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# Pointwise asymptotic behavior of modulated periodic reaction–diffusion waves

Soyeun Jung [1]

*Indiana University, Bloomington, IN 47405, United States*

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ABSTRACT

By working with the periodic resolvent kernel and the Bloch-decomposition, we establish pointwise bounds for the Green function of the linearized equation associated with spatially periodic traveling waves of a system of reaction–diffusion equations. With our linearized estimates together with a nonlinear iteration scheme developed by Johnson–Zumbrun, we obtain $L^p$-behavior ($p \geqslant 1$) of a nonlinear solution to a perturbation equation of a reaction–diffusion equation with respect to initial data in $L^1 \cap H^2$ recovering and slightly sharpening results obtained by Schneider using weighted energy and renormalization techniques. We obtain also pointwise nonlinear estimates with respect to two different initial perturbations $|u_0| \leqslant E_0 e^{-|x|^2/M}$, $|u_0|_{H^2} \leqslant E_0$ and $|u_0| \leqslant E_0(1+|x|)^{-r}$, $r > 2$, $|u_0|_{H^2} \leqslant E_0$ respectively, $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large, showing that behavior is that of a heat kernel. These pointwise bounds have not been obtained elsewhere, and do not appear to be accessible by previous techniques.



## 1. Introduction

In this paper, we obtain pointwise bounds for the Green function of the linearized equations associated with a spatially periodic traveling wave of a system of reaction–diffusion equations, and use this to obtain pointwise bounds on decay and asymptotic behavior, sharping bounds of [12], and [18,19], of perturbations of a periodic traveling wave of a system of reaction–diffusion equations. Suppose that $u(x,t) = \bar{u}(x - at)$ is a spatially periodic wave of a system of reaction–diffusion equa-

*E-mail address:* soyjung@indiana.edu.

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tions of form $u_t = u_{xx} + f(u)$, where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $u \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth: equivalently, $u(x,t) = \bar{u}(x)$ is a spatially periodic standing-wave solution of

$$u_t - au_x = u_{xx} + f(u). \tag{1.1}$$

Throughout our analysis, we assume the existence of an $X$-periodic solution $\bar{u}(x)$ of (1.1). Without loss of generality, we assume that $\bar{u}$ is 1-periodic, that is, $\bar{u}(x+1) = \bar{u}(x)$ for all $x \in \mathbb{R}$. A different pointwise Green function approach was carried out in [16] in the context of parabolic conservation laws by direct inverse Laplace transform computations not using the standard Bloch decomposition into periodic waves. In this paper we work from the Bloch representation and in the process we develop an interesting new formula for the high-frequency description of the resolvent of an operator with periodic boundary conditions on $[0,1]$.

Linearizing (1.1) about a standing-wave solution $\bar{u}(x)$ gives the eigenvalue equation

$$\lambda v = Lv := \big(\partial_x^2 + a\partial_x + df(\bar{u})\big)v. \tag{1.2}$$

As coefficients of $L$ are 1-periodic, Floquet theory implies that the $L^2$ spectrum is purely continuous and corresponds to the union of $\lambda$ such that (1.2) admits a bounded eigenfunction of the form

$$v(x) = e^{i\xi x} w(x), \quad \xi \in \mathbb{R} \tag{1.3}$$

where $w(x+1) = w(x)$, that is, the eigenvalues of the family of associated Floquet, or Bloch, operators

$$L_\xi := e^{-i\xi x} L e^{i\xi x} = (\partial_x + i\xi)^2 + a(\partial_x + i\xi) + df(\bar{u}), \quad \text{for } \xi \in [-\pi, \pi), \tag{1.4}$$

considered as acting on $L^2$ periodic functions on $[0,1]$.

Recall that any function $g \in L^2(\mathbb{R})$ admits an inverse Bloch–Fourier representation

$$g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \check{g}(\xi, x)\, d\xi, \tag{1.5}$$

where $\check{g}(\xi, x) = \sum_{j\in\mathbb{Z}} e^{i2\pi jx} \hat{g}(\xi + 2\pi j)$ is a 1-periodic functions of $x$, and $\hat{g}(\cdot)$ denotes the Fourier transform of $g$ with respect to $x$. Indeed, using the Fourier transform we have

$$2\pi\, g(x) = \int_{-\infty}^{\infty} e^{i\xi x} \hat{g}(\xi)\, d\xi = \sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i(\xi+2\pi j)x} \hat{g}(\xi + 2\pi j)\, d\xi = \int_{-\pi}^{\pi} e^{i\xi x} \check{g}(\xi, x)\, d\xi. \tag{1.6}$$

Since $L(e^{i\xi x} f) = e^{i\xi x}(L_\xi f)$ for $f$ periodic, the Bloch–Fourier transform diagonalizes the periodic-coefficient operator $L$, yielding the inverse Bloch–Fourier transform representation

$$e^{Lt} g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \check{g}(\xi, x)\, d\xi. \tag{1.7}$$

By the translation invariance of (1.1), the function $\bar{u}'(x)$ is a 1-periodic solution of the differential equation $L_0 v = 0$. Hence, it follows that $\lambda = 0$ is an eigenvalue of the Bloch operator $L_0$. Define following [18,19,12] the *diffusive spectral stability* conditions:

(D1) $spec_{L^2(\mathbb{R})}(L) \subset \{\lambda \in \mathbb{C}:\ R(\lambda) < 0\} \cup \{0\}$.
(D2) $\lambda = 0$ is a simple eigenvalue of $L_0$.
(D3) There exists a $\theta > 0$ such that $R\sigma(L_\xi) \leqslant -\theta|\xi|^2$ for all real $\xi \in [-\pi, \pi)$.

Assumption (D1) says the only “neutrally stable” point in the $L^2$-spectrum of $L$ is at the origin and (D2) corresponds to transversality of $\bar{u}$ as a solution of the associated traveling-wave ODE, while assumption (D3) guarantees that the spectrum of $L$ only touches the orgin when $\xi = 0$ and it also corresponds to “dissipativity” of the large-time behavior of the linearized system; see [18,19,12].

**Remark 1.1.** (See [12].) By standard spectral perturbation theory [14], (D2) implies that the eigenvalue $\lambda(\xi)$ bifurcating from $\lambda = 0$ at $\xi = 0$ is analytic at $\xi = 0$, with $\lambda(\xi) = \lambda_1\xi + \lambda_2\xi^2 + O(|\xi|^3)$, from which we find from the necessary stability condition $Re\,\lambda(\xi) \leqslant 0$ that $Re\,\lambda_1 = 0$ and $Re\,\lambda_2 \leqslant 0$. Assumption (D3) thus amounts to the nondegeneracy condition $Re\,\lambda_2 \neq 0$ together with the strict stability condition $Re\,\sigma(L_\xi) < 0$ for $\xi \neq 0$.

**Remark 1.2.** The condition (D3) may be readily verified by direct numerical Evans function analysis as described in [1,2]. Alternatively, it could be expressed through spectral perturbation analysis as a sign condition on a certain inner product of certain generalized eigenfunctions, as done for example in [6,3,4]. However, this involves an additional layer of analysis and to us does not appear to add further illumination.

Rewriting the eigenvalue equation (1.2) as a first-order system

$$V' = \mathbb{A}(\lambda, x)V, \tag{1.8}$$

where

$$V = \begin{pmatrix} v \\ v' \end{pmatrix}, \qquad \mathbb{A} = \begin{pmatrix} 0 & I \\ \lambda I - df(\bar{u}) & -aI \end{pmatrix},$$

denote by $\mathcal{F}^{y\to x} \in \mathbb{C}^{2n\times 2n}$ the solution operator of (1.8), defined by $\mathcal{F}^{y\to y} = I$, $\partial_x\mathcal{F} = \mathbb{A}\mathcal{F}$. That is, $\mathcal{F}^{y\to x} = \Phi(x)\Phi(y)^{-1}$, for any fundamental matrix solution $\Phi$ of the (1.8).

By the definition of Bloch operators (1.4), for each $\xi \in [-\pi, \pi)$, we have a second-order eigenvalue equation

$$\lambda u = L_\xi u = u'' - A_\xi u' - C_\xi u, \tag{1.9}$$

where $A_\xi = -(a + 2i\xi)I \in \mathbb{C}^{n\times n}$ a constant matrix and $C_\xi(x) = -df(\bar{u}) - (ia\xi - \xi^2)I \in \mathbb{C}^{n\times n}$ a matrix depending on $x$, and $u \in \mathbb{C}^n$ is a vector.

Rewriting (1.9) as a first-order system

$$U' = \mathbb{A}_\xi(x, \lambda)U, \tag{1.10}$$

where

$$U = \begin{pmatrix} u \\ u' \end{pmatrix}, \qquad \mathbb{A}_\xi = \begin{pmatrix} 0 & I \\ \lambda I + C_\xi & A_\xi \end{pmatrix}, \tag{1.11}$$

similarly, denote by $\mathcal{F}_\xi^{y\to x} \in \mathbb{C}^{2n\times 2n}$ the solution operator of (1.10), defined by $\mathcal{F}_\xi^{y\to y} = I$, $\partial_x\mathcal{F}_\xi = \mathbb{A}_\xi\mathcal{F}_\xi$. That is, $\mathcal{F}_\xi^{y\to x} = \Phi_\xi(x)\Phi_\xi(y)^{-1}$, for any fundamental matrix solution $\Phi_\xi$ of the (1.10).

### 1.1. Main result

With these preparations, we now state our two main results.

**Theorem 1.3.** *The Green function $G(x,t;y)$ for Eq.* (1.2) *satisfies the estimates*:

$$G(x,t;y)=\frac{1}{\sqrt{4\pi bt}}e^{-\frac{|x-y-at|^2}{4bt}}q(x,0)\tilde{q}(y,0)+\tilde{G}(x,t;y), \tag{1.12}$$

*where*

$$\begin{aligned}\big|\tilde{G}(x,t;y)\big| &\lesssim \big((1+t)^{-1}+t^{-\frac{1}{2}}e^{-\eta t}\big)e^{-\frac{|x-y-at|^2}{Mt}},\\ \big|\tilde{G}_y(x,t;y)\big| &\lesssim t^{-1}e^{-\frac{|x-y-at|^2}{Mt}},\end{aligned} \tag{1.13}$$

*uniformly on $t \geqslant 0$, for some sufficiently large constants $M>0$ and $\eta>0$, where $q$ and $\tilde{q}$ are the periodic right and left eigenfunctions of $L_0$, respectively, at $\lambda=0$. In particular $q(x,0)=\bar{u}'(x)$.*

**Theorem 1.4.** *Define the nonlinear perturbation $u:=\tilde{u}-\bar{u}$, where $\tilde{u}$ satisfies* (1.1). *Then the asymptotic behavior of $u$ with respect to three kinds of initial data (denoted by $u_0$):*

(1) $|u_0(x)|_{L^1\cap H^2} \leqslant E_0$ *and* $|xu_0|_{L^1} \leqslant E_0$,
(2) $|u_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$ *and* $|u_0(x)|_{H^2} \leqslant E_0$,
(3) $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r>2$ *and* $|u_0(x)|_{H^2} \leqslant E_0$,

*where $E_0>0$ sufficiently small and $M>1$ sufficiently large, converges to a heat kernel with the following estimates, respectively*

(a) $|u(x,t)-\bar{U}_*\bar{u}'\bar{k}(x,t)|_{L^p(x)} \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+\ln(1+t))$, *for* $1\leqslant p\leqslant\infty$,
(b) $|u(x,t)-\bar{U}_*\bar{u}'\bar{k}(x,t)| \leqslant CE_0(1+t)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}(1+\ln(1+t))$,
(c) $|u(x,t)-\bar{U}_*\bar{u}'\bar{k}(x,t)| \leqslant CE_0[(1+t)^{-\frac{1}{2}}(1+|x-at|+\sqrt{t})^{-r+1}+(1+t)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}(1+\ln(1+t))]$,

*for $\bar{k}(x,t)=\frac{1}{\sqrt{4\pi bt}}e^{-\frac{|x-at|^2}{4bt}}$, $M''>M$ and $C>0$ sufficiently large and some constant $\bar{U}_*$ (defined in Section* 7*).*

**Remark 1.5.** Integrating bounds (b) and (c) with respect to $x$ recovers the same $L^p$ bound as in (a) for all $1\leqslant p\leqslant\infty$. Note that it is clear for (b) and 2nd term of (c), and slightly harder for 1st term of (c). For $p=1$, $|(1+|x-at|+\sqrt{t})^{-r+1}|_{L^1(x)} \leqslant C$ and for $p=\infty$, $|(1+|x-at|+\sqrt{t})^{-r+1}|_{L^\infty(x)} \leqslant C(1+t)^{-\frac{1}{2}}$, so we have $|(1+|x-at|+\sqrt{t})^{-r+1}|_{L^p(x)} \leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$, implies that $|(1+t)^{-\frac{1}{2}}(1+|x-at|+\sqrt{t})^{-r+1}|_{L^p(x)} \leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$ for $1\leqslant p\leqslant\infty$.

**Remark 1.6.** The initial condition $|u_0|_{L^1\cap H^2}$, $|xu_0|_{L^1}$ sufficiently small is compared with Schneider's [19] initial assumption. By Fourier transform, we can roughly consider $|(1+|x|^2)u_0|_{H^2}$ as Schneider's initial condition with weight $(1+|x|^2)$ (see Schneider [19, pp. 690–691]). This implies that our initial data roughly satisfies $|u_0| \lesssim |x|^{-2}$ whereas Schneider's initial data roughly satisfies $|u_0| \lesssim |x|^{-\frac{5}{2}}$. Our $L^p$ bounds on asymptotic behavior for all $p\geqslant 1$ are compared with Schneider's $L^\infty$ bound. In particular, our $L^\infty$ bound $t^{-1}\ln(1+t)$ is roughly equivalent to but slightly sharper than Schneider's $L^\infty$ bound $t^{-1+\varepsilon}$ for $\varepsilon>0$. Though Schneider does not state $L^p$ bounds, his renormalized $H^2(2)$ bounds (see Theorem 15 [19]) by a simple scaling argument yield $L^p$ bounds $\sim t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\eta}$ for any $\eta>0$, for all $p\geqslant 1$, again roughly equivalent to but slightly less sharp than ours.

### 1.2. Discussion and open problems

Pointwise Green function bounds have been obtained by Oh and Zumbrun previously for systems of conservation laws, by somewhat different methods, without use of the Bloch representation. Those methods would work here as well; however, we find the present method proceeding from the Bloch transform both more direct and more connected to other literature in the area; in particular, it makes a direct connection between the Oh–Zumbrun analysis and other works, filling in the previously missing link of pointwise Green function bounds for periodic-coefficient operators on a bounded periodic domain, a topic that seems of interest in its own right. In addition, the analysis has a flavor of explicit, spatial domain computation that illuminates the arguments of Schneider, Johnson–Zumbrun, and others by weighted energy estimates, Hausdorff–Young inequality, and other frequency domain techniques.

