{6}} \left\| f(\tau) \right\|_{\mathbf{H}_1^{0,2}}\, d\tau \end{aligned}$$

$$+ Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}} t^{-1}\|f(\tau)\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{7-m}{6}}\|f(\tau)\|_{\mathbf{H}_1^{0,2}}\,d\tau$$

$$+ C\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\|f(\tau)\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{2}{3}}\,d\tau$$

$$+ C\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\|f(\tau)\|_{\mathbf{H}_1^{0,\frac{m}{2}+\frac{3}{4}}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{3}{4}}\,d\tau$$

$$+ C\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\|f(\tau)\|_{\mathbf{L}^1}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{3-m}{6}}\,d\tau$$

and

$$\left\|\int_0^t x^{\frac{m}{2}} g^{-1}(\tau)\Phi(t-\tau)f(\tau)\,d\tau\right\| \leqslant C\int_0^{\sqrt{t}} t^{-\frac{2}{3}}\|f(\tau)\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{5-m}{6}}\|f(\tau)\|_{\mathbf{H}_1^{0,2}}\,d\tau$$

$$+ Cg^{-1}(t)\int_{\sqrt{t}}^{\frac{t}{2}} t^{-\frac{2}{3}}\|f(\tau)\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{5-m}{6}}\|f(\tau)\|_{\mathbf{H}_1^{0,2}}\,d\tau$$

$$+ C\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\|f(\tau)\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{2}{3}}\,d\tau$$

$$+ C\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\|f(\tau)\|_{\mathbf{H}_1^{0,2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{5-m}{6}}\,d\tau.$$

By the assumption we have

$$\|x^{\frac{m}{2}}f(t)\|_{\mathbf{L}^1} \leqslant C\varepsilon^{\beta}(1+t)^{-\frac{5}{6}}t^{\frac{m-3}{6}}g(t)^{\alpha}, \quad 0\leqslant m\leqslant 5,$$

hence

$$\left\|\int_0^t x^{\frac{m}{2}} g^{-1}(\tau)\partial_x\Phi(t-\tau)f(\tau)\,d\tau\right\|$$

$$\leqslant Cg^{\alpha}(t)\int_0^{\sqrt{t}} t^{-1}(1+\tau)^{-\frac{5}{6}}\tau^{\frac{m}{6}} + t^{-\frac{7-m}{6}}(1+\tau)^{-\frac{5}{6}}\tau^{\frac{1}{6}}\,d\tau$$

$$+ Cg^{-1+\alpha}(t)\int_{\sqrt{t}}^{\frac{t}{2}} t^{-1}(1+\tau)^{-\frac{5}{6}}\tau^{\frac{m}{6}} + t^{-\frac{7-m}{6}}(1+\tau)^{-\frac{5}{6}}\tau^{\frac{1}{6}}\,d\tau$$

$$+ Ct^{\frac{1}{3}}\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\left\|f(\tau)\right\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}} + Ct^{\frac{1}{4}}\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\left\|f(\tau)\right\|_{\mathbf{H}_1^{0,\frac{m+1}{2}+\frac{1}{4}}}$$

$$+ Ct^{\frac{3+m}{6}}\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\left\|f(\tau)\right\|_{\mathbf{L}^1}$$

$$\leqslant C\varepsilon^{\beta}g^{-1+\alpha}(t)t^{-\frac{5-m}{6}}$$

and

$$\left\|x^{\frac{m}{2}}\int_0^t g^{-1}(\tau)\Phi(t-\tau)f(\tau)\,d\tau\right\|$$

$$\leqslant C\int_0^{\sqrt{t}} t^{-\frac{2}{3}}(1+\tau)^{-\frac{5}{6}}\tau^{\frac{m}{6}} + t^{-\frac{5-m}{6}}(1+\tau)^{-\frac{5}{6}}\tau^{\frac{1}{6}}\,d\tau$$

$$+ C\int_{\sqrt{t}}^{\frac{t}{2}} t^{-\frac{2}{3}}g^{-1+\alpha}(t)(1+\tau)^{-\frac{5}{6}}\tau^{\frac{m}{6}} + t^{-\frac{5-m}{6}}g^{-1+\alpha}(t)(1+\tau)^{-\frac{5}{6}}\tau^{\frac{1}{6}}\,d\tau$$

$$+ Ct^{\frac{1}{3}}\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\left\|f(\tau)\right\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + Ct^{\frac{m+1}{6}}\sup_{\frac{t}{2}<\tau<t} g^{-1}(\tau)\left\|f(\tau)\right\|_{\mathbf{H}_1^{0,2}}$$

$$\leqslant \beta^{\beta}Cg^{-1+\alpha}(t)t^{-\frac{3-m}{6}}.$$

Lemma 3.5 is proved. □

Lemma 3.6. *We assume that $u_0 \in \mathbf{H}_1^{0,(m+3)/2}$ and $\|u_0\|_{\mathbf{H}_1^{0,(m+3)/2}} = \varepsilon$ is small enough and*

$$\lambda\int_0^{\infty} xu_0(x)\,dx = \lambda\theta < 0,$$

a function $v(x,t)$ satisfies the estimate for all $t > 0$

$$\|xvv_x\|_{\mathbf{L}_1} \leqslant C\varepsilon^{\beta}(1+t)^{-1}$$

and has the asymptotic representation for all $t > 0$, $m \geqslant 0$

$$\left\|\partial_x^{(n)}\big(v(t) - \Phi(t)u_0\big)\right\|_{\frac{m}{2}} \leqslant C\varepsilon^{\beta}g^{-1+\alpha}(t)(1+t)^{-\frac{3+2n-m}{6}},$$