A novel aspect of the present work is to obtain pointwise bounds also on the nonlinear solution, and thereby sharp $L^p$ bounds for all $1 \leqslant p \leqslant \infty$. Schneider's weighted $H^2$ estimates, obtained by renormalization techniques, yield $L^p$ bounds for $1 \leqslant p \leqslant \infty$ of $(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\eta}$ for any $\eta > 0$, just slightly weaker than ours; however, the estimates of Johnson–Zumbrun, obtained by Hausdorff–Young's inequality appear limited to $2 \leqslant p \leqslant \infty$. The more detailed pointwise bounds we obtain here do not seem to be accessible by either of these previous two techniques.

An important advantage of our approach over the renormalization techniques used by Schneider and others, is that, being based rather on the nonlinear tracking scheme of Johnson–Zumbrun, it should apply in principle also to situations, such as periodic solutions of conservation laws like the Kuramoto–Sivashinsky equations and others, for which the asymptotic behavior consists of multiple signals convecting with distinct speeds; see for example the analysis of [11,13,10]. By contrast, renormalization techniques appear limited to situations of a single signal. The extension of our results to the conservation law case is an interesting open problem.

Finally, we mention that the techniques used here extend to general quasilinear parabolic or even mixed, partially parabolic problems, so that our analysis could in principle extend to these more general settings; see, for example, the related analyses in [9,17,13]. This would be another very interesting direction to carry out.

### 1.3. Plan of the paper

The paper is divided mainly into two parts. In the first part (Sections 2–5), we obtain pointwise bounds on the Green function $G(x,t;y)$ for Eq. (1.2). In the second part (Sections 6–7), we show the asymptotic behavior of perturbations of spatially periodic traveling waves converges to heat kernel. More precisely, in Section 2, we recall the definition of resolvent kernel and then we construct resolvent kernels of the linear operator $L_\xi$ for the whole line and for the periodic condition on $[0,1]$, respectively. In Section 3, we estimate pointwise high frequency periodic resolvent kernel bounds using the formula we obtained in Section 2. As we see the spectral resolution formula (see (4.1)), high frequency resolvent kernel bounds is the first step for pointwise bounds on the Green function $G(x,t;y)$ for Eq. (1.2) in Section 4. In Section 5, we give the simplest case of a scalar, constant coefficient equation. By a direction calculation, we construct resolvent kernels of the simplest operator for the whole line and for the periodic condition on $[0,1]$ respectively, and we show how those two resolvent kernels are related. In Section 6, as a practice, we show the asymptotic behavior of a solution of $u_t = u_{xx} + u^p$ for $p \geqslant 4$. This will give the idea how to show the asymptotic behavior of perturbations of spatially periodic traveling waves which we want to show mainly in Section 7. The 3 parts of Theorem 1.4 are established in Theorems 7.7, 7.13 and 7.22, respectively.

## 2. The resolvent kernel

In this section, we develop an interesting formula for the resolvent kernel on the whole line and for periodic boundary conditions on $[0,1]$ using solution operators and projections. Those formulas are motivated by a constant-coefficient scalar case (see Section 5). Here, "whole-line" means the kernel of periodic-coefficient operator considered as acting on $L^2(\mathbb{R})$. As we mentioned, the main difference between Oh–Zumbrun analysis and this paper is using the Bloch transform more directly. In this sense, together with the spectral resolution formula for $L_\xi$, it is natural to construct the periodic resolvent kernel and compare this to the whole line resolvent kernel. We use this periodic resolvent kernel to obtain a high-frequency description of the resolvent (that is, $|\lambda| > R$, for sufficiently large $R$) for periodic boundary conditions $[0,1]$ in Section 3. We start with the definition of resolvent kernel.

For $\lambda$ in the resolvent set of $L$, we denote by $G_\lambda(x, y)$ the resolvent kernel defined by

$$(L-\lambda I)G_\lambda(\cdot, y) := \delta_y \cdot I,$$

$\delta_y$ denoting the Dirac delta distribution centered at $y$, or equivalently

$$(L-\lambda I)^{-1} f(x) = \int G_\lambda(x, y) f(y)\, dy.$$

In this paper, for each $\xi \in [-\pi, \pi)$ and for $\lambda$ in the resolvent set of $L_\xi$, we denote by $\mathcal{G}_{\xi,\lambda}(x, y)$ and $G_{\xi,\lambda}(x, y)$ the resolvent kernels of $L_\xi$ on the whole line and on $[0,1]$ with periodic boundary conditions, respectively.

**Remark 2.1.** The spectrum of each $L_\xi$ may alternatively be characterized as the zero set for fixed $\xi$ of the periodic Evans function introduced by Gardner in [7] and [8],

$$D(\lambda, \xi) = det\big(\Psi(\lambda) - e^{i\xi} I\big),$$

where $\Psi$ is the monodromy matrix of (1.8), and $D(\lambda, \xi)$ is analytic in each argument $\lambda$ and $\xi$; likewise, the spectrum of $L$ may be described as the set of all $\lambda$ such that $D(\lambda, \xi)$ vanished for some real $\xi$. So if $\lambda$ is in the resolvent set of $L$, then

$$det\big(\Psi(\lambda) - e^{i\xi} I\big) \neq 0 \quad \text{for all } \xi \in \mathbb{R}, \tag{2.1}$$

that is, $\mathcal{F}^{y\to y+1} - e^{i\xi} I$ is invertible for all $\xi \in \mathbb{R}$. Using decomposition

$$\mathcal{F}^{y\to y+1} = e^{i\xi} \begin{pmatrix} I & 0 \\ i\xi I & I \end{pmatrix} \mathcal{F}_\xi^{y\to y+1} \begin{pmatrix} I & 0 \\ i\xi I & I \end{pmatrix}^{-1}, \tag{2.2}$$

$I - \mathcal{F}_\xi^{y\to y+1}$ is invertible for all $\xi \in \mathbb{R}$. Also (2.1) implies the existence of $\Pi^\pm$ and $\Pi_\xi^\pm$ because $\Psi(\lambda)$ does not have eigenvalue of norm 1.

### 2.1. The whole line case

**Lemma 2.2.** *For all $\xi \in [-\pi, \pi)$, the whole line kernel (see the definition above) satisfies*

$$\begin{pmatrix} \mathcal{G}_{\xi,\lambda} \\ \mathcal{G}'_{\xi,\lambda} \end{pmatrix}(x, y) = \begin{cases} \mathcal{F}_\xi^{y\to x} \Pi_\xi^+(y) \binom{0}{I}, & x > y, \\ -\mathcal{F}_\xi^{y\to x} \Pi_\xi^-(y) \binom{0}{I}, & x \leqslant y, \end{cases} \tag{2.3}$$

*where $\Pi_\xi^\pm$ are projections onto the manifolds of solutions decaying as $x \to \pm\infty$.*

**Proof.** We must only check the jump condition $[\binom{\mathcal{G}_{\xi,\lambda}}{\mathcal{G}'_{\xi,\lambda}}]|_y = \binom{0}{I}$, which follows from $\mathcal{F}_\xi^{y\to y} = I$ and $\Pi_\xi^+ + \Pi_\xi^- = I$, and the fact that $\mathcal{G}_{\xi,\lambda}(x, y) \to 0$ as $x \to \pm\infty$, which is clear by inspection. □

### 2.2. The periodic case

**Lemma 2.3.** *For $\lambda$ in the resolvent set of $L$ and all $\xi \in [-\pi, \pi)$, the periodic kernel satisfies*

$$\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(x,y) = \begin{cases} \mathcal{F}_\xi^{y\to x} M_\xi^+(y)\binom{0}{I}, & x > y, \\ -\mathcal{F}_\xi^{y\to x} M_\xi^-(y)\binom{0}{I}, & x \leqslant y, \end{cases} \tag{2.4}$$

*where $M_\xi^+(y) = (I - \mathcal{F}_\xi^{y\to y+1})^{-1}$ and $M_\xi^-(y) = -(I - \mathcal{F}_\xi^{y\to y+1})^{-1}\mathcal{F}_\xi^{y\to y+1}$.*

(*Note*: *Remark 2.1 implies the existence of $M_\xi^+$ and $M_\xi^-$.*)

**Proof.** We must check the jump condition $[(\begin{smallmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{smallmatrix})]|_y = \binom{0}{I}$, which follows from $\mathcal{F}_\xi^{y\to y} = I$ and $M_\xi^+ + M_\xi^- = I$, and the periodicity, $(\begin{smallmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{smallmatrix})(0,y) = (\begin{smallmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{smallmatrix})(1,y)$. By the periodicity of the solution operator, $\mathcal{F}_\xi^{0\to y}\mathcal{F}_\xi^{y\to 1} = \mathcal{F}_\xi^{1\to y+1}\mathcal{F}_\xi^{y\to 1} = \mathcal{F}_\xi^{y\to y+1}$. By a direct computation, we obtain $\mathcal{F}_\xi^{y\to 1}(I - \mathcal{F}_\xi^{y\to y+1})^{-1} = \mathcal{F}_\xi^{y\to 0}(I - \mathcal{F}_\xi^{y\to y+1})^{-1}\mathcal{F}_\xi^{y\to y+1}$ which gives us $(\begin{smallmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{smallmatrix})(0,y) = (\begin{smallmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{smallmatrix})(1,y)$. □

## 3. Pointwise bounds on $G_{\xi,\lambda}$ for $|\lambda| > R$, $R$ sufficiently large

We now estimate pointwise bounds on periodic resolvent kernel $G_{\xi,\lambda}$ for $|\lambda| > R$, for sufficiently large $R$ with the formula (2.4). From the decomposition

$$\mathcal{F}_\xi^{y\to x} M_\xi^\pm(y) = \mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y) M_\xi^\pm(y) + \mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y) M_\xi^\pm(y),$$

we start with estimates of $\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)$ and $\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)$ for sufficiently large $|\lambda|$.

For the proof of Lemma 3.1, we follow the proof of high frequency bounds which come from Zumbrun and Howard [20].

**Lemma 3.1.** *For each $|\xi| \leqslant \pi$ and for sufficiently large $|\lambda|$,*

$$\begin{aligned} \mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y) &= e^{-\beta^{-1/2}|\lambda^{1/2}|(x-y)} N_1 O(1) N_2, \quad \textit{for } x > y, \\ \mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y) &= e^{-\beta^{-1/2}|\lambda^{1/2}|(y-x)} N_1 O(1) N_2, \quad \textit{for } x \leqslant y, \end{aligned} \tag{3.1}$$

*where $N_1 = \begin{pmatrix} |\lambda^{-1/2}| I & 0 \\ 0 & I \end{pmatrix}$, $N_2 = \begin{pmatrix} |\lambda^{1/2}| I & 0 \\ 0 & I \end{pmatrix}$ and $\Pi_\xi^\pm$ projections onto the manifolds of solutions decaying as $x \to \pm\infty$, and here $\beta^{-1/2} \sim \min_{\{\lambda:\, Re\,\lambda \geqslant \eta_1 - \eta_2 |Im\,\lambda|\}} Re(\sqrt{\lambda/|\lambda|})$.*[2]

**Proof.** Setting $\bar{x} = |\lambda^{\frac{1}{2}}|x$, $\bar{\lambda} = \lambda/|\lambda|$, $\bar{u}(\bar{x}) = u(\bar{x}/|\lambda^{\frac{1}{2}}|)$, $\bar{C}(\bar{x}) = C(\bar{x}/|\lambda^{\frac{1}{2}}|)$, in (1.9), we obtain

$$\bar{u}'' = \bar{\lambda}\bar{u} + \left|\lambda^{-\frac{1}{2}}\right| A_\xi \bar{u}' + \left|\lambda^{-1}\right| \bar{C}_\xi \bar{u}, \tag{3.2}$$

or

$$\bar{U}' = \bar{\mathbb{A}}\bar{U} + \Theta_\xi \bar{U}, \tag{3.3}$$

[2] Here and elsewhere in this section, $O(1)$ is matrix-valued, denoting a matrix with bounded coefficients.

where $\bar U = \binom{\bar u}{\bar u'}$, $\bar{\mathbb{A}} = \begin{pmatrix} 0 & I \\ \bar\lambda I & 0 \end{pmatrix}$, $\Theta_\xi = \begin{pmatrix} 0 & 0 \\ |\lambda^{-1}|\bar C_\xi & |\lambda^{-\frac12}|A_\xi \end{pmatrix}$ and $|\bar\lambda| = 1$. Denote by $\bar{\mathcal{F}}_\xi^{\bar y \to \bar x}$ the solution operator of (3.3) and by $\bar\Pi_\xi^\pm$ projections onto the manifolds of solutions decaying as $x \to \pm\infty$.

It is easily computed that the eigenvalues of $\bar{\mathbb{A}}$ are $\mp\sqrt{\bar\lambda}$ and

$$Re\sqrt{\bar\lambda} > \beta^{-1/2} \tag{3.4}$$

for all $\lambda \in \{Re\,\lambda \geqslant \eta_1 - \eta_2|Im\,\lambda|\}$ for some $\beta > 0$ and $\eta_1, \eta_2 > 0$, hence the stable and unstable subspaces of each $\bar{\mathbb{A}}$ are both of dimension $n$, and separated by a spectral gap of more than $2\beta$. Let $P = \binom{P_+}{P_-}$, where rows of $P_\pm$ are left eigenvectors corresponding $\mp\sqrt{\bar\lambda}$, respectively.

Introducing new coordinates $w_\pm = P_\pm \bar U$ and using $P\bar{\mathbb{A}}P^{-1} = \begin{pmatrix} -\sqrt{\bar\lambda}I & 0 \\ 0 & \sqrt{\bar\lambda}I \end{pmatrix}$, we obtain a block diagonal system

$$\begin{pmatrix} w_+ \\ w_- \end{pmatrix}' = \begin{pmatrix} -\sqrt{\bar\lambda}I & 0 \\ 0 & \sqrt{\bar\lambda}I \end{pmatrix}\begin{pmatrix} w_+ \\ w_- \end{pmatrix} + \bar\Theta_\xi \begin{pmatrix} w_+ \\ w_- \end{pmatrix}, \tag{3.5}$$

where

$$\begin{aligned} \bar\Theta_\xi &= P\Theta_\xi P^{-1} \\ &= \frac12 \begin{pmatrix} I & -\sqrt{\bar\lambda}^{-1} \\ I & \sqrt{\bar\lambda}^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ |\lambda^{-1}|\bar C_\xi & |\lambda^{-\frac12}|A_\xi \end{pmatrix} \begin{pmatrix} I & I \\ -\sqrt{\bar\lambda} & \sqrt{\bar\lambda} \end{pmatrix} \\ &= \frac12 |\lambda^{-\frac12}| \begin{pmatrix} -\lambda^{-\frac12}\bar C_\xi + A_\xi & -\lambda^{-\frac12}\bar C_\xi - A_\xi \\ \lambda^{-\frac12}\bar C_\xi - A_\xi & \lambda^{-\frac12}\bar C_\xi + A_\xi \end{pmatrix} \\ &= |\lambda^{-\frac12}| \begin{pmatrix} \theta_{\xi 11} & \theta_{\xi 12} \\ \theta_{\xi 21} & \theta_{\xi 22} \end{pmatrix}. \end{aligned} \tag{3.6}$$

Since $|\lambda^{-\frac12}|$ is sufficiently small for $|\lambda|$ sufficiently large, by using the tracking lemma (see [15, p. 20]), there is a unique linear transformation

$$S = \begin{pmatrix} I & \Phi_+ \\ \Phi_- & I \end{pmatrix} \quad \text{with } |\Phi_\pm| \leqslant |\lambda^{-\frac12}| \tag{3.7}$$

so that new coordinates $w_\pm = Sz_\pm$ generate an exact block diagonal system

$$\begin{pmatrix} z_+ \\ z_- \end{pmatrix}' = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}\begin{pmatrix} z_+ \\ z_- \end{pmatrix}, \tag{3.8}$$

where $A_+ = -\sqrt{\bar\lambda}I + |\lambda^{-\frac12}|(\theta_{\xi_{11}} + \theta_{\xi_{12}}\Phi_-)$, and $A_- = \sqrt{\bar\lambda}I + |\lambda^{-\frac12}|(\theta_{\xi 21}\Phi_+ + \theta_{\xi_{22}})$.

For any $|\xi| \leqslant \pi$ and for $i, j = 1, 2$, $|\theta_{\xi_{ij}}| = O(|\lambda^{-\frac12}(C - (ia\xi + \xi^2)I) + (a - 2i\xi)I|)$, and so $\theta_{\xi_{11}} + \theta_{\xi_{12}}\Phi_- = O(1) = \theta_{\xi 21}\Phi_+ + \theta_{\xi_{22}}$ for sufficiently large $|\lambda|$.

Now we have $z_+' = (-\sqrt{\bar\lambda}I + O(|\lambda^{-\frac12}|))z_+$ and $z_-' = (\sqrt{\bar\lambda}I + O(|\lambda^{-\frac12}|))z_-$. From this we obtain the energy estimate,

$$\begin{aligned} \langle z_\pm, z_\pm\rangle' &= \langle z_\pm, \mp Re\sqrt{\bar\lambda}Iz_\pm\rangle + O(|\lambda^{-\frac12}|)\langle z_\pm, z_\pm\rangle \\ &\lessgtr (\mp\beta^{-1/2} + O(|\lambda^{-\frac12}|))\langle z_\pm, z_\pm\rangle. \end{aligned}$$

So we find that

$$\big(|z_\pm|^2\big)' \leqslant \big(\mp\beta^{-1/2} + O\big(\big|\lambda^{-\frac{1}{2}}\big|\big)\big)|z_\pm|^2,$$

hence

$$\begin{aligned}
\frac{|z_+(\bar{x})|}{|z_+(\bar{y})|} &\leqslant e^{-\beta^{-1/2}(\bar{x}-\bar{y})}, \quad \text{for } \bar{x} > \bar{y}, \\
\frac{|z_-(\bar{x})|}{|z_-(\bar{y})|} &\leqslant e^{-\beta^{-1/2}(\bar{y}-\bar{x})}, \quad \text{for } \bar{x} \leqslant \bar{y},
\end{aligned} \tag{3.9}$$

provided $|\lambda|$ is sufficiently large. Since $|S| = O(1 + |\lambda^{-\frac{1}{2}}|)$ and $|P| = O(1)$, translating the bound (3.9) back to (3.3), we obtain for any $|\xi| \leqslant \pi$,

$$\begin{aligned}
\bar{\mathcal{F}}_\xi^{\bar{y}\to\bar{x}}\bar{\Pi}_\xi^+(\bar{y}) &= O(1)e^{-\beta^{-1/2}(\bar{x}-\bar{y})}, \quad \text{for } \bar{x} > \bar{y}, \\
\bar{\mathcal{F}}_\xi^{\bar{y}\to\bar{x}}\bar{\Pi}_\xi^-(\bar{y}) &= O(1)e^{-\beta^{-1/2}(\bar{y}-\bar{x})}, \quad \text{for } \bar{x} \leqslant \bar{y},
\end{aligned} \tag{3.10}$$

provided $|\lambda|$ is sufficiently large.

The operators $\mathcal{F}_\xi^{y\to x}\Pi_\xi^\pm(y)$ are evidently related to the corresponding operators $\bar{\mathcal{F}}_\xi^{\bar{y}\to\bar{x}}\bar{\Pi}_\xi^\pm(y)$ for the rescaled system by the scaling transformation

$$\mathcal{F}_\xi^{y\to x}\Pi_\xi^\pm(y) = \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix} \bar{\mathcal{F}}_\xi^{|\lambda^{1/2}|y\to|\lambda^{1/2}|x}\bar{\Pi}_\xi^\pm(y) \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}. \tag{3.11}$$

From (3.10) and $\bar{\Pi}_\xi^\pm(y) = O(1)$, we thus have

$$\begin{aligned}
\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y) &= e^{-\beta^{-1/2}|\lambda^{1/2}|(x-y)} \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix} O(1) \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}, \quad \text{for } x > y, \\
\mathcal{F}_\xi^{y\to x}\Pi_{\xi_-}(y) &= e^{-\beta^{-1/2}|\lambda^{1/2}|(y-x)} \begin{pmatrix} |\lambda^{-1/2}|I & 0 \\ 0 & I \end{pmatrix} O(1) \begin{pmatrix} |\lambda^{1/2}|I & 0 \\ 0 & I \end{pmatrix}, \quad \text{for } x \leqslant y,
\end{aligned} \tag{3.12}$$

provided $|\lambda|$ is sufficiently large. □

Now we are ready to obtain high frequency periodic resolvent bounds.

**Proposition 3.2.** *For any $|\xi| \leqslant \pi$ and any $0 \leqslant x, y \leqslant 1$,*

$$\begin{aligned}
\big|G_{\xi,\lambda}(x,y)\big| &\leqslant C\big|\lambda^{-1/2}\big|\big(e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|} + e^{-\beta^{-1/2}|\lambda^{1/2}|(1-|x-y|)}\big), \\
\big|(\partial/\partial_x)G_{\xi,\lambda}(x,y)\big| &\leqslant C\big(e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|} + e^{-\beta^{-1/2}|\lambda^{1/2}|(1-|x-y|)}\big)
\end{aligned} \tag{3.13}$$

*provided $|\lambda|$ is sufficiently large and $C > 0$, that is, $|G_{\xi,\lambda}|$ is uniformly bounded as $|\lambda| \to \infty$.*

**Proof.** We note that, by the periodicity of the resolvent kernel,

$$\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^\pm(y) = \Pi_\xi^\pm(y+1)\mathcal{F}_\xi^{y\to y+1} = \Pi_\xi^\pm(y)\mathcal{F}_\xi^{y\to y+1}, \tag{3.14}$$

which implies

$$\begin{aligned}
&\Pi_\xi^\pm(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\big)\big(I-\Pi_\xi^\pm(y)\mathcal{F}_\xi^{y\to y+1}\big)\\
&\quad=\big(I-\Pi_\xi^\pm(y)\mathcal{F}_\xi^{y\to y+1}\big)\Pi_\xi^\pm(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\big).
\end{aligned}\tag{3.15}$$

Now, recall the resolvent kernel for the periodic case as

$$\begin{pmatrix} G_{\xi,\lambda}\\ G'_{\xi,\lambda}\end{pmatrix}(x,y)=\begin{cases}\mathcal{F}_\xi^{y\to x}M_\xi^+(y)\binom{0}{I}, & x>y,\\ -\mathcal{F}_\xi^{y\to x}M_\xi^-(y)\binom{0}{I}, & x\leqslant y,\end{cases}$$

where $M_\xi^+(y)=(I-\mathcal{F}_\xi^{y\to y+1})^{-1}$ and $M_\xi^-(y)=-(I-\mathcal{F}_\xi^{y\to y+1})^{-1}\mathcal{F}_\xi^{y\to y+1}$.

Let's consider the case of $x>y$ first. Since $\Pi_\xi^+ + \Pi_\xi^- = I$,

$$\mathcal{F}_\xi^{y\to x}M_\xi^+(y)=\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)M_\xi^+(y)+\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)M_\xi^+(y).$$

From (3.12) and (3.15) and recalling that $N_1=\begin{pmatrix}|\lambda^{-1/2}|I & 0\\ 0 & I\end{pmatrix}$, $N_2=\begin{pmatrix}|\lambda^{1/2}|I & 0\\ 0 & I\end{pmatrix}$, we have for $x>y$,

$$\begin{aligned}
&\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)M_\xi^+(y)\\
&\quad=\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1}\big(I-\mathcal{F}_\xi^{y\to y+1}\big)^{-1}\\
&\quad=\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\big)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1}\big(I-\mathcal{F}_\xi^{y\to y+1}\big)^{-1}\\
&\quad=\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\big)\big(I-\mathcal{F}_\xi^{y\to y+1}\big)^{-1}\\
&\quad=\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1}\Pi_\xi^+(y)\\
&\quad=e^{-\beta^{-1/2}|\lambda^{1/2}|(x-y)}N_1O(1)N_2,
\end{aligned}\tag{3.16}$$

where we have used the fact that $\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)$ is decaying for $|\lambda|$ sufficiently large. Similarly, we have

$$\begin{aligned}
\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)M_\xi^+(y)&=\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^-(y)\big)^{-1}\Pi_\xi^-(y)\\
&\approx\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)\big(\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^-(y)\big)^{-1}\Pi_\xi^-(y)\\
&=\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)\Pi_\xi^-(y)\mathcal{F}_\xi^{y+1\to y}\\
&=\mathcal{F}_\xi^{y+1\to x}\Pi_\xi^-(y)\\
&=e^{-\beta^{-1/2}|\lambda^{1/2}|(y+1-x)}N_1O(1)N_2,
\end{aligned}\tag{3.17}$$

here, the above approximation is from the fact that $\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^-(y)$ is growing for $|\lambda|$ sufficiently large.

So, for $x>y$,

$$\begin{aligned}
\begin{pmatrix} G_{\xi,\lambda}\\ G'_{\xi,\lambda}\end{pmatrix}(x,y)&=\big(e^{-\beta^{-1/2}|\lambda^{1/2}|(x-y)}N_1O(1)N_2+e^{-\beta^{-1/2}|\lambda^{1/2}|(y+1-x)}N_1O(1)N_2\big)\begin{pmatrix}0\\ I\end{pmatrix}\\
&=\big(e^{-\beta^{-1/2}|\lambda^{1/2}|(x-y)}+e^{-\beta^{-1/2}|\lambda^{1/2}|(y+1-x)}\big)\begin{pmatrix}O(|\lambda^{-1/2}|)I\\ O(1)I\end{pmatrix}.
\end{aligned}\tag{3.18}$$

Now, we consider the case of $x \leqslant y$. From (3.15) and the calculation of (3.16), we have for $x \leqslant y$,

$$\begin{aligned}
\mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)M_\xi^-(y) &= \mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\big)^{-1}\mathcal{F}_\xi^{y\to y+1} \\
&= \mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1}\Pi_\xi^+(y)\mathcal{F}_\xi^{y\to y+1} \\
&= \mathcal{F}_\xi^{y\to x}\Pi_\xi^+(y)\mathcal{F}_\xi^{y\to y+1}\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1} \\
&= \mathcal{F}_\xi^{y\to x}\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big)^{-1} \\
&= \mathcal{F}_\xi^{y+1\to x+1}\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^+(y)\big(I-\Pi_\xi^+(y)\mathcal{F}_\xi^{y\to y+1}\big)^{-1} \\
&= \mathcal{F}_\xi^{y\to x+1}\Pi_\xi^+(y)\big(I-\Pi_\xi^+(y)\mathcal{F}_\xi^{y\to y+1}\big)^{-1} \\
&= e^{-\beta^{-1/2}|\lambda^{1/2}|(x+1-y)}N_1O(1)N_2.
\end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned}
\mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)M_\xi^-(y) &= \mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\big)^{-1}\mathcal{F}_\xi^{y\to y+1} \\
&= \mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)\big(I-\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^-(y)\big)^{-1}\Pi_\xi^-(y)\mathcal{F}_\xi^{y\to y+1} \\
&\approx \mathcal{F}_\xi^{y\to x}\Pi_\xi^-(y)\big(\mathcal{F}_\xi^{y\to y+1}\Pi_\xi^-(y)\big)^{-1}\Pi_\xi^-(y)\mathcal{F}_\xi^{y\to y+1} \\
&= \Pi_\xi^-(x)\mathcal{F}_\xi^{y\to x} \\
&= e^{-\beta^{-1/2}|\lambda^{1/2}|(y-x)}N_1O(1)N_2.
\end{aligned} \tag{3.20}$$

So, for $x \leqslant y$,

$$\begin{aligned}
\begin{pmatrix} G_{\xi,\lambda} \\ G'_{\xi,\lambda} \end{pmatrix}(x,y) &= \big(e^{-\beta^{-1/2}|\lambda^{1/2}|(x+1-y)}N_1O(1)N_2 + e^{-\beta^{-1/2}|\lambda^{1/2}|(y-x)}N_1O(1)N_2\big)\begin{pmatrix} 0 \\ I \end{pmatrix} \\
&= \big(e^{-\beta^{-1/2}|\lambda^{1/2}|(x+1-y)} + e^{-\beta^{-1/2}|\lambda^{1/2}|(y-x)}\big)\begin{pmatrix} O(|\lambda^{-1/2}|)I \\ O(1)I \end{pmatrix}.
\end{aligned} \tag{3.21}$$

This completes the proof of the proposition. □

**Remark 3.3.** We can express (3.13) as

$$G_{\xi,\lambda}(x,y) = O\big(\big|\lambda^{-1/2}\big|\big)\big(e^{-\beta^{-1/2}|\lambda^{1/2}|\min|x-y_i|}\big), \tag{3.22}$$

where $y_j = y + j$.

**Remark 3.4.** The aliasing between $y$, $y-1$ and $y+1$ indicates why the periodic resolvent formula possesses always a "$y < x$" type piece even when $y > x$. This comes from the influence of $y-1$.