where $1 < \beta < 2$. *We also assume that the function* $g(t)$ *is such that for all* $t > 0$

$$\frac{1}{2}\big(1+\eta\log(1+t)\big) < g(t) < 2\big(1+\eta\log(1+t)\big),$$

where $0 < \eta = -9\theta\lambda\int_0^\infty Ai'^2(z)\,dz \leqslant C\varepsilon$. *Then the following inequality is valid for all* $t > 0$

$$\frac{1}{2}\big(1+\eta\log(1+t)\big) < 1 + \frac{\lambda}{\theta}\int_0^t d\tau\int_0^\infty xvv_x\,dx < 2\big(1+\eta\log(1+t)\big). \tag{3.12}$$

Proof. For $t < 1$ the estimate (3.12) is trivial. Using (3.6) we have

$$\Phi(t)u_0 = 3\theta t^{-\frac{2}{3}} Ai'\left(\frac{x}{\sqrt[3]{t}}\right) + R(x,t),$$

where

$$R(x,t) = \int_0^\infty u_0(y)\big(\mathcal{G}_1(x,y,t) + \mathcal{F}_1(x,y,t)\big)\,dy.$$

We again use estimates (3.7)–(3.9) to find that for $t > 1$ and $n = 0, 1$

$$\begin{aligned}\left\|x^{\frac{m}{2}}\partial_x^{(n)}R(t)\right\| &\leqslant C\Big(t^{-\frac{5-m+2n}{6}}\|u_0\|_{\mathbf{H}_1^{0,2}} + t^{-\frac{4+2n}{6}}\|u_0\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}}\Big)\\ &\leqslant C\varepsilon t^{-\frac{3-m+2n}{6}}\max\big\{t^{-\frac{1}{3}}, t^{-\frac{m+1}{6}}\big\}.\end{aligned}$$

We have

$$\left\|Ai^{(n)}\left(\frac{\cdot}{\sqrt[3]{t}}\right)\right\|_{\frac{1}{2}}^2 = \int_0^\infty x\left|Ai^{(n)}\left(\frac{x}{\sqrt[3]{t}}\right)\right|^2 dx \leqslant Ct^{\frac{2}{3}}.$$

Therefore we have by a direct computation with

$$\begin{aligned}
&r = v - \Phi(t)u_0 = v - \frac{3\theta}{2\pi}t^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{t}}\right) - R(x,t),\\
&\left\|xvv_x - \frac{\theta^2}{4\pi^2}t^{-\frac{5}{3}}xAi'\left(\frac{x}{\sqrt[3]{t}}\right)Ai''\left(\frac{x}{\sqrt[3]{t}}\right)\right\|_1\\
&\quad = \left\|\frac{\theta}{2\pi}t^{-\frac{2}{3}}xAi'\left(\frac{x}{\sqrt[3]{t}}\right)(R_x + r_x)\right.\\
&\qquad \left. + \frac{\theta}{2\pi}t^{-1}xAi''\left(\frac{x}{\sqrt[3]{t}}\right)(R+r) + xRR_x + xrr_x + xRr_x + xR_xr\right\|_1\\
&\quad \leqslant C\theta t^{-\frac{1}{3}}\big(\|R_x\|_{\frac{1}{2}} + \|r_x\|_{\frac{1}{2}}\big) + C\theta t^{-\frac{2}{3}}\big(\|R\|_{\frac{1}{2}} + \|r\|_{\frac{1}{2}}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\|R\|_{\frac12}\|r_x\|_{\frac12}+\|r\|_{\frac12}\|r_x\|_{\frac12}+\|R\|_{\frac12}\|R_x\|_{\frac12}+\|r\|_{\frac12}\|R_x\|_{\frac12}\\
&\leqslant C\varepsilon^2 t^{-\frac43}+C\varepsilon^{1+\beta}t^{-1}g^{-1+\alpha}(t)+C\varepsilon^2 t^{-\frac43}+C\varepsilon^{1+\beta}t^{-1}g^{-1+\alpha}(t)\\
&\quad+C\varepsilon^{1+\beta}t^{-\frac43}g^{-1+\alpha}(t)+C\varepsilon^{1+\beta}t^{-1}g^{-2+2\alpha}(t)+C\varepsilon^2 t^{-\frac53}+C\varepsilon^{1+\beta}t^{-\frac43}g^{-1+\alpha}(t)\\
&\leqslant C\varepsilon^2 t^{-\frac43}+C\varepsilon^{1+\beta}t^{-1}g^{-1+\alpha}(t)
\end{aligned}
$$

and

$$
9\theta^2 t^{-\frac53}\int_0^\infty x\,Ai'\left(\frac{x}{\sqrt[3]{t}}\right)Ai''\left(\frac{x}{\sqrt[3]{t}}\right)dx=-9\theta^2 t^{-1}\int_0^\infty Ai'^2(z)\,dz.
$$

Therefore by virtue of the estimate $g^{-1}(t)<C(1+\eta\log(1+t))^{-1}$ we get for $t>1$

$$
\left|\frac{\lambda}{\theta}\int_0^\infty x v v_x\,dx+9\theta\lambda(1+t)^{-1}\int_0^\infty Ai'^2(z)\,dz\right|\leqslant C\varepsilon t^{-\frac{3+\delta}{3}}+C\varepsilon^2 t^{-1}\big(1+\eta\log(1+t)\big)^{-1+\alpha}.
$$