**Remark 3.5.** The periodic resolvent kernel $G_{\xi,\lambda}$ may also be obtained in indirect fashion from the whole-line version $\mathcal{G}_{\xi,\lambda}$ by the *method of images*

$$\big[G_{\xi,\lambda}(x,y)\big] = \sum_{j\in\mathbb{Z}}\mathcal{G}_{\xi,\lambda}(x,y+j), \tag{3.23}$$

which is readily seen to converge (by exponential decay in $|x-y|$) for $\lambda$ in the resolvent set, and clearly is periodic and satisfies the resolvent equation on $[0,1]$. Likewise, the periodic Green function $G_\xi$ may be expressed in terms of the whole-line version $\mathcal{G}_\xi$, as

$$\big[G_\xi(x,t;y)\big] = \sum_{j\in\mathbb{Z}} \mathcal{G}_\xi(x,t;y+j). \tag{3.24}$$

See (5.14)–(5.15) for an illustrative computation in the scalar constant-coefficient case. This clarifies the results obtained above by a direct computation, and the relation between the periodic and whole-line kernels. Here, by the "whole-line" version, we mean the kernel of periodic-coefficient operator considered as acting on $L^2(\mathbb{R})$.

**Remark 3.6.** By a similar method, we have the bounds of $(G_{\xi,\lambda}(x,\cdot), \partial_y G_{\xi,\lambda}(x,\cdot))$. See [20] more details.

## 4. Pointwise bounds on G

Now we start pointwise bounds on $G(x,t;y)$ of Eq. (1.2). Let's first define the sector

$$\Omega := \big\{\lambda:\ Re(\lambda) \geqslant -\theta_1 - \theta_2\big|Im(\lambda)\big|\big\},$$

where $\theta_1, \theta_2 > 0$ are small constants.

**Proposition 4.1.** *(See [20].) The parabolic operator $\partial_t - L$ has a Green function $G(x,t;y)$ for each fixed $y$ and $(x,t) \neq (y,0)$ given by*

$$G(x,t;y) = \frac{1}{2\pi i} \int_{\Gamma := \partial(\Omega\setminus B(0,R))} e^{\lambda t} G_\lambda(x,y)\,d\lambda \tag{4.1}$$

*for $R > 0$ sufficiently large and $\theta_1$, $\theta_2 > 0$ sufficiently small. This is the standard spectral resolution (inverse Laplace transform) formula.*

The standard spectral resolution formula (4.1), together with high frequency periodic resolvent bounds given previous section, will be the starting point for the proof of Theorem 1.3.

**Proof of Theorem 1.3.** *Case* (i). $\frac{|x-y|}{t}$ *large*. We first consider the case that $|x-y|/t \geqslant S$, $S$ sufficiently large. For this case, it is hard to estimate $G$ through $|[G_\xi(x,t;y)]|$, directly, because of the problem of aliasing; see Remark 4.2. Instead we estimate $|G_\lambda(x,y)|$ first and we estimate $|G(x,t;y)|$ by (4.1). This is treated by exactly the same argument as in [20]. By [20], notice that

$$\big|G_\lambda(x,y)\big| \leqslant C\big|\lambda^{-1/2}\big|e^{-\beta^{-1/2}|\lambda^{1/2}||x-y|},$$

for all $\lambda \in \Omega\setminus B(0,R)$ and $R > 0$ sufficiently large, and here, $\beta^{-1/2} \sim \min_{\lambda\in\Omega\cap\{|\lambda|>R\}} Re\sqrt{\lambda/|\lambda|}$.

Finally we have

$$\big|G(x,t;y)\big| \leqslant C\left|\int_\Gamma e^{\lambda t} G_\lambda(x,y)\,d\lambda\right| \leqslant t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{|x-y-at|^2}{Mt}},$$

for some $\eta > 0$ and $M > 0$ sufficiently large. (See [20] for a detail proof.)

*Case* (ii). $\frac{|x-y|}{t} < S$ *bounded*. To begin, notice that by standard spectral perturbation theory [14], the total eigenprojection $P(\xi)$ onto the eigenspace of $L_\xi$ associated with the eigenvalues $\lambda(\xi)$ bifurcating from the $(\xi, \lambda(\xi)) = (0,0)$ state is well defined and analytic in $\xi$ for $\xi$ sufficiently small, since the discreteness of the spectrum of $L_\xi$ implies that the eigenvalue $\lambda(\xi)$ is separated at $\xi = 0$ from the remainder of the spectrum of $L_0$. By (D2), there exists an $\varepsilon > 0$ such that $Re\,\sigma(L_\xi) \leqslant -\theta|\xi|^2$ for $0 < |\xi| < 2\varepsilon$. With this choice of $\varepsilon$, we first introduce a smooth cut off function $\phi(\xi)$ such that

$$\phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leqslant \varepsilon, \\ 0, & \text{if } |\xi| \geqslant 2\varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a sufficiently small parameter. Now from the inverse Bloch–Fourier transform representation, we split the Green function

$$G(x,t;y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \check{\delta}_y(\xi, x)\, d\xi$$

into its low-frequency part

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \phi(\xi) P(\xi) e^{L_\xi t} \check{\delta}_y(\xi, x)\, d\xi$$

and high frequency part

$$II = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \big(1 - \phi(\xi) P(\xi)\big) e^{L_\xi t} \check{\delta}_y(\xi, x)\, d\xi.$$

Let's start by considering the second part *II*. Noting first that

$$\check{\delta}_y(\xi, x) = \sum_{j \in \mathbb{Z}} e^{j2\pi i x} \check{\delta}_y(\xi + j2\pi) = \sum_{j \in \mathbb{Z}} e^{j2\pi i x} e^{-(\xi + j2\pi) y} = e^{-i\xi y} \sum_{j \in \mathbb{Z}} e^{j2\pi i (x-y)} = e^{-i\xi y} \big[\delta_y(x)\big],$$

we have for $|\xi| \geqslant 2\varepsilon$, $\phi(\xi) = 0$ and

$$\begin{aligned} \int_{2\varepsilon \leqslant |\xi| \leqslant \pi} e^{i\xi x} \big(1 - \phi(\xi) P(\xi)\big) e^{L_\xi t} \check{\delta}_y(\xi, x)\, d\xi &= \int_{2\varepsilon \leqslant |\xi| \leqslant \pi} e^{i\xi x} e^{L_\xi t} \check{\delta}_y(\xi, x)\, d\xi \\ &= \int_{2\varepsilon \leqslant |\xi| \leqslant \pi} e^{i\xi (x-y)} e^{L_\xi t} \big[\delta_y(x)\big]\, d\xi \\ &= \int_{2\varepsilon \leqslant |\xi| \leqslant \pi} e^{i\xi (x-y)} \big[G_\xi(x,t;y)\big]\, d\xi, \end{aligned}$$

where the brackets $[\cdot]$ denote the periodic extensions of the given function onto the whole line. Assuming that $Re\,\sigma(L_\xi) \leqslant -\eta < 0$ for $|\xi| \geqslant 2\varepsilon$, we have

$$\left[G_\xi(x,t;y)\right] = \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}\left[G_{\xi,\lambda}(x,y)\right]d\lambda,$$

here, we fix $\Gamma_1 = \partial(\Omega \cap \{Re\,\lambda \leqslant -\eta\})$ independent of $\xi$. Parameterizing $\Gamma_1$ by $Im\,\lambda := k$, and applying the bounds of $\sup_{|\xi|\leqslant\pi} |[G_{\xi,\lambda}(x,y)]| < O(|\lambda^{-\frac{1}{2}}|)$ for large $|\lambda|$ in Section 3, we have

$$\begin{aligned}\left|\left[G_\xi(x,t;y)\right]\right| &\leqslant C\int_{\Gamma_1} e^{Re\,\lambda t}\left|\left[G_{\xi,\lambda}(x,y)\right]\right|d\lambda \\ &\leqslant Ce^{-\eta t}\int_0^\infty k^{-\frac{1}{2}}e^{-\theta_2 kt}\,dk \\ &\leqslant Ct^{-\frac{1}{2}}e^{-\eta t} \\ &\leqslant Ct^{-\frac{1}{2}}e^{-\frac{\eta}{2}t}e^{-\frac{|x-y-at|^2}{Mt}},\end{aligned}$$

here, the last inequality is from $\frac{|x-y-at|}{t} \leqslant S_1$ bounded. Indeed, for large $M > 0$,

$$e^{-\frac{|x-y-at|^2}{Mt}} = e^{-(\frac{|x-y-at|}{t})^2\frac{t}{M}} \geqslant e^{-\frac{S_1}{M}t} \geqslant e^{-\frac{\eta}{2}t},$$

and so,

$$\begin{aligned}\left|\int_{2\varepsilon\leqslant|\xi|\leqslant\pi} e^{i\xi x}\big(1-\phi(\xi)P(\xi)\big)e^{L_\xi t}\check{\delta}_y(\xi,x)\,d\xi\right| &\leqslant C\sup_{2\varepsilon\leqslant|\xi|\leqslant\pi}\left|\left[G_\xi(x,t;y)\right]\right| \\ &\leqslant Ct^{-\frac{1}{2}}e^{-\frac{\eta}{2}t}e^{-\frac{|x-y-at|^2}{Mt}}. \end{aligned} \tag{4.2}$$

For $|\xi|$ sufficiently small, on the other hand, $\phi(\xi) = 1$, and $I - \phi(\xi)P = I - P = Q$, where $Q$ is the eigenprojection of $L_\xi$ associated with eigenvalues complementary to $\lambda(\xi)$ bifurcating from $(\xi,\lambda(\xi)) = (0,0)$, which have real parts strictly less than zero. So we can estimate for $|\xi| \leqslant \varepsilon$ in the same way as in (4.2). Combining these observations, we have the estimate

$$|II| \leqslant Ct^{-\frac{1}{2}}e^{-\frac{\eta}{2}t}e^{-\frac{|x-y-at|^2}{Mt}},$$

for some $\eta > 0$ and sufficiently large $M > 0$.

Next, we consider the first part $I$

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{|\xi|\leqslant 2\varepsilon} e^{i\xi x}\phi(\xi)P(\xi)e^{L_\xi t}\check{\delta}_y(\xi,x)\,d\xi \\ &= \frac{1}{2\pi}\int_{|\xi|\leqslant 2\varepsilon} e^{i\xi x}\phi(\xi)e^{\lambda(\xi)t}q(x,\xi)\tilde{q}(y,\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(x-y)}e^{(-ia\xi-b\xi^2)t}q(x,0)\tilde{q}(y,0)\,d\xi \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2\pi} \int\limits_{|\xi|\geqslant 2\varepsilon} e^{i\xi(x-y)} e^{(-ia\xi-b\xi^2)t} q(x,0)\tilde{q}(y,0)\, d\xi \\
&+ \frac{1}{2\pi} \int\limits_{|\xi|\leqslant 2\varepsilon} e^{i\xi(x-y)} e^{(-ia\xi-b\xi^2)t} \bigl(e^{O(|\xi^3|)t}\phi(\xi)q(x,\xi)\tilde{q}(y,\xi) - q(x,0)\tilde{q}(y,0)\bigr)\, d\xi \\
= {} & \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} q(x,0)\tilde{q}(y,0) + II' + III'.
\end{aligned} \tag{4.3}
$$

View $II'$ and $III'$ as complex contour integrals in the variable $\xi$ and define

$$
\bar{\alpha} := \left| \frac{x-y-at}{2bt} \right| \tag{4.4}
$$

which is bounded because $|x-y|/t$ is bounded. Using the Cauchy's Theorem and writing $\xi_1 = \xi + i\bar{\alpha}$ and $\xi_2 = \varepsilon + iz$, we have the estimate

$$
\begin{aligned}
|II'| &\leqslant C \left| \int_{\varepsilon}^{\infty} e^{i\xi_1(x-y-at)} e^{-b\xi_1^2 t}\, d\xi_1 \right| + C \left| \int_0^{\bar{\alpha}} e^{i\xi_2(x-y-at)} e^{-b\xi_2^2 t}\, d\xi_2 \right| \\
&= C \int_{\varepsilon}^{\infty} \bigl| e^{i(\xi+i\bar{\alpha})2bt\bar{\alpha}} e^{-b(\xi+i\bar{\alpha})^2 t} \bigr|\, d\xi + C \int_0^{\bar{\alpha}} \bigl| e^{i(\varepsilon+zi)2bt\bar{\alpha}} e^{-b(\varepsilon+zi)^2 t} \bigr|\, dz \\
&= C e^{-bt\bar{\alpha}^2} \int_{\varepsilon}^{\infty} e^{-b\xi^2 t}\, d\xi + C e^{-b\varepsilon^2 t} \int_0^{\bar{\alpha}} e^{btz^2 - 2bt\bar{\alpha}z}\, dz \\
&\leqslant C e^{-\frac{|x-y-at|^2}{4bt}} t^{-\frac{1}{2}} e^{-\eta t} + C e^{-b\varepsilon^2 t} \int_0^{\bar{\alpha}} e^{-btz^2}\, dz \\
&\leqslant C e^{-\frac{|x-y-at|^2}{4bt}} t^{-\frac{1}{2}} e^{-\eta t} + C e^{-b\varepsilon^2 t} t^{-\frac{1}{2}} \\
&\leqslant C t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{|x-y-at|^2}{Mt}},
\end{aligned}
$$

for some positive $\eta$ and $M > 0$ sufficiently large.