Whence integrating with respect to time we obtain

$$
\begin{aligned}
\left|\frac{\lambda}{\theta}\int_1^t d\tau\int_0^\infty x v v_x\,dx-\eta\log(1+t)\right|&\leqslant C\varepsilon\int_1^t \tau^{-\frac{3+\delta}{3}}\,d\tau+C\varepsilon^\beta\int_1^t\tau^{-1}(1+\eta\log\tau)^{-1+\alpha}\,d\tau\\
&\leqslant C\varepsilon+C\varepsilon^\beta\eta^{-1}\big(1+\eta\log(2+t)\big)^\alpha,
\end{aligned}
$$

where $\eta=-9\theta\lambda\int_0^\infty Ai'^2(z)\,dz$. Therefore we get

$$
\begin{aligned}
1+\eta\log(1+t)-C\varepsilon^\beta\eta^{-1}(1+\eta\log t)^\alpha&\leqslant 1+\frac{1}{\theta}\int_1^t d\tau\int_0^\infty x v v_x\,dx\\
&\leqslant 1+\eta\log(1+t)+C\varepsilon^\beta\eta^{-1}(1+\eta\log t)^\alpha.
\end{aligned}
$$

Since $C\varepsilon^\beta\eta^{-1}<\frac12$, Lemma 3.6 is proved. □

4. Local existence

In this section we prove the following theorem.

Theorem 4.1. *Let $u_0(x)\in\mathbf{H}_2^{1,7/2}$. Then there exist a positive time T and a unique solution of the problem* (1.1) *such that*

$$
u(t)\in\mathbf{C}\big([0,T];\mathbf{H}_2^{1,7/2}\big)\cap\mathbf{L}^2\big(0,T;\mathbf{H}_2^{2,3}\big).
$$

Proof. We consider the linearized equation of (1.1) such that

$$\begin{cases} u_t + u_{xxx} = -\lambda v u_x, & t > 0,\ x > 0, \\ u(x,0) = u_0(x), & x > 0, \\ u_x(0,t) = 0, & t > 0. \end{cases} \tag{4.1}$$

For simplicity, we let $\lambda = 1$. Let

$$\mathbf{Z}_T = \left\{ f \in \mathbf{C}\big([0,T], \mathbf{H}_2^{1,\frac{7}{2}}\big);\ \|f\|_{\mathbf{Z}_T} = \sup_{t\in[0,T]} \big\|f(t)\big\|_{1,\frac{7}{2}} + \left(\int_0^T \big\|\partial_x f(t)\big\|_{1,3}^2 \, dt \right)^{\frac{1}{2}} < \infty \right\}$$

and the closed ball

$$\mathbf{Z}_{T,\rho} = \big\{ f \in \mathbf{Z}_T;\ \|f\|_{\mathbf{Z}_T} \leqslant \rho \big\}.$$

We define the mapping M by $u = Mv$, where $v \in \mathbf{Z}_{T,\rho}$. Applying both sides of Eq. (4.1) by ∂_x, multiplying the resulting equation by $x^{7-3j}\partial_x u$, and integrating over $\mathbf{R}^+$, we get

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{2} \frac{d}{dt}\big\|x^{\frac{7-3j}{2}}\partial_x u(t)\big\|^2 - \sum_{j=0}^{1} \frac{(7-3j)(6-3j)(5-3j)}{2}\big\|x^{\frac{4-3j}{2}}\partial_x u(t)\big\|^2 \\ &\quad + \sum_{j=0}^{2} \frac{3(7-3j)}{2}\big\|x^{\frac{6-3j}{2}}\partial_x^2 u(t)\big\|^2 \\ &= -\sum_{j=0}^{2}\big(x^{7-3j}\partial_x(vu_x), \partial_x u\big). \end{aligned} \tag{4.2}$$

By the boundary condition

$$\frac{1}{2}\frac{d}{dt}\big\|\partial_x u(t)\big\|^2 + \frac{1}{2}\big(\partial_x^2 u(0,t)\big)^2 = -\big(\partial_x(vu_x), \partial_x u\big). \tag{4.3}$$

In the same way as in the proof of (4.2), (4.3)

$$\begin{aligned} &\frac{1}{2}\sum_{j=0}^{2} \frac{d}{dt}\big\|x^{\frac{7-3j}{2}} u(t)\big\|^2 - \sum_{j=0}^{1} \frac{(7-3j)(6-3j)(5-3j)}{2}\big\|x^{\frac{4-3j}{2}} u(t)\big\|^2 \\ &\quad + \sum_{j=0}^{2} \frac{3(7-3j)}{2}\big\|x^{\frac{6-3j}{2}}\partial_x u(t)\big\|^2 \\ &= -\sum_{j=0}^{2}\big(x^{7-3j}(vu_x), u\big) \end{aligned} \tag{4.4}$$

and

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 - \left(\partial_x^2 u(0,t)\right)\left(u(0,t)\right) = -\left((vu_x), u\right). \tag{4.5}$$

Via (4.2)–(4.5) we find that there exists a positive constant C_1 such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{1,\frac{7}{2}} + C_1\|\partial_x u(t)\|^2_{1,3} \\
&\quad \leqslant C\|u(t)\|^2_\infty + C\|v(t)\|_{1,\frac{7}{2}}\|u(t)\|_{1,\frac{7}{2}}\left(\|\partial_x u(t)\|_{1,3} + \|u(t)\|_{1,\frac{7}{2}}\right)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{1,\frac{7}{2}} + \frac{C_1}{2}\|\partial_x u(t)\|^2_{1,3} &\leqslant C\|u(t)\|^2_\infty + C\|v(t)\|_{1,\frac{7}{2}}\|u(t)\|^2_{1,\frac{7}{2}}\left(1 + \|v(t)\|_{1,\frac{7}{2}}\right) \\
&\leqslant C\|u(t)\|\|\partial_x u(t)\| + C\rho(1+\rho)\|u(t)\|^2_{1,\frac{7}{2}}. \qquad (4.6)
\end{aligned}$$

The integral equation associated with (4.1) is written as

$$u(x,t) = \Phi(x,t)u_0(x) + \lambda\int_0^t \Phi(x,t-\tau)vu_x\,d\tau.$$

By Lemma 3.4

$$\begin{aligned}
\|u(t)\| &\leqslant Ct^{-\frac{1}{2}}\|u_0\|_{\mathbf{H}_1^{0,1}} + C\int_0^t (t-\tau)^{-\frac{1}{2}}\|vu_x\|_{\mathbf{H}_1^{0,1}}\,d\tau \\
&\leqslant Ct^{-\frac{1}{2}}\|u_0\|_{\mathbf{H}_1^{0,1}} + C\|v\|_{0,1}\|u\|t^{\frac{1}{2}}. \qquad (4.7)
\end{aligned}$$

Integrating (4.6) with respect to time t, using (4.7) we have for $\varepsilon > 0$

$$\begin{aligned}
&\|u(t)\|^2_{1,\frac{7}{2}} + C_1\int_0^t \|\partial_x u(\tau)\|^2_{1,3}\,d\tau \\
&\quad \leqslant \|u_0\|^2_{1,\frac{7}{2}} + \int_0^t \|u(\tau)\|\|\partial_x u(\tau)\|\,d\tau + C\rho(1+\rho)\int_0^t \|u(\tau)\|^2_{1,\frac{7}{2}}\,d\tau \\
&\quad \leqslant C\|u_0\|^2_{1,\frac{7}{2}} + C\int_0^t \left(\tau^{-\frac{1}{2}}\|u_0\|_{\mathbf{H}_1^{0,1}} + \|v\|_{0,1}\|u\|\tau^{\frac{1}{2}}\right)\|\partial_x u(\tau)\|\,d\tau
\end{aligned}$$

$$+ C\rho(1+\rho)\int_0^t \|u(\tau)\|_{1,\frac{7}{2}}^2\, d\tau$$

$$\leqslant C\|u_0\|_{1,\frac{7}{2}}^2 + C\rho(1+\rho)(1+t)^{\frac{1}{2}}\int_0^t \|u(\tau)\|_{1,\frac{7}{2}}^2\, d\tau$$

hence

$$\sup_{t\in[0,T]}\|u(t)\|_{1,\frac{7}{2}} \leqslant C\|u_0\|_{1,\frac{7}{2}} + C\rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}(1+T)^{\frac{1}{4}}T^{\frac{1}{2}}\sup_{t\in[0,T]}\|u(t)\|_{1,\frac{7}{2}}$$

and

$$\int_0^T \|\partial_x u(t)\|_{1,3}^2\, dt \leqslant \frac{1}{C_1}\|u_0\|_{1,\frac{7}{2}}^2 + C\rho(1+\rho)(1+T)^{\frac{1}{2}}T\Big(\sup_{t\in[0,T]}\|u(t)\|_{1,\frac{7}{2}}\Big)^2.$$

If we choose T such that $C\rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}(1+T)^{\frac{1}{4}}T^{\frac{1}{2}} \leqslant \frac{1}{2}$, then

$$\|u\|_{\mathbf{Z}_T} \leqslant C\|u_0\|_{1,\frac{7}{2}}. \tag{4.8}$$

We let $u_j = Mv_j$, $j = 1, 2$. Then in the same way as in the proof of (4.8) we find that there exists a T such that

$$\|Mv_1 - Mv_2\|_{\mathbf{Z}_T} \leqslant \frac{1}{2}\|v_1 - v_2\|_{\mathbf{Z}_T}.$$

Therefore there exists a unique solution the problem (1.1) in the set $\mathbf{Z}_{T,\rho}$. Theorem is proved. □

The above theorem says that the solutions have a smoothing property if the data decay rapidly at infinity. Therefore we have the following theorem. We do not give a proof since we do not need the result of a smoothing property to prove asymptotic behavior of solutions.