Similarly, setting

$$
\tilde{\alpha} = \min\{\varepsilon, \bar{\alpha}\},
$$

we can estimate $|III'|$ which is

$$
\begin{aligned}
|III'| &= C \left| \int\limits_{|\xi|\leqslant\varepsilon} e^{i\xi(x-y)} e^{(-ia\xi-b\xi^2)t} \bigl(e^{O(|\xi|^3)t} - 1 + O(|\xi|)\bigr)\, d\xi \right| \\
&\leqslant C \left| \int_{-\varepsilon}^{\varepsilon} e^{i(\xi+i\tilde{\alpha})(x-y-at)} e^{-b(\xi+i\tilde{\alpha})^2 t} \bigl(e^{O(|\xi|^3)t + O(|\tilde{\alpha}|^3)t} - 1 + O(|\xi|) + O(|\tilde{\alpha}|)\bigr)\, d\xi \right|
\end{aligned}
$$

$$+ C\left| \int_0^{\tilde{\alpha}} e^{i(\varepsilon+iz)(x-y-at)} e^{-b(\varepsilon+iz)^2 t} \left(e^{O(|\varepsilon|^3)t+O(|z|^3)t} - 1 + O\left(|\varepsilon|\right) + O\left(|z|\right)\right) dz \right|$$

$$\leqslant Ce^{-bt\tilde{\alpha}^2} \int_{-\varepsilon}^{\varepsilon} e^{-b\xi^2 t} \left(e^{O(|\xi|^3)t+O(|\tilde{\alpha}|^3)t} - 1 + O\left(|\xi|\right) + O(\tilde{\alpha})\right) d\xi$$

$$+ Ce^{-b\varepsilon^2 t} \int_0^{\tilde{\alpha}} e^{bz^2 t - 2bt\tilde{\alpha}z} \left(e^{O(|\varepsilon|^3)t+O(|z|^3)t} - 1 + O\left(|\varepsilon|\right) + O\left(|z|\right)\right) dz$$

$$\leqslant Ce^{-\frac{bt\tilde{\alpha}^2}{2}} \int_{-\varepsilon}^{\varepsilon} e^{-\frac{b\xi^2 t}{2}} \left(O\left(|\xi|\right) + 1\right) d\xi + Ce^{-\frac{b\varepsilon^2 t}{2}} \int_0^{\tilde{\alpha}} e^{-\frac{bz^2 t}{2}} \left(O\left(|z|\right) + 1\right) dz$$

$$\leqslant Ce^{-\frac{|x-y-at|^2}{M_2 t}} \left((t+1)^{-1} + t^{-\frac{1}{2}} e^{-\eta t}\right),$$

for some $\eta > 0$ and $M > 0$ sufficiently large. A similar argument yields the corresponding result for $\tilde{G}_y$. This completes the proof of the theorem. □

**Remark 4.2.** From (3.24), we see that estimating $G$ using $|[G_\xi]|$ would result rather in the sum of aliased versions of the Green functions on the whole line, centered at all $y + j$, which for small $|x-y|/t$ would lead to non-negligible errors. That is, in the "small-time" regime $|x-y|/t$ large there is considerable cancellation in the inverse Bloch transform involving the integration with respect to $\xi$, that cannot be detected by modulus bounds alone. It is for this reason that we compute in this regime using direct inverse Laplace transform estimates as in [20]. That is, this part of our analysis has a very different flavor from the rest of the estimates using the Bloch decomposition. For short time, these estimates may be obtained from standard parametrix estimates as in [5]; indeed, we conjecture that with further effort one might recover by parametrix methods the same bounds for all $|x-y|/t$ sufficiently large.

## 5. Example (constant-coefficient scalar case)

In this section, we illustrate the previous analysis by a simple example. Consider the constant-coefficient scalar case

$$u_t + au_x = u_{xx}, \quad a > 0 \text{ constant}. \tag{5.1}$$

This gives an eigenvalue equation for each $\xi \in [-\pi, \pi)$,

$$u'' - (a - i2\xi)u' - \left(\xi^2 + ia\xi\right)u = \lambda u. \tag{5.2}$$

Rewriting as a first-order system

$$U' = \mathbb{A}_\xi(x, \lambda)U, \tag{5.3}$$

where

$$U = \begin{pmatrix} u \\ u' \end{pmatrix}, \qquad \mathbb{A}_\xi = \begin{pmatrix} 0 & 1 \\ \lambda + \xi^2 + ia\xi & a - i2\xi \end{pmatrix}. \tag{5.4}$$

By a direct calculation we can find two eigenvalues of $\mathbb{A}_\xi$,

$$\mu_\pm = \frac{a - i2\xi \pm \sqrt{a^2 + 4\lambda}}{2}, \tag{5.5}$$

which are solutions of the characteristic equation

$$\mu^2 - (a - i2\xi)\mu - \lambda - \xi^2 - ia\xi = 0. \tag{5.6}$$

Without of loss generality we assume $Re\,\mu_- < 0$ and $Re\,\mu_+ > 0$.

Let's construct $G_{\xi,\lambda}(x, y)$ and $\mathcal{G}_{\xi,\lambda}(x, y)$. To find $\mathcal{G}_{\xi,\lambda}(x, y)$, set

$$\mathcal{G}_{\xi,\lambda}(x, y) = \begin{cases} A(y)e^{\mu_- x}, & x > y, \\ B(y)e^{\mu_+ x}, & x \leqslant y, \end{cases} \tag{5.7}$$

which satisfies the jump condition $[\binom{\mathcal{G}_{\xi,\lambda}}{\mathcal{G}'_{\xi,\lambda}}]|_y = \binom{0}{1}$. By a direct calculation, we have

$$\mathcal{G}_{\xi,\lambda}(x, y) = \begin{cases} \frac{e^{\mu_-(x-y)}}{\mu_- - \mu_+}, & x > y, \\ \frac{e^{\mu_+(x-y)}}{\mu_- - \mu_+}, & x \leqslant y. \end{cases} \tag{5.8}$$

In this case, the projections are

$$\Pi_\xi^+ = \begin{pmatrix} -\frac{\mu_+}{\mu_- - \mu_+} & \frac{1}{\mu_- - \mu_+} \\ -\frac{\mu_-\mu_+}{\mu_- - \mu_+} & \frac{\mu_-}{\mu_- - \mu_+} \end{pmatrix}, \qquad \Pi_\xi^- = \begin{pmatrix} \frac{\mu_-}{\mu_- - \mu_+} & -\frac{1}{\mu_- - \mu_+} \\ \frac{\mu_-\mu_+}{\mu_- - \mu_+} & -\frac{\mu_+}{\mu_- - \mu_+} \end{pmatrix}, \tag{5.9}$$

and the solution operator of (5.3) is

$$\mathcal{F}_\xi^{y\to x} = e^{\mathbb{A}_\xi(x-y)} = e^{\mu_-(x-y)}\Pi_\xi^+ + e^{\mu_+(x-y)}\Pi_\xi^-, \tag{5.10}$$

and hence the formula (2.3) is exactly the same as (5.8).

Similarly, we find $G_{\xi,\lambda}(x, y)$ by setting

$$G_{\xi,\lambda}(x, y) = \begin{cases} A(y)e^{\mu_- x} + B(y)e^{\mu_+ x}, & x > y, \\ C(y)e^{\mu_- x} + D(y)e^{\mu_+ x}, & x \leqslant y. \end{cases} \tag{5.11}$$

We need to find $A(y)$, $B(y)$, $C(y)$ and $D(y)$ which satisfy the periodicity $\binom{G_{\xi,\lambda}}{G'_{\xi,\lambda}}(0, y) = \binom{G_{\xi,\lambda}}{G'_{\xi,\lambda}}(1, y)$ and the jump condition $[\binom{G_{\xi,\lambda}}{G'_{\xi,\lambda}}]|_y = \binom{0}{1}$. By a direct calculation, we find for each $\xi \in [-\pi, \pi)$,

$$G_{\xi,\lambda}(x, y) = \begin{cases} \frac{e^{\mu_-(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} - \frac{e^{\mu_+(x-y)}}{(\mu_- - \mu_+)(1 - e^{\mu_+})}, & x > y, \\ \frac{e^{\mu_-(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_-})} - \frac{e^{\mu_+(x-y+1)}}{(\mu_- - \mu_+)(1 - e^{\mu_+})}, & x \leqslant y. \end{cases} \tag{5.12}$$

To verify (2.4), we first check

$$\left(I - e^{\mathbb{A}_\xi}\right)\left(\frac{1}{1 - e^{\mu_-}}\Pi_\xi^+ + \frac{1}{1 - e^{\mu_+}}\Pi_\xi^-\right) = I.$$

So

$$M_\xi^+ = \left(I - \mathcal{F}_\xi^{y\to y+1}\right)^{-1} = \left(I - e^{\mathbb{A}_\xi}\right)^{-1} = \frac{1}{1-e^{\mu_-}}\Pi_\xi^+ + \frac{1}{1-e^{\mu_+}}\Pi_\xi^-,$$

and

$$M_\xi^- = -\left(I - \mathcal{F}_\xi^{y\to y+1}\right)^{-1}\mathcal{F}_\xi^{y\to y+1} = -\frac{e^{\mu_-}}{1-e^{\mu_-}}\Pi_\xi^+ - \frac{e^{\mu_+}}{1-e^{\mu_+}}\Pi_\xi^-.$$

This implies (2.4) is exactly the same as (5.12).

Now let's show that

$$G_{\xi,\lambda}(x,y) = \sum_{j\in\mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x, y+j). \tag{5.13}$$

We first consider the case of $0 \leqslant y \leqslant x \leqslant 1$. For $j \leqslant 0$, $x > y+j$, and for $j \geqslant 1$, $x < y+j$. Thus we have, by the geometric series,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x,y+j) &= \sum_{j\leqslant 0} \mathcal{G}_{\xi,\lambda}(x,y+j) + \sum_{j\geqslant 1} \mathcal{G}_{\xi,\lambda}(x,y+j) \\
&= \frac{1}{\mu_- - \mu_+}\sum_{j\leqslant 0} e^{\mu_-(x-y-j)} + \frac{1}{\mu_- - \mu_+}\sum_{j\geqslant 1} e^{\mu_+(x-y-j)} \\
&= \frac{e^{\mu_-(x-y)}}{\mu_- - \mu_+}\sum_{j\geqslant 0}\left(e^{\mu_-}\right)^j + \frac{e^{\mu_+(x-y)}}{\mu_- - \mu_+}\sum_{j\geqslant 1}\left(e^{-\mu_+}\right)^j \\
&= \frac{e^{\mu_-(x-y)}}{(\mu_- - \mu_+)(1-e^{\mu_-})} + \frac{e^{\mu_+(x-y-1)}}{(\mu_- - \mu_+)(1-e^{-\mu_+})} \\
&= \frac{e^{\mu_-(x-y)}}{(\mu_- - \mu_+)(1-e^{\mu_-})} - \frac{e^{\mu_+(x-y)}}{(\mu_- - \mu_+)(1-e^{\mu_+})} \\
&= G_{\xi,\lambda}(x,y).
\end{aligned} \tag{5.14}$$

Similarly, we consider the case of $0 \leqslant x \leqslant y \leqslant 1$. For $j \leqslant -1$, $x > y+j$, and for $j \geqslant 0$, $x \leqslant y+j$

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x,y+j) &= \sum_{j\leqslant -1} \mathcal{G}_{\xi,\lambda}(x,y+j) + \sum_{j\geqslant 0} \mathcal{G}_{\xi,\lambda}(x,y+j) \\
&= \frac{1}{\mu_- - \mu_+}\sum_{j\leqslant -1} e^{\mu_-(x-y-j)} + \frac{1}{\mu_- - \mu_+}\sum_{j\geqslant 0} e^{\mu_+(x-y-j)} \\
&= \frac{e^{\mu_-(x-y)}}{\mu_- - \mu_+}\sum_{j\geqslant 1}\left(e^{\mu_-}\right)^j + \frac{e^{\mu_+(x-y)}}{\mu_- - \mu_+}\sum_{j\geqslant 0}\left(e^{-\mu_+}\right)^j \\
&= \frac{e^{\mu_-(x-y+1)}}{(\mu_- - \mu_+)(1-e^{\mu_-})} + \frac{e^{\mu_+(x-y)}}{(\mu_- - \mu_+)(1-e^{-\mu_+})}
\end{aligned}$$

$$= \frac{e^{\mu_-(x-y+1)}}{(\mu_- - \mu_+)(1-e^{\mu_-})} - \frac{e^{\mu_+(x-y+1)}}{(\mu_- - \mu_+)(1-e^{\mu_+1})}$$
$$= G_{\xi,\lambda}(x,y). \tag{5.15}$$

Thus, $[G_{\xi,\lambda}(x,y)] = \sum_{j\in\mathbb{Z}} \mathcal{G}_{\xi,\lambda}(x, y+j)$, and so $[G_\xi(x,t;y)] = \sum_{j\in\mathbb{Z}} \mathcal{G}_\xi(x,t;y+j)$ for all $x, y \in \mathbb{R}$.

## 6. Behavior of $u$ for $u_t = u_{xx} + u^q$, $q \geqslant 4$

In this section, we start with the nonlinear analysis of a perturbed heat equation $u_t = u_{xx} + u^q$, $q \geqslant 4$ as practice for our later analysis of $u_t = Lu + O(|u|^2)$ for the linear operator $L$ of (1.2). This allows us to illustrate the main ideas in a simple setting without the arguments of the more complicated actual system (7.11)–(7.12). In Section 6.4 we indicate in more detail the relation between $u_t = u_{xx} + u^q$ and (7.11)–(7.12), which relates more to $u_t - u_{xx} = u_x^{q/2}$. We show the behavior of $u$ satisfying $u_t = u_{xx} + u^q$, $q \geqslant 4$ for three cases of initial data ($u_0(x) = u(x,0)$):

(1) $|u_0|_{L^1\cap L^\infty}$, $|xu_0|_{L^1} \leqslant E_0$,
(2) $|u_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$,
(3) $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$,

where $E_0 > 0$ is sufficiently small and $M > 0$ sufficiently large. It is very natural to consider only $q \geqslant 4$ because for the heat kernel $k$, $u^q \sim k^q \sim t^{-\frac{(q-1)}{2}} k$ and $u_t$, $u_{xx} \sim t^{-1}k$ implies that $\frac{(q-1)}{2} > 1$ is the criterion that the nonlinear part be asymptotically negligible; see [18,19] for further discussion.

To get the asymptotic behavior of $u$, we estimate $|u(x,t) - U_* k(x,t)|_{L^p}$ with respect to the initial data (1) and estimate pointwise $|u(x,t) - U_* k(x,t)|$ with respect to (2) and (3), respectively, where

$$U_* = \int_0^\infty \int_{-\infty}^\infty u^q(y,s)\,dy\,ds + \int_{-\infty}^\infty u_0(y)\,dy.$$

For each initial data, we show that the above difference decays faster than a heat kernel so that the asymptotic behavior of $u$ converges to heat kernel (Theorems 6.7, 6.14 and 6.23).

The process in each Sections 6.1, 6.2 and 6.3 is exactly same. The main idea of the estimate is that we separate $u(x,t) - U_* k(x,t)$ into 4 parts. Setting

$$U_0 := \int_{-\infty}^\infty u_0(y)\,dy \quad \text{and} \quad U(s) = \int_{-\infty}^\infty u^q(y,s)\,dy, \tag{6.1}$$

we have, by Duhamel's principle,

$$\begin{aligned} u(x,t) - U_* k(x,t) &= \int_{-\infty}^\infty k(x-y,t)u_0(y)\,dy - U_0 k(x,t) + \int_t^\infty U(s)k(x,t)\,ds \\ &\quad + \int_0^t \left[ \int_{-\infty}^\infty k(x-y,t-s)u^q(y,s)\,dy - U(s)k(x,t-s) \right] ds \\ &\quad + \int_0^t U(s)\big(k(x,t-s) - k(x,t)\big)\,ds \\ &= I + II + III + IV. \end{aligned} \tag{6.2}$$

The parts $I$ and $III$ are exactly linear and nonlinear estimates, respectively. We begin each subsection with estimates $u$ and $xu$. We use these estimates for $U(s)$ and for terms from the Mean Value Theorem. Here, we use the Mean Value Theorem for integral,

$$f(x) - f(y) = (x-y)\int_0^1 f'\big(wx + (1-w)y\big)\,dw. \tag{6.3}$$

### 6.1. Behavior for initial data $|u_0|_{L^1\cap L^\infty}, |xu_0|_{L^1} \leqslant E_0$

In this section, we take $E_0 > 0$ sufficiently small and $q \geqslant 4$. We first estimate $|u(x,t)|_{L^p(x)}$ and $|xu(x,t)|_{L^1(x)}$ in Corollaries 6.2 and 6.4, respectively, for $|u_0|_{L^1\cap L^\infty}, |xu_0|_{L^1} \leqslant E_0$. For these two estimates, we start with Lemmas 6.1 and 6.3, respectively.