Theorem 4.2. *Let* $u_0(x) \in \mathbf{H}_2^{1,\frac{7}{2}}$. *Then there exist a positive time* T *and a unique solution of the problem* (1.1) *such that*

$$u(t) \in \mathbf{C}\big([0,T]; \mathbf{H}_2^{1,\frac{7}{2}}\big)$$

and

$$t^{\frac{7-j}{2}}\partial_x^{7-j}u(t) \in \mathbf{C}\big((0,T]; \mathbf{H}_2^{0,\frac{j}{2}}\big), \quad 0 \leqslant j \leqslant 6.$$

5. Proof of Theorem 1.1

We suppose that for a sufficiently small $\varepsilon > 0$

$$\|u_0\|_{\mathbf{H}_1^{0,\frac{21}{4}}} + \|u_0\|_{\mathbf{H}_2^{1,\frac{7}{2}}} \leqslant \varepsilon$$

and

$$\lambda \int_0^\infty x u_0(x)\,dx = \lambda\theta < 0.$$

Denote

$$\eta = -9\theta\lambda \int_0^\infty Ai'^2(z)\,dz > 0.$$

We let $u(x,t) = e^{-\phi(t)} v(x,t)$. Then we get from (1.1)

$$\begin{cases} v_t - \phi_t v + \lambda e^{-\phi(t)} v v_x + v_{xxx} = 0, & t > 0,\ x > 0, \\ v(x,0) = e^{\phi(0)} u_0(x), & x > 0, \\ v_x(0,t) = 0, & t > 0. \end{cases} \tag{5.1}$$

Now we assume that real-valued function $\phi(t)$ satisfies the following condition

$$\int_0^\infty x\left(-\phi_t v + \lambda e^{-\phi(t)} v v_x\right) dx = 0. \tag{5.2}$$

Then via (5.1) we have for all $t > 0$

$$\frac{d}{dt} \int_0^\infty x v(x,t)\,dx = -\int_0^\infty x v_{xxx}(x,t)\,dx = v_x(0,t) = 0.$$

Therefore choosing $\phi(0) = 0$ we get by (5.2)

$$\phi_t e^{\phi(t)} = \theta^{-1}\lambda \int_0^\infty x v v_x\,dx.$$

Integrating with respect to time we get

$$e^{\phi(t)} = g(t) = 1 + \theta^{-1}\lambda \int_0^t d\tau \int_0^\infty x v v_x\,dx = 1 - \frac{1}{2}\theta^{-1}\lambda \int_0^t \left\|v(\tau)\right\|^2 d\tau.$$

Thus we get the following problem

$$\begin{cases} v_t + \lambda g(t)^{-1}\left(\frac{v}{2\theta}\left\|v(t)\right\|^2 + vv_x\right) + v_{xxx} = 0, & t > 0,\ x > 0,\\ g(t) = 1 - \frac{\lambda}{2\theta}\int_0^t \|v(\tau)\|^2\,d\tau, \\ v(x,0) = u_0(x), \quad x > 0, \\ v_x(0,t) = 0, \quad g(0) = 1, \quad t > 0. \end{cases} \tag{5.3}$$

We consider the integral equation associated with (5.3) which is written as

$$v(x,t) = \Phi(x,t)u_0(x) + \lambda\int_0^t g(\tau)^{-1}\Phi(x,t-\tau)\left(\frac{v}{2\theta}\left\|v(\tau)\right\|^2 + vv_x\right)d\tau.$$

Making a change of variables

$$v(x,t) = \Phi(x,t)u_0(x) + r(x,t) \tag{5.4}$$

we get the system of integral equations $(r, g) = (M_1(r, g), M_2(r, g))$ for the first approximation of perturbation theory

$$\begin{cases} M_1(r,g) = \lambda\int_0^t g(\tau)^{-1}\Phi(x,t-\tau)\left(\frac{v}{2\theta}\|v(\tau)\|^2 + vv_x\right)d\tau, \\ M_2(r,g) = 1 - \lambda\frac{1}{2\theta}\int_0^t \|v(\tau)\|^2\,d\tau. \end{cases} \tag{5.5}$$

We prove that $(M_1(r, g), M_2(r, g))$ is contraction mapping in the set

$$\mathbf{X}_\varepsilon = \Big\{ r \in \mathbf{C}\big([0,\infty),\mathbf{X}\big);\ g \in \mathbf{C}(0,\infty),\ |||r|||_{\mathbf{X}_1} \leqslant \varepsilon,\ |||r|||_{\mathbf{X}_2} \leqslant \varepsilon^{\frac{3}{4}},$$
$$r_x(0,t) = 0,\ \frac{1}{2}(1+\eta\log t) < g(t) < 2(1+\eta\log t),\ t > 0\Big\}$$

with

$$|||r|||_{\mathbf{X}} = |||r|||_{\mathbf{X}_1} + |||r|||_{\mathbf{X}_2},$$
$$|||r|||_{\mathbf{X}_1} = \sum_{n=0}^{1}\sum_{m=0}^{2}\sup_{t>0} g(t)^{\frac{1}{5}}(1+t)^{\frac{3+2n-m}{6}}\left\|x^{\frac{m}{2}}\partial_x^{(n)}r(t)\right\|,$$
$$|||r|||_{\mathbf{X}_2} = \sum_{n=0}^{1}\sum_{m=3}^{7}\sup_{t>0} g(t)^{-1}(1+t)^{\frac{3+2n-m}{6}}\left\|x^{\frac{m}{2}}\partial_x^{(n)}\big(\Phi(t)u_0 + r(t)\big)\right\|.$$

We assume that

$$|||r|||_{\mathbf{X}_1} \leqslant \varepsilon, \qquad |||r|||_{\mathbf{X}_2} \leqslant \varepsilon^{\frac{3}{4}}. \tag{5.6}$$

First we prove that the mapping transforms the set $\mathbf{X}_\varepsilon$ into itself if $\varepsilon > 0$ is small. By the local existence theorem we have $|||M_1(r,g)|||_{\mathbf{X}_1} \leqslant \varepsilon$, $|||M_1(r,g)|||_{\mathbf{X}_2} \leqslant \varepsilon^{3/4}$ for $t < 1$ since data are sufficiently small. Now we consider the case $t > 1$. From Lemma 3.4 we have for $t > 1$ and $0 \leqslant m \leqslant 7$

$$\left\| x^{\frac{m}{2}} \partial_x^{(n)} \Phi(t) u_0(x) \right\| \leqslant C\varepsilon t^{-\frac{3-m+2n}{6}}$$

since $u_0 \in \mathbf{H}_1^{0,\frac{21}{4}}$. So via (5.4) and (5.6) we have

$$\begin{aligned}\sum_{n=0}^{1}\sum_{m=0}^{2} \sup_{t>0}(1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} v(t) \right\| &\leqslant \sum_{n=0}^{1}\sum_{m=0}^{2} \sup_{t>0}(1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} \Phi(t) u_0(x) \right\| \\ &\quad + \sum_{n=0}^{1}\sum_{m=0}^{2} \sup_{t>0}(1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} r(t) \right\| \\ &\leqslant C\varepsilon\end{aligned}$$

and

$$\sum_{n=0}^{1}\sum_{m=3}^{7} \sup_{t>0} g(t)^{-1}(1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} v(t) \right\| \leqslant C\varepsilon^{\frac{3}{4}}, \tag{5.7}$$

and therefore using that for $\gamma > 0$ is small $\|v\|_{\mathbf{L}^1} \leqslant C\|xv\|^{\frac{1}{2}} \|x^{-\gamma}\langle x\rangle^{2\gamma} v\|^{\frac{1}{2}}$ we get for $t > 0$

$$\begin{aligned}&\left\| x^{\frac{m+3}{2}} \left(-\frac{v}{\theta} \int_0^\infty y v v_y \, dy + v v_x \right) \right\|_{\mathbf{L}^1} \\ &\quad \leqslant C\theta^{-1} \left\| x^{\frac{m+5}{2}} v(t) \right\|^{\frac{1}{2}} \left\| x^{\frac{m+3}{2}-\gamma} \langle x\rangle^{2\gamma} v(t) \right\|^{\frac{1}{2}} \left\| v(t) \right\|^2 + C \left\| x^{\frac{m+1}{2}} v \right\| \|x v_x\| \\ &\quad \leqslant C\theta^{-1} \left\| \langle x\rangle^{\frac{m+5}{2}} v(t) \right\|^{\frac{1}{2}+\frac{1}{2}\frac{m+1}{m+3}} \left\| \langle x\rangle^{1+\gamma_1} v(t) \right\|^{\frac{1}{2}\frac{2}{m+3}} \left\| v(t) \right\|^2 + C \left\| x^{\frac{m+1}{2}} v \right\| \|x v_x\| \\ &\quad \leqslant C\varepsilon^{\frac{9}{5}} (1+t)^{-\frac{5-m}{6}} \left(\log(1+t)\right)^{\frac{4}{5}}\end{aligned}$$

for $m = 0, 1, 2$, $\gamma_1 > 0$ is small. Also by construction of v

$$\int_0^\infty x \left(-\frac{v}{\theta} \int_0^\infty y v v_y \, dy + v v_x \right) dx = 0.$$

So using results of Lemma 3.5 we have for $t > 1$, $\alpha = 4/5$, $m = 0, 1, 2$

$$\begin{aligned}&\sum_{n=0}^{1}\sum_{m=0}^{2} \sup_{t>1} g(t)^{1-\alpha} t^{\frac{3-m+2n}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} M_1(r, g) \right\| \\ &\quad \leqslant C \sum_{n=0}^{1}\sum_{m=0}^{2} \sup_{t>1} g(t)^{1-\alpha} t^{\frac{3-m+2n}{6}}\end{aligned}$$

$$\times \left\| x^{\frac{m}{2}} \int_0^t g(\tau)^{-1} \partial_x^{(n)} \Phi(x, t-\tau) \left(-\frac{v}{\theta} \int_0^\infty y v v_y \, dy + v v_x \right) d\tau \right\| \leqslant C\varepsilon^{\frac{9}{5}} \leqslant \varepsilon. \tag{5.8}$$

For higher order m, we turn to the original equation

$$u(x,t) = \Phi(x,t) u_0(x) + \lambda \int_0^t \Phi(x, t-\tau) u u_x \, d\tau.$$

By Lemma 3.4

$$\left\| x^{\frac{m}{2}} \Phi(t) f \right\| \leqslant C \Big(t^{-\frac{2}{3}} \| f \|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{3-m}{6}} \| f \|_{\mathbf{H}_1^{0,1}} + t^{-\frac{3}{4}} \| f \|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}} \Big)$$

and

$$\left\| x^{\frac{m}{2}} \partial_x \Phi(t) f \right\| \leqslant C \begin{cases} t^{-1} \| f \|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + t^{-\frac{5-m}{6}} \| f \|_{\mathbf{H}_1^{0,1}}, \\ t^{-\frac{2}{3}} \| f \|_{\mathbf{H}_1^{0,\frac{m+1}{2}}} + t^{-\frac{3-m}{6}} \| f \|_{\mathbf{L}^1} + t^{-\frac{5}{6}+\frac{1}{12}} \| f \|_{\mathbf{H}_1^{0,\frac{m+3}{2}-\frac{3}{4}}} + t^{-\frac{5-m}{6}} \| f \|_{\mathbf{H}_1^{0,1}}. \end{cases}$$