**Lemma 6.1.** *Suppose that $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1\cap L^\infty} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $q \geqslant 4$. Define*

$$\zeta(t) := \sup_{0\leqslant s\leqslant t,\,1\leqslant p\leqslant\infty} |u|_{L^p}(s)(1+s)^{\frac{1}{2}(1-\frac{1}{p})}.$$

*Then, for all $t \geqslant 0$ for which $\zeta(t)$ is finite, some $C > 0$,*

$$\zeta(t) \leqslant C\big(E_0 + \zeta^4(t)\big). \tag{6.4}$$

**Proof.** Noting, because of $q \geqslant 4$, that

$$|u|_{L^\infty}(s) \leqslant \zeta(t)(1+s)^{-\frac{1}{2}} \quad \text{and} \quad \big|u^q\big|_{L^1(x)}(s) \leqslant \big|u^{q-1}\big|_{L^\infty}|u|_{L^1} \leqslant \zeta^4(t)(1+s)^{-\frac{3}{2}},$$

we obtain

$$\begin{aligned}
\big|u(\cdot,t)\big|_{L^p(x)} &\leqslant \left|\int_{-\infty}^{\infty} k(x-y,t)u_0(y)\,dy\right|_{L^p(x)} + \left|\int_0^t\int_{-\infty}^{\infty} k(x-y,t-s)u^q(y,s)\,dy\,ds\right|_{L^p(x)} \\
&\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} + C\zeta^4(t)\int_0^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})}(1+s)^{-\frac{3}{2}}\,ds \\
&\leqslant C\big(E_0 + \zeta^4(t)\big)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.
\end{aligned}$$

Rearranging, we obtain (6.4). □

**Corollary 6.2.** *Suppose that $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1\cap L^\infty} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $q \geqslant 4$. Then*

$$\big|u(x,t)\big|_{L^p(x)} \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{6.5}$$

**Proof.** Recalling that $\zeta(t)$ is continuous so long as it remains finite, it follows by continuous induction that $\zeta(t) \leqslant 2CE_0$ for all $t \geqslant 0$ provided $E_0 < (\frac{1}{2c})^{\frac{4}{3}}$ and (as holds without loss of generality) $C \geqslant 1$, and hence (6.4) implies (6.5). □

We now estimate $|xu(x,t)|_{L^1}$.

**Lemma 6.3.** *Let $u(x,t)$ satisfy $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leqslant E_0$. Define*

$$\zeta(t) := \sup_{0 \leqslant s \leqslant t} \left|xu(x,s)\right|_{L^1(x)} (1+s)^{-\frac{1}{2}}.$$

*Then, for all $t \geqslant 0$ for which $\zeta(t)$ is finite, some $C > 0$,*

$$\zeta(t) \leqslant C\big(E_0 + \zeta^2(t)\big). \tag{6.6}$$

**Proof.** Noting, by (6.5) and $q \geqslant 4$, that

$$\left|xu^q(x,t)\right|_{L^1(x)} \leqslant \left|u^{q-1}(x,t)\right|_{L^\infty} \left|xu(x,t)\right|_{L^1} \leqslant CE_0\zeta(t)(1+t)^{-\frac{q-1}{2}+\frac{1}{2}} \leqslant CE_0\zeta(t)(1+t)^{-1},$$

we obtain the estimate

$$\begin{aligned}
\left|xu(x,t)\right|_{L^1(x)} &\leqslant \left|\int_{-\infty}^{\infty} \frac{x}{\sqrt{t}} e^{-\frac{|x-y|^2}{t}} u_0(y)\,dy\right|_{L^1(x)} + \left|\int_0^t \int_{-\infty}^{\infty} \frac{x}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{t-s}} u^q(y,s)\,dy\,ds\right|_{L^1(x)} \\
&\leqslant \left|\int_{-\infty}^{\infty} \left(\frac{x-y}{\sqrt{t}} e^{-\frac{|x-y|^2}{t}} u_0(y) + \frac{y}{\sqrt{t}} e^{-\frac{|x-y|^2}{t}} u_0(y)\right) dy\right|_{L^1(x)} \\
&\quad + \left|\int_0^t \int_{-\infty}^{\infty} \frac{x-y}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{t-s}} u^q(y,s) + \frac{y}{\sqrt{t-s}} e^{-\frac{|x-y|^2}{t-s}} u^q(y,s)\,dy\,ds\right|_{L^1(x)} \\
&\leqslant C\big((1+t)^{\frac{1}{2}} |u_0|_{L^1} + |xu_0|_{L^1}\big) + C\int_0^t \big((1+t-s)^{\frac{1}{2}} \left|u^q(x,s)\right|_{L^1} + \left|xu^q(x,s)\right|_{L^1}\big)\,ds \\
&\leqslant C\big(E_0 + \zeta^2(t)\big)(1+t)^{\frac{1}{2}}.
\end{aligned}$$

Rearranging, we obtain (6.6). □

**Corollary 6.4.** *Let $u(x,t)$ satisfy $u_t = u_{xx} + u^q$ and $|u_0|_{L^1 \cap L^\infty}, |xu_0|_{L^1} \leqslant E_0$, for $E_0 > 0$ sufficiently small, and $q \geqslant 4$. Then*

$$\left|xu(x,t)\right|_{L^1} \leqslant CE_0(1+t)^{\frac{1}{2}}, \quad \textit{for all } t \geqslant 0. \tag{6.7}$$

**Proof.** Recalling that $\zeta(t)$ is continuous so long as it remains finite, it follows by continuous induction that $\zeta(t) \leqslant 2CE_0$ for all $t \geqslant 0$ provided $E_0 < \frac{1}{4C^2}$ and (as holds without loss of generality) $C \geqslant 1$, and hence (6.6) implies (6.7). □

In the following two lemmas, we see the behavior of linear part and nonlinear part of $u$, respectively. These estimates are the fundamental decay estimates to get the behavior of $u$ in Theorem 6.7.

**Lemma 6.5** *(Linear estimate). Suppose that $u(x,t)$ solves $u_t = u_{xx}$ and $|u_0|_{L^1\cap L^\infty}, |xu_0|_{L^1} \leqslant E_0$. Then*

$$\big|u(x,t) - U_0 k(x,t)\big|_{L^p(x)} \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \tag{6.8}$$

*where $U_0 := \int_{-\infty}^{\infty} u_0(x)\,dx$ and $k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$.*

**Proof.** Setting $e(x,t) := u(x,t) - U_0 k(x,t)$, we have

$$e_t(x,t) = e_{xx}(x,t) \quad \text{and} \quad \int_{-\infty}^{\infty} e_0(x)\,dx = 0,$$

so that, for any $t \geqslant 0$,

$$\begin{aligned}\big|e(x,t)\big|_{L^p(x)} &\leqslant \int_{-\infty}^{\infty} \big|k(x-y,t)\big|_{L^p(x)}\big|u_0(y)\big|\,dy + |U_0|\big|k(x,t)\big|_{L^p(x)} \\ &\leqslant 2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}|u_0|_{L^1}.\end{aligned} \tag{6.9}$$

For $t \leqslant 1$, $\sqrt{2}(1+t)^{-\frac{1}{2}} > 1$, and hence, (6.9) implies

$$\big|u(x,t) - U_0 k(x,t)\big|_{L^p(x)} \leqslant 2(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}|u_0|_{L^1} \leqslant 2\sqrt{2}E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

Now we consider the case of $t > 1$. Noting, by the Mean Value Theorem, that

$$\big|k(x-y,t) - k(x,t)\big|_{L^p(x)} \leqslant |y| \int_0^1 \big|k_x(x-wy,t)\big|_{L^p(x)}\,dw \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}|y|,$$

we obtain

$$\begin{aligned}\big|u(x,t) - U_0 k(x,t)\big|_{L^p(x)} &\leqslant Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_{-\infty}^{\infty} |y|\big|u_0(y)\big|\,dy \\ &\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \qquad \square\end{aligned}$$

**Lemma 6.6** *(Nonlinear estimate). Suppose $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1\cap L^\infty}, |xu_0|_{L^1} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $q \geqslant 4$. Then*

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} k(x-y,t-s)u^q(y,s)\,dy - U(s)k(x,t-s)\right|_{L^p(x)} \\ &\quad\leqslant CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+s)^{-1},\end{aligned} \tag{6.10}$$

*where $U(s) = \int_{-\infty}^{\infty} u^q(y,s)\,dy$.*

**Proof.** Noting first, by (6.5) and (6.7), that

$$\left|xu^q(x,t)\right|_{L^1(x)} \leqslant \left|u^{q-1}\right|_{L^\infty(x)}\left|xu(x,t)\right|_{L^1(x)} \leqslant CE_0(1+t)^{-\frac{3}{2}}(1+t)^{\frac{1}{2}} \leqslant CE_0(1+t)^{-1},$$

we have

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} k(x-y,t-s)u^q(y,s)\,dy - U(s)k(x,t-s)\right|_{L^p(x)} \\
&\quad\leqslant \int_{-\infty}^{\infty} \left|k(x-y,t-s)-k(x,t-s)\right|_{L^p(x)}\left|u^q(y,s)\right|dy \\
&\quad\leqslant \left|k_x\big(x-y^*,t-s\big)\right|_{L^p(x)}\left|yu^q(y,s)\right|_{L^1(y)} \\
&\quad\leqslant CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+s)^{-1}. \qquad \square
\end{aligned} \tag{6.11}$$

With all previous preparations, we now obtain the asymptotic behavior of $u$. We show that the difference $|u(x,t) - U_*k(x,t)|_{L^p(x)}$ decays faster than a heat kernel with respect to $t$, so that the asymptotic behavior converges to a heat kernel.

**Theorem 6.7** *(Behavior). Suppose $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0|_{L^1\cap L^\infty}$, $|xu_0|_{L^1} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $q \geqslant 4$. Set*

$$U_* = \int_0^\infty U(s)\,ds + U_0 = \int_0^\infty \int_{-\infty}^\infty u^q(y,s)\,dy\,ds + \int_{-\infty}^\infty u_0(y)\,dy.$$

*Then $|U_*| < \infty$ and*

$$\left|u(x,t) - U_*k(x,t)\right|_{L^p(x)} \leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\big(1+\ln(1+t)\big). \tag{6.12}$$

**Proof.** Noting first, by (6.5) and $q \geqslant 4$, that

$$\left|U(s)\right| = |u^q|_{L^1} = |u|^q_{L^q} \leqslant CE_0(1+s)^{-\frac{3}{2}}, \tag{6.13}$$

we obtain

$$|U_*| \leqslant CE_0\int_0^\infty (1+s)^{-\frac{3}{2}} + |u_0|_{L^1} < \infty.$$

Now as I mentioned in (6.2), we break $|u(x,t) - U_*k(x,t)|_{L^p(x)}$ into four parts.

$$\begin{aligned}
&\left|u(x,t) - U_*k(x,t)\right|_{L^p(x)} \\
&\quad\leqslant \left|\int_{-\infty}^{\infty} k(x-y,t)u_0(y)\,dy - \int_{-\infty}^{\infty} k(x,t)u_0(y)\,dy\right|_{L^p(x)} + \int_t^\infty \left|U(s)\right|\left|k(x,t)\right|_{L^p(x)}\,ds
\end{aligned}$$

$$+\int_0^t \left| \int_{-\infty}^{\infty} k(x-y,t-s)u^q(y,s)\,dy - k(x,t-s)U(s) \right|_{L^p(x)} ds$$

$$+\int_0^t \left|U(s)\right| \left|k(x,t-s)-k(x,t)\right|_{L^p(x)} ds$$

$$= I + II + III + IV.$$

By (6.8), we already have $I \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$. By (6.13),

$$II \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_t^{\infty} (1+s)^{-\frac{3}{2}}\,ds \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$

By (6.10), we have

$$III \leqslant CE_0 \int_0^t (1+s)^{-1}(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\,ds$$

$$\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \int_0^{t/2} (1+s)^{-1}\,ds + CE_0(1+t)^{-1} \int_{t/2}^{t} (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\,ds$$

$$\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\bigl(1+\ln(1+t)\bigr).$$

By (6.13) and by the Mean Value Theorem, for some $s^* \in (0, t/2)$, we have

$$IV \leqslant CE_0\left[\int_{t/2}^{t} (1+s)^{-\frac{3}{2}}\left|k(x,t-s)-k(x,t)\right|_{L^p(x)} ds + \int_0^{t/2} (1+s)^{-\frac{3}{2}} s\left|k_t\bigl(x,t-s^*\bigr)\right|_{L^p(xi)} ds\right]$$

$$\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_{t/2}^{t} (1+s)^{-\frac{3}{2}}\,ds + CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-1} \int_0^{t/2} (1+s)^{-\frac{1}{2}}\,ds$$

$$\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \qquad \square$$

### *6.2. Behavior for initial data $|u_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$*

In this section, we take $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large and $q \geqslant 4$. The process of this section is exactly same as the previous section. We start with the following lemma which is a very useful calculation for following sections.

**Lemma 6.8.** *For all* $0 < s < t$,

$$\int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} s^{-\frac{1}{2}} e^{-\frac{|y|^2}{s}}\,dy \leqslant t^{-\frac{1}{2}} e^{-\frac{|x|^2}{t}}. \tag{6.14}$$

**Proof.** Noting first that

$$\frac{|x-y|^2}{t-s}+\frac{|y|^2}{s}=\frac{s(x^2-2xy+y^2)+(t-s)y^2}{s(t-s)}=\frac{t(y-\frac{sx}{t})^2+sx^2\frac{(t-s)}{t}}{s(t-s)},$$

we obtain

$$\int_{-\infty}^{\infty}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}s^{-\frac{1}{2}}e^{-\frac{|y|^2}{s}}\,dy\leqslant e^{-\frac{|x|^2}{t}}\int_{-\infty}^{\infty}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}e^{-\frac{t(y-sx/t)^2}{s(t-s)}}\,dy\leqslant t^{-\frac{1}{2}}e^{-\frac{|x|^2}{t}}. \qquad \square$$

We now estimate pointwise bounds of $|u(x,t)|$ and $|xu(x,t)|$ from Lemma 6.9.