Therefore by the local existence theorem we get for $3 \leqslant m \leqslant 7$

$$\begin{aligned} \left\| x^{\frac{m}{2}} u(t) \right\| &\leqslant C(1+t)^{-\frac{3-m}{6}} \| u_0 \|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}} \\ &\quad + C \int_0^t \Big((t-\tau)^{-\frac{2}{3}} \| u u_x \|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + (t-\tau)^{-\frac{3}{4}} \| u u_x \|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}} \\ &\quad + (t-\tau)^{-\frac{3-m}{6}} \| u u_x \|_{\mathbf{H}_1^{0,1}} \Big) d\tau \leqslant C(1+t)^{-\frac{3-m}{6}} \| u_0 \|_{\mathbf{H}_1^{0,\frac{m+2}{2}+\frac{3}{4}}} \\ &\quad + C\varepsilon^{\frac{3}{2}} \int_0^t \Big((t-\tau)^{-\frac{2}{3}} (1+\tau)^{-\frac{5-m}{6}} + (t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{5-m}{6}+\frac{1}{12}} \Big) d\tau \\ &\quad + C\varepsilon^2 \int_0^t (t-\tau)^{-\frac{3-m}{6}} (1+\tau)^{-1} \big(1 + \eta \log(1+\tau) \big)^{-2} d\tau \\ &\leqslant C(1+t)^{-\frac{3-m}{6}} \| u_0 \|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + C\varepsilon(1+t)^{-\frac{3-m}{6}} \leqslant C\varepsilon(1+t)^{-\frac{3-m}{6}}. \end{aligned}$$

In the same way

$$\begin{aligned}
\left\| x^{\frac{m}{2}} \partial_x u(t) \right\| &\leqslant C(1+t)^{-\frac{5-m}{6}} \|u_0\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} \\
&\quad + C \int_0^{\frac{t}{2}} \Big((t-\tau)^{-1} \|uu_x\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + (t-\tau)^{-\frac{5-m}{6}} \|uu_x\|_{\mathbf{H}_1^{0,1}} \Big)\, d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} \Big((t-\tau)^{-\frac{2}{3}} \|uu_x\|_{\mathbf{H}_1^{0,\frac{m+1}{2}}} + (t-\tau)^{-\frac{3-m}{6}} \|uu_x\|_{\mathbf{L}^1} \\
&\quad + (t-\tau)^{-\frac{5}{6}+\frac{1}{12}} \|uu_x\|_{\mathbf{H}_1^{0,\frac{m+3}{2}-\frac{3}{4}}} + (t-\tau)^{-\frac{5-m}{6}} \|uu_x\|_{\mathbf{H}_1^{0,1}} \Big)\, d\tau \\
&\leqslant C(1+t)^{-\frac{5-m}{6}} \|u_0\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + C\varepsilon^2 \int_0^{\frac{t}{2}} (t-\tau)^{-1} (1+\tau)^{-\frac{5-m}{6}}\, d\tau \\
&\quad + C\varepsilon^2 \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{5-m}{6}} (1+\tau)^{-1} \big(1+\eta \log(1+\tau)\big)^{-2}\, d\tau \\
&\quad + C\varepsilon^{\frac{3}{2}} \int_{\frac{t}{2}}^{t} \Big((t-\tau)^{-\frac{2}{3}} (1+\tau)^{-\frac{3-m}{6}} + (t-\tau)^{-\frac{3-m}{6}} (1+\tau)^{-\frac{4}{3}} \\
&\quad + (t-\tau)^{-\frac{5}{6}+\frac{1}{12}} (1+\tau)^{-1-\frac{1}{12}+\frac{m}{6}} \Big)\, d\tau \\
&\quad + \varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{5-m}{6}} (1+\tau)^{-1} \big(1+\eta \log(1+\tau)\big)^{-2}\, d\tau \\
&\leqslant C(1+t)^{-\frac{5-m}{6}} \|u_0\|_{\mathbf{H}_1^{0,\frac{m+3}{2}}} + C\varepsilon (1+t)^{-\frac{5-m}{6}} \leqslant C\varepsilon (1+t)^{-\frac{5-m}{6}}.
\end{aligned}$$

Thus we get

$$\sum_{n=0}^{1} \sum_{m=3}^{7} \sup_{t>0} (1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} u(t) \right\| \leqslant C\varepsilon$$

which implies

$$\sum_{n=0}^{1} \sum_{m=3}^{7} \sup_{t>0} g(t)^{-1} (1+t)^{\frac{3+2n-m}{6}} \left\| x^{\frac{m}{2}} \partial_x^{(n)} M_2(r,g) \right\| \leqslant C\varepsilon \leqslant \varepsilon^{\frac{3}{4}}.$$

Then applying (5.8) to Lemma 3.6 we get for $t > 0$

$$\frac{1}{2} \big(1+\eta \log(1+t)\big) < g(t) < 2 \big(1+\eta \log(1+t)\big).$$

Thus $(M_1(r,g), M_2(r,g))$ transform the set $\mathbf{X}_\varepsilon$ into itself. Similarly, we can prove that the transformation $(M_1(r,g), M_2(r,g))$ is a contraction mapping. Hence there exists a unique solution (r,g) of the system of integral equations (5.5) in the set $\mathbf{X}$ and for $t>1$ and

$$\|r(t)\|_\infty = \|v(t) - \Phi(t)u_0\|_\infty \leqslant C\|r(t)\|^{\frac{1}{2}}\|r_x(t)\|^{\frac{1}{2}} \leqslant Ct^{-\frac{2}{3}}g^{-1+\frac{4}{5}}(t).$$

From Lemma 3.4 we have

$$\Phi(t)u_0 = 3\theta t^{-\frac{2}{3}} Ai'\left(\frac{x}{\sqrt[3]{t}}\right) + R(x,t),$$

where for $t>1$, $0<\delta<1$

$$\left\|\partial_x^{(n)} R(t)\right\|_{\frac{m}{2}} \leqslant C\varepsilon t^{-\frac{3-m+2n+2\delta}{6}}$$

and therefore

$$\|R\|_\infty \leqslant C\|R\|^{\frac{1}{2}}\|R_x\|^{\frac{1}{2}} \leqslant Ct^{-\frac{2+\delta}{3}}.$$

From the proof of Lemma 3.6 we have for $t>1$

$$g(t) = \eta\log t + O\big(\varepsilon(\eta\log t)^{\frac{4}{5}}\big)$$

which gives

$$g^{-1}(t) = (\eta\log t)^{-1}\big(1 + O\big(\varepsilon(\eta\log t)^{-\frac{1}{5}}\big)\big).$$

So we obtain the following asymptotics of solutions for $t\to\infty$ uniformly with respect to $x>0$ such that

$$\begin{aligned} u(x,t) &= g^{-1}(t)v(x,t) = g^{-1}(t)\big(r(x,t) + \Phi(t)u_0\big) \\ &= 3\theta(\eta\log t)^{-1}t^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{t}}\right) + g^{-1}(t)R(x,t) + g^{-1}(t)r(x,t) \\ &= 3\theta(\eta\log t)^{-1}t^{-\frac{2}{3}}Ai'\left(\frac{x}{\sqrt[3]{t}}\right) + O\big(t^{-\frac{2}{3}}(\eta\log t)^{-\frac{6}{5}}\big). \end{aligned}$$

Theorem 1.1 is now proved.

References

[1] M.J. Ablowitz, H. Segur, Solitons and Inverse Scattering Transform, SIAM, 1980.

[2] J.L. Bona, S.M. Sun, B.Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg–de Vries equation in a quarter plane, Trans. Amer. Math. Soc. 354 (2) (2001) 427–490.

[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations, Part II: The KdV equation, Geom. Funct. Anal. 3 (1993) 107–156.

[4] F.M. Christ, M.I. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg–de Vries equation, J. Funct. Anal. 100 (1991) 87–109.

[5] T. Colin, M. Gisclon, An initial-boundary-value problem that approximate the quarter-plane problem for the Korteweg–de Vries equation, Nonlinear Anal. 46 (2001) 869–892.
[6] A.F. Fokas, B. Polloni, The solution of certain initial boundary-value problems for the linearized Korteweg–de Vries equation, Proc. Roy. Soc. London Ser. A 454 (1998) 645–657.
[7] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Large time behavior of solutions to the dissipative nonlinear Schrödinger equation, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 1029–1043.
[8] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Large time behavior of solutions to the Landau–Ginzburg type equations, Funkcial. Ekvac. 44 (1) (2001) 171–200.
[9] N. Hayashi, E.I. Kaikina, R.F. Paredes, Korteweg–de Vries–Burgers equation on half-line, Nonlinear Differential Equations Appl. 8 (2001) 439–463.
[10] N. Hayashi, E.I. Kaikina, L. Gurdado Zavala, On the boundary-value problem for Korteweg–de Vries equation, Proc. Roy. Soc. London Ser. A 459 (2003) 2861–2884.
[11] N. Hayashi, E.I. Kaikina, I.A. Shishmarev, Large time asymptotics for the Korteweg–de Vries–Burgers equation on a half-line, J. Math. Anal. Appl. 265 (2002) 343–370.
[12] N. Hayashi, P.I. Naumkin, Large time asymptotics of solutions to the generalized Korteweg–de Vries equation, J. Funct. Anal. 156 (1998) 110–136.
[13] N. Hayashi, P.I. Naumkin, Large time behavior of solutions for the modified Korteweg–de Vries equation, Int. Math. Res. Not. (1999) 395–418.
[14] N. Hayashi, P.I. Naumkin, On the modified Korteweg–de Vries equation, Math. Phys. Anal. Geom. 4 (2001) 197–227.
[15] E.I. Kaikina, P.I. Naumkin, I.A. Shishmarev, Schrödinger equation on half-line, SUT J. Math. 35 (1) (1999) 37–79.
[16] C.E. Kenig, G. Ponce, L. Vega, The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces in negative indices, Duke Math. J. 71 (1993) 1–21.
[17] C.E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996) 573–603.
[18] P.I. Naumkin, I.A. Shishmarev, Nonlinear Nonlocal Equations in the Theory of Waves, Transl. Math. Monogr., vol. 133, Amer. Math. Soc., Providence, RI, 1994.
[19] G. Ponce, L. Vega, Nonlinear small data scattering for the generalized Korteweg–de Vries equation, J. Funct. Anal. 90 (1990) 445–457.
[20] N.J. Zabusky, C.J. Galvin, Shallow-water waves, the Korteweg–de Vries equation and solitons, J. Fluid Mech. 47 (1971) 811–824.