**Lemma 6.9.** *Suppose $u(x,t)$ satisfies $u_t=u_{xx}+u^q$ and $|u_0(x)|\leqslant E_0e^{-\frac{|x|^2}{M}}$, for $E_0>0$ sufficiently small and $q\geqslant 4$. Define*

$$\zeta(t):=\sup_{0\leqslant s\leqslant t,\,x\in\mathbb{R}}\big|u(x,s)\big|(1+s)^{\frac{1}{2}}e^{\frac{|x|^2}{M(1+s)}},$$

*with $M>0$ sufficiently large. Then, for all $t\geqslant 0$ for which $\zeta(t)$ is finite,*

$$\zeta(t)\leqslant C\big(E_0+\zeta^2(t)\big). \tag{6.15}$$

**Proof.** By $|u^q|=|u^{q-2}|_{L^\infty}|u^2|\leqslant\zeta^2(t)(1+s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M(1+s)}}$ and (6.14), we obtain

$$\begin{aligned}\big|u(x,t)\big|&\leqslant\int_{-\infty}^{\infty}k(x-y,t)\big|u_0(y)\big|\,dy+\int_0^t\int_{-\infty}^{\infty}k(x-y,t-s)\big|u^q(y,s)\big|\,dy\,ds\\&\leqslant CE_0\int_{-\infty}^{\infty}t^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{t}}e^{-\frac{|y|^2}{M}}\,dy+C\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}(1+s)^{-\frac{3}{2}}e^{-\frac{|x|^2}{M(1+s)}}\,dy\,ds\\&\leqslant C\big(E_0+\zeta^2(t)\big)(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M(1+t)}}.\end{aligned}$$

Rearranging, we have (6.15). $\square$

**Corollary 6.10.** *Suppose $u(x,t)$ satisfies $u_t=u_{xx}+u^q$ and $|u_0(x)|\leqslant E_0e^{-\frac{|x|^2}{M}}$, for $E_0>0$ sufficiently small, $M>0$ sufficiently large, and $q\geqslant 4$. Then*

$$\big|u(x,t)\big|\leqslant CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M(1+t)}}. \tag{6.16}$$

**Proof.** Same proof as Corollary 6.4. $\square$

**Lemma 6.11.** *Suppose $u(x,t)$ satisfies $u_t=u_{xx}+u^q$ and $|u_0(x)|\leqslant E_0e^{-\frac{|x|^2}{M}}$, for $E_0>0$ sufficiently small, $M>0$ sufficiently large, and $q\geqslant 4$. Then for $M'>M$,*

$$\big|xu(x,t)\big|\leqslant CE_0e^{-\frac{|x|^2}{M'(1+t)}}. \tag{6.17}$$

**Proof.** Notice first that $|x|e^{-|x|^2} \leqslant Ce^{-|x|^2/r}$ for $r > 1$. Then by (6.16), we have

$$\big|xu(x,t)\big| \leqslant CE_0|x|(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M(1+t)}} \leqslant CE_0e^{-\frac{|x|^2}{M'(1+t)}}. \qquad \square$$

With the above estimates, we now estimate of linear and nonlinear part of $u$.

**Lemma 6.12** *(Linear estimate). Suppose $u(x,t)$ satisfies $u_t = u_{xx}$ and $|u_0(x)| \leqslant E_0e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geqslant 4$. Then for some sufficiently large $M'' > M' > M$,*

$$\big|u(x,t) - U_0k(x,t)\big| \leqslant CE_0(1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}, \tag{6.18}$$

*where $U_0 = \int_{-\infty}^{\infty} u_0(y)\,dy$ and $k(x,t) = (1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{(1+t)}}$. (Note: $|U_0| \leqslant E_0\sqrt{M}$.)*

**Proof.** Noting, by the Mean Value Theorem, that

$$\big|k(x-y,t) - k(x,t)\big| \leqslant |y| \int_0^1 \big|k_x(x-wy,t)\big|\,dw$$

we obtain, by (6.17),

$$\begin{aligned}
\big|u(x,t) - U_0k(x,t)\big| &\leqslant \int_{-\infty}^{\infty} \big|k(x-y,t) - k(x,t)\big|\big|u_0(y)\big|\,dy \\
&\leqslant E_0 \int_{-\infty}^{\infty}\int_0^1 (1+t)^{-\frac{3}{2}}|x-wy|e^{-\frac{|x-wy|^2}{(1+t)}}|y|e^{-\frac{|y|^2}{M}}\,dw\,dy \\
&\leqslant E_0 \int_0^1\int_{-\infty}^{\infty} (1+t)^{-1}e^{-\frac{|x-wy|^2}{M'(1+t)}}e^{-\frac{|y|^2}{M'}}\,dy\,dw \\
&\leqslant CE_0(1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}. \qquad \square
\end{aligned}$$

**Lemma 6.13** *(Nonlinear estimate). Suppose $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leqslant E_0e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geqslant 4$. Then for some sufficiently large $M'' > M' > M$,*

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} k(x-y,t-s)u^q(y,s)\,dy - U(s)k(x,t-s)\right| \\
&\quad\leqslant E_0(1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x|^2}{M''(1+t)}},
\end{aligned} \tag{6.19}$$

*where $U(s) = \int_{-\infty}^{\infty} u^q(y,s)\,dy$.*

**Proof.** Noting first that by $q \geqslant 4$,

$$\left|xu^q(x,s)\right| \leqslant \left|u^{p-1}\right|_{L^\infty}\left|xu(x,s)\right| \leqslant CE_0(1+s)^{-\frac{3}{2}}e^{-\frac{|x|^2}{M'(1+s)}},$$

we have, by (6.14)

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} k(x-y,t-s)u^q(y,s)\,dy - U(s)k(x,t-s)\right| \\
&\quad\leqslant E_0\int_{-\infty}^{\infty}\int_0^1 (1+t-s)^{-\frac{3}{2}}|x-wy|e^{-\frac{|x-wy|^2}{(1+t-s)}}\left|yu^q(y,s)\right|dw\,dy \\
&\quad\leqslant E_0\int_0^1\int_{-\infty}^{\infty}(1+t-s)^{-1}(1+s)^{-\frac{3}{2}}e^{-\frac{|x-wy|^2}{M'(1+t-s)}}e^{-\frac{|y|^2}{M'(1+s)}}\,dy\,dw \\
&\quad\leqslant CE_0(1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}. \qquad \square
\end{aligned}$$

**Theorem 6.14** *(Behavior). Suppose $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leqslant E_0e^{-\frac{|x|^2}{M}}$, for $E_0 > 0$ sufficiently small, $M > 0$ sufficiently large, and $q \geqslant 4$. Set*

$$U_* = \int_0^{\infty} U(s)\,ds + U_0 = \int_0^{\infty}\int_{-\infty}^{\infty} u^q(y,s)\,dy\,ds + \int_{-\infty}^{\infty} u_0(y)\,dy.$$

*Then $|U_*| < \infty$ and for some sufficiently large $M'' > M' > M$,*

$$\left|u(x,t) - U_*k(x,t)\right| \leqslant E_0(1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}\big(1+\ln(1+t)\big). \tag{6.20}$$

**Proof.** Recalling (6.5) and $q \geqslant 4$, $|U(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$ and so

$$|U_*| \leqslant CE_0\int_0^{\infty}(1+s)^{-\frac{3}{2}}\,ds + |u_0|_{L^1} < \infty.$$

Now we break $|u(x,t) - U_*k(x,t)|$ into four parts like (6.14). Then

$$II \leqslant CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{(1+t)}}\int_t^{\infty}(1+s)^{-\frac{3}{2}}\,ds \leqslant CE_0(1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}. \tag{6.21}$$

By (6.19), we have

$$III \leqslant CE_0 e^{-\frac{|x|^2}{2M'(1+t)}} \int_0^t (1+t-s)^{-1}(1+s)^{-1}\,ds$$
$$\leqslant CE_0(1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} \ln(1+t). \tag{6.22}$$

By $|U(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$ and by the Mean Value Theorem, we have, for some $s^* \in (0, t/2)$,

$$\begin{aligned}
IV &\leqslant \int_0^t |U(s)|\big|k(x,t-s) - k(x,t)\big|\,ds \\
&\leqslant CE_0 \int_{t/2}^t (1+s)^{-\frac{3}{2}}\big[(1+t-s)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t-s)}} + (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}}\big]\,ds \\
&\quad + CE_0 \int_0^{t/2} (1+s)^{-\frac{3}{2}} |s|\big|k_t\big(x, t-s^*\big)\big|\,ds \\
&\leqslant E_0(1+t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{(1+t)}} \int_{t/2}^t (1+t-s)^{-\frac{1}{2}}\,ds + E_0(1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}} \int_{t/2}^t (1+s)^{-\frac{3}{2}}\,ds \\
&\quad + E_0 e^{-\frac{|x|^2}{(1+t)}} \int_0^{t/2} (1+s)^{-\frac{1}{2}}\big(1+t-s^*\big)^{-\frac{3}{2}}\,ds \\
&\leqslant E_0(1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}.
\end{aligned} \tag{6.23}$$

By (6.18) and (6.21)–(6.23), we have (6.20). □

### 6.3. *Behavior for initial data $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$*

In this section, we take $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large and $q \geqslant 4$. Before we estimate $|u|$, we require the following information about the effects on algebraically decaying data. Corollary 6.16 is used throughout this section.

**Lemma 6.15.** *For all $t \geqslant 0$, $x \in \mathbb{R}$ and $r > 1$,*

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy \leqslant C\big[t^{-\frac{1}{2}} \wedge \big(1+|y|\big)^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x|^2}{Mt}}\big]. \tag{6.24}$$

**Proof.** We need only consider $\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} (1+|y|)^{-r}\,dy$ by symmetry.

Notice first that

$$\int_0^\infty \big(1+|y|\big)^{-r}\,dy \leqslant \frac{1}{r-1} < \infty.$$

If $x=0$, it is trivial, from

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|y|^2}{t}} \big(1+|y|\big)^{-r}\,dy \leqslant C(1+\sqrt{t}\,)^{-1}.$$

For $x \neq 0$, we break the integration into two parts

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy = \int_0^{|x|/2} + \int_{|x|/2}^\infty = I + II.$$

For the first integral $I$, if $t \leqslant 1$, we have

$$\int_0^{|x|/2} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy \leqslant C\frac{|x|}{\sqrt{t}} e^{-\frac{|x|^2}{t}} \leqslant Ce^{-\frac{|x|^2}{Mt}} \leqslant C(1+\sqrt{t}\,)^{-1} e^{-\frac{|x|^2}{Mt}},$$

and if $t \geqslant 1$, we have

$$\begin{aligned}\int_0^{|x|/2} t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy &\leqslant C(1+\sqrt{t}\,)^{-1} e^{-\frac{|x|^2}{Mt}} \int_0^{|x|/2} \big(1+|y|\big)^{-r}\,dy \\ &\leqslant C(1+\sqrt{t}\,)^{-1} e^{-\frac{|x|^2}{Mt}}.\end{aligned}$$

For the second integral $II$, we have

$$\int_{|x|/2}^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy \leqslant t^{-\frac{1}{2}} \int_{|x|/2}^\infty \big(1+|y|\big)^{-r}\,dy \leqslant Ct^{-\frac{1}{2}},$$

or

$$\int_{|x|/2}^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy \leqslant \big(1+|x|\big)^{-r} \int_{|x|/2}^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}}\,dy \leqslant C\big(1+|x|\big)^{-r}. \qquad \square$$

**Corollary 6.16.** *For all $t \geqslant 0$, $x \in \mathbb{R}$ and $r > 1$,*

$$\int_{-\infty}^\infty t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{t}} \big(1+|y|\big)^{-r}\,dy \leqslant C\Big[\big(1+|x|+\sqrt{t}\,\big)^{-r} + (1+\sqrt{t}\,)^{-1} e^{-\frac{|x|^2}{Mt}}\Big]. \tag{6.25}$$

**Proof.** By (6.24), it is enough to show that for all $x \geqslant 0$ and $t \geqslant 0$, and any $r > 1$,

$$t^{-\frac{1}{2}} \wedge \big(1+|x|\big)^{-r} \leqslant C\Big[\big(1+|x|+\sqrt{t}\,\big)^{-r} + (1+\sqrt{t}\,)^{-1} e^{-\frac{|x|^2}{Mt}}\Big].$$

For $t \leqslant 1$, we have

$$t^{-\frac{1}{2}} \wedge \left(1+|x|\right)^{-1} = \left(1+|x|\right)^{-1} \leqslant C\left(1+|x|+1\right)^{-1} \leqslant C\left(1+|x|+\sqrt{t}\right)^{-1}.$$

For $t > 1$ and $|x| \leqslant \sqrt{t}$, we have $e^{-\frac{|x|^2}{Mt}} \geqslant e^{-\frac{1}{M}} > 0$, and so

$$t^{-\frac{1}{2}} \wedge \left(1+|x|\right)^{-r} \leqslant C(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{Mt}}.$$

For $t > 1$ and $|x| \geqslant \sqrt{t}$,

$$\left(1+|x|\right)^{-r} \leqslant |x|^{-r} \leqslant |x|^{-1} \leqslant t^{-\frac{1}{2}}$$

and so

$$t^{-\frac{1}{2}} \wedge \left(1+|x|\right)^{-r} = \left(1+|x|\right)^{-r} \leqslant C\left(1+|x|+|x|\right)^{-r} \leqslant C\left(1+|x|+\sqrt{t}\right)^{-r}. \qquad \square$$

From the above corollary, we estimate $|u|$.

**Lemma 6.17.** *Suppose $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 1$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geqslant 4$. Define*

$$\zeta(t) := \sup_{0 \leqslant s \leqslant t,\, x \in \mathbb{R}} \left|u(x,s)\right| \left[\left(1+|x|+\sqrt{s}\right)^{-r} + (1+\sqrt{s})^{-1} e^{-\frac{|x|^2}{M(1+s)}}\right]^{-1}.$$

*Then for all $t \leqslant 0$ for which $\zeta(t)$ is finite, some $C > 0$,*

$$\zeta(t) \leqslant C\left(E_0 + \zeta^2(t)\right). \tag{6.26}$$

**Proof.** By Duhamel's formula, we have

$$\left|u(x,t)\right| \leqslant \int_{-\infty}^{\infty} k(x-y,t)\left|u_0(y)\right| dy + \int_0^t \int_{-\infty}^{\infty} k(x-y,t-s)\left|u^q(y,s)\right| dy\, ds = I + II.$$

By (6.25), we already have $I \leqslant CE_0[(1+|x|+\sqrt{t})^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x|^2}{M(1+t)}}]$. Now we break $II$ into three parts. Recalling (6.5) and $q \geqslant 4$, $|u^{p-2}|_{L^\infty} \leqslant (1+s)^{-1}$, we have

$$\begin{aligned} II &\leqslant \int_0^t \int_{-\infty}^{\infty} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} \left|u^{p-2}\right|_{L^\infty} \left|u^2(y,s)\right| dy\, ds \\ &\leqslant \zeta^2(t) \int_0^t \int_{-\infty}^{\infty} (1+s)^{-1} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{(t-s)}} \left(1+|y|+\sqrt{s}\right)^{-2r} dy\, ds \end{aligned}$$

$$+\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}\big(1+\sqrt{s}\big)^{-2}e^{-\frac{|y|^2}{M(1+s)}}\,dy\,ds$$

$$+\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}\big(1+|y|+\sqrt{s}\big)^{-r}(1+\sqrt{s})^{-1}e^{-\frac{|y|^2}{M(1+s)}}\,dy\,ds$$

$$=I'+II'+III'.$$

Sine $III'\leqslant CII'$, we only need to estimate two parts $I'$ and $II'$. Recalling (6.14), we have

$$\begin{aligned}II'&\leqslant\zeta^2(t)\int_0^t(1+s)^{-2}\int_{-\infty}^{\infty}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}e^{-\frac{|y|^2}{M(1+s)}}\,dy\,ds\\&\leqslant\zeta^2(t)(1+\sqrt{t})^{-1}e^{-\frac{|x|^2}{M(1+t)}}\int_0^t(1+s)^{-2}(1+s)^{\frac{1}{2}}\,ds\\&\leqslant\zeta^2(t)(1+\sqrt{t})^{-1}e^{-\frac{|x|^2}{M(1+t)}}.\end{aligned}$$

By (6.25), we break $I'$ into two parts,

$$\begin{aligned}I'&=\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}\big(1+|y|+\sqrt{s}\big)^{-r}\big(1+|y|+\sqrt{s}\big)^{-r}\,dy\,ds\\&\leqslant\zeta^2(t)\int_0^t(1+s)^{-1}(1+\sqrt{s})^{-r}\int_{-\infty}^{\infty}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{(t-s)}}\big(1+|y|\big)^{-r}\,dy\,ds\\&\leqslant\zeta^2(t)\int_0^t(1+\sqrt{s})^{-r-2}\Big[\big(1+|x|+\sqrt{t-s}\big)^{-r}+(1+\sqrt{t-s})^{-1}e^{-\frac{|x|^2}{M(t-s)}}\Big]\,ds\\&=I''+II''.\end{aligned}$$

We now estimate $I''$ and $II''$,

$$\begin{aligned}I''&\leqslant C\zeta^2(t)\left[\big(1+|x|+\sqrt{t}\big)^{-r}\int_0^{t/2}(1+\sqrt{s})^{-r-2}\,ds+\big(1+|x|\big)^{-r}\int_{t/2}^{t}(1+\sqrt{s})^{-r-2}\,ds\right]\\&\leqslant C\zeta^2(t)\big(1+|x|+\sqrt{t}\big)^{-r}+C\zeta^2(t)\big[\big(1+|x|\big)(1+\sqrt{t})\big]^{-r}\\&\leqslant C\zeta^2(t)\big(1+|x|+\sqrt{t}\big)^{-r},\end{aligned}$$

and

$$II'' \leqslant C\zeta^2(t)e^{-\frac{|x|^2}{Mt}} \int_0^t (1+\sqrt{s})^{-r-2}(1+\sqrt{t-s})^{-1}\,ds$$

$$\leqslant C\zeta^2(t)e^{-\frac{|x|^2}{Mt}} \left[ (1+\sqrt{t})^{-1} \int_0^{t/2} (1+\sqrt{s})^{-r-2}\,ds + (1+\sqrt{t})^{-r-2} \int_{t/2}^t (1+\sqrt{t-s})^{-1}\,ds \right]$$

$$\leqslant C\zeta^2(t)(1+\sqrt{t})^{-1}e^{-\frac{|x|^2}{M(1+t)}}. \qquad \square$$

**Corollary 6.18.** *Suppose $u(x,t)$ satisfies $u_t = u_{xx} + u^q$ and $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 1$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geqslant 4$. Then for all $t \geqslant 0$ and $x \in \mathbb{R}$*

$$\left|u(x,t)\right| \leqslant CE_0\left[\left(1+|x|+\sqrt{t}\right)^{-r} + (1+\sqrt{t})^{-1}e^{-\frac{|x|^2}{M(1+t)}}\right]. \tag{6.27}$$

**Proof.** Same proof as for Corollary 6.4. $\square$

Now we need more information about the effects on algebraically decaying data. The following Lemmas 6.19 and 6.20 are used in the proof of linear and nonlinear estimates of $u$ when we use Mean Value Theorem.

**Lemma 6.19.** *For all $t \geqslant 0$, $x \in \mathbb{R}$, $r > 1$ and all $0 < w < 1$,*

$$\int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy|^2}{M(1+t)}}\left(1+|y|\right)^{-r}dy \leqslant C\left[\left(1+|x|+\sqrt{t}\right)^{-r} + (1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M'(1+t)}}\right], \tag{6.28}$$

*for some sufficiently large $M' > M$.*

**Proof.** We first consider the case of $|x| \leqslant \sqrt{1+t}$ which implies $e^{-\frac{|x|^2}{M(1+t)}} > e^{-\frac{1}{M}}$. Then

$$\int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy|^2}{M(1+t)}}\left(1+|y|\right)^{1-r}dy \leqslant (1+t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left(1+|y|\right)^{1-r}dy$$

$$\leqslant C(1+t)^{-\frac{1}{2}}$$

$$\leqslant C(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{M(1+t)}}.$$

For the case of $|x| > \sqrt{1+t}$, we break the integration into two parts.

$$\int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy|^2}{M(1+t)}}\left(1+|y|\right)^{-r}dy = \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{M(1+t)}}\left(1+\frac{|y|}{w}\right)^{-r}\frac{1}{w}\,dy$$

$$= \int_0^{|x|/2} + \int_{|x|/2}^{\infty} = I + II.$$

For part $I$, we have

$$I \leqslant (1+t)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4M(1+t)}} \int_0^{|x|/2} \left(1+\frac{|y|}{w}\right)^{-r} \frac{1}{w}\, dy \leqslant C(1+t)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{4M(1+t)}}.$$

For part $II$, we have

$$II \leqslant C\left(1+\frac{|x|}{w}\right)^{-r} \frac{1}{w} \int_{|x|/2}^{\infty} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{M(1+t)}}\, dy \leqslant C\left(1+\frac{|x|}{w}\right)^{-r} \frac{1}{w}.$$

Define a function

$$f(w) = \left(1+\frac{|x|}{(r-1)w}\right)^{-r} \frac{1}{w}.$$

We easily show that $f(1) = (1+\frac{|x|}{r-1})^{-r}$ and $f(w)$ is increasing for $|x|>1$ which implies that if $|x| > \sqrt{1+t} > 1$, for all $0<w<1$, we have

$$II \leqslant Cf(w) \leqslant Cf(1) \leqslant C\big(1+|x|\big)^{-r}. \qquad \square$$

**Lemma 6.20.** *For all $t>s>0$, $x\in\mathbb{R}$, $r>1$ and all $0<w<1$,*

$$\int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} \big(1+|y|+\sqrt{s}\big)^{-r}\, dy$$
$$\leqslant C\Big[\big(1+|x|+\sqrt{t-s}+\sqrt{s}\big)^{-r} + (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{(r-1)}{2}} e^{-\frac{|x|^2}{M'(1+t)}}\Big] \tag{6.29}$$

*for some sufficiently large $M'>M$.*

**Proof.** We consider first the case of $|x| \leqslant \sqrt{1+t}$ which implies $e^{-\frac{|x|^2}{M(1+t)}} \geqslant e^{-\frac{1}{M}}$, and so

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} \big(1+|y|+\sqrt{s}\big)^{-r}\, dy \\
&\leqslant (1+t-s)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \big(1+|y|+\sqrt{s}\big)^{-r}\, dy \\
&\leqslant C(1+t-s)^{-\frac{1}{2}}(1+\sqrt{s})^{-r+1} \\
&\leqslant C(1+t-s)^{-\frac{1}{2}}(1+\sqrt{s})^{-r+1} e^{-\frac{|x|^2}{M(1+t)}}.
\end{aligned}$$

For the case of $|x| > \sqrt{1+t}$, we separate the integration into two parts

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t-s)}} \left(1+|y|+\sqrt{s}\right)^{-r} dy \\
&= \int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{M(1+t-s)}} \left(1+\frac{|y|}{w}+\sqrt{s}\right)^{-r} \frac{1}{w}\, dy \\
&= \int_{0}^{|x|/2} + \int_{|x|/2}^{\infty} = I + II.
\end{aligned}$$

For $I$, we have

$$\begin{aligned}
I &\leqslant (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4M(1+t-s)}} \int_{0}^{|x|/2} \left(1+\frac{|y|}{w}+\sqrt{s}\right)^{-r} \frac{1}{w}\, dy \\
&\leqslant C(1+t-s)^{-\frac{1}{2}} (1+\sqrt{s})^{-r+1} e^{-\frac{|x|^2}{M'(1+t)}}.
\end{aligned}$$

For $II$, we have

$$II \leqslant C\left(1+\frac{|x|}{w}+\sqrt{s}\right)^{-r} \frac{1}{w} \int_{|x|/2}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{M(1+t-s)}}\, dy \leqslant C\left(1+\frac{|x|}{w}+\sqrt{s}\right)^{-r} \frac{1}{w}.$$

Since $|x| > \sqrt{1+t} > \sqrt{t-s}$,

$$II \leqslant C\left(1+\frac{2|x|}{w}+\sqrt{s}\right)^{-r} \frac{1}{w} \leqslant C\left(1+\frac{|x|+\sqrt{t-s}}{w}+\sqrt{s}\right)^{-r} \frac{1}{w}.$$

Define a function

$$f(w) = \left(1+\frac{2(|x|+\sqrt{t-s})}{(r-1)w}+\sqrt{s}\right)^{-r} \frac{1}{w}.$$

Then $f(1) = (1+\frac{2(|x|+\sqrt{t-s})}{(r-1)}+\sqrt{s})^{-r}$ and $f$ is increasing. Indeed,

$$f'(w) = \left(1+\frac{2(|x|+\sqrt{t-s})}{(r-1)w}+\sqrt{s}\right)^{-r-1} \frac{1}{w^3}\left[2\left(|x|+\sqrt{t-s}\right) - w(1+\sqrt{s})\right].$$

Since $|x| > \sqrt{1+t}$, $|x| > 1$ and $|x| > \sqrt{s}$, that is, $f'(w) > 0$. Thus if $|x| > \sqrt{1+t}$, for all $0 < w < 1$, we have

$$II \leqslant Cf(w) \leqslant Cf(1) \leqslant C\left(1+|x|+\sqrt{t-s}+\sqrt{s}\right)^{-r}. \qquad \square$$

With these preparations, we are in a suitable position to estimate linear and nonlinear parts of $u$ and finally obtain the asymptotic behavior of $u$ in Theorem 6.23.

**Lemma 6.21** *(Linear estimate). Suppose $u(x,t)$ satisfies that $u_t = u_{xx}$ and $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geqslant 4$. Then for some sufficiently large $M' > M$,*

$$\bigl|u(x,t) - U_0 k(x,t)\bigr| \leqslant C E_0\bigl[(1+t)^{-\frac{1}{2}}\bigl(1+|x|+\sqrt{t}\bigr)^{-r+1} + (1+t)^{-1} e^{-\frac{|x|^2}{M'(1+t)}}\bigr], \tag{6.30}$$

*where $U_0 = \int_{-\infty}^{\infty} u_0(y)\,dy$ and $k(x,t) = (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}}$.*

**Proof.** By the Mean Value Theorem, (6.28) and $r-1>1$, we have

$$\begin{aligned}
\bigl|u(x,t) - U_0 k(x,t)\bigr| &\leqslant \int_{-\infty}^{\infty}\int_0^1 \bigl|k_x(x-wy,t)\bigr||y|\bigl(1+|y|\bigr)^{-r}\,dw\,dy \\
&\leqslant C E_0 (1+t)^{-\frac{1}{2}} \int_0^1 \int_{-\infty}^{\infty} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-wy|^2}{M(1+t)}} \bigl(1+|y|\bigr)^{-r+1}\,dy\,dw \\
&\leqslant C E_0\bigl[(1+t)^{-\frac{1}{2}}\bigl(1+|x|+\sqrt{t}\bigr)^{-r+1} + (1+t)^{-1} e^{-\frac{|x|^2}{M'(1+t)}}\bigr]. \qquad \square
\end{aligned}$$

**Lemma 6.22** *(Nonlinear estimate). Suppose $u(x,t)$ satisfies that $u_t = u_{xx} + u^q$ and $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$, for $E_0 > 0$ sufficiently small, $M > 1$ sufficiently large, and $q \geqslant 4$. Then for some sufficiently large $M'' > M' > M$,*

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} k(x-y,t-s) u^q(y,s)\,ds - U(s)k(x,t-s)\right| \\
&\quad \leqslant C E_0 (1+s)^{-1}\bigl[(1+t-s)^{-\frac{1}{2}}\bigl(1+|y|+\sqrt{t-s}+\sqrt{s}\bigr)^{-2r+1} \\
&\qquad + (1+t-s)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}\bigr],
\end{aligned} \tag{6.31}$$

*where $U(s) = \int_{-\infty}^{\infty} u^q(y,s)\,dy$ and $k(x,t) = (1+t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{(1+t)}}$.*

**Proof.** Noting, by (6.27) and $q \geqslant 4$, that

$$\bigl|y u^q(y,s)\bigr| = \bigl|u^{p-2}\bigr|\bigl|y u^2(y,s)\bigr| \leqslant C E_0 (1+s)^{-1}\bigl[\bigl(1+|y|+\sqrt{s}\bigr)^{-2r+1} + (1+s)^{-\frac{1}{2}} e^{-\frac{|y|^2}{M(1+s)}}\bigr],$$

we obtain, by Mean Value Theorem again and by (6.29),

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} k(x-y,t-s) u^q(y,s)\,ds - U(s)k(x,t-s)\right| \\
&\quad \leqslant C E_0 \int_0^1 \int_{-\infty}^{\infty} (1+t-s)^{-1} e^{-\frac{|x-wy|^2}{M(1+t-s)}} (1+s)^{-1}\bigl(1+|y|+\sqrt{s}\bigr)^{-2r+1}\,dy\,dw
\end{aligned}$$

$$+CE_0\int_0^1\int_{-\infty}^{\infty}(1+t-s)^{-1}e^{-\frac{|x-wy|^2}{M(1+t-s)}}(1+s)^{-\frac{3}{2}}e^{-\frac{|x|^2}{M'(1+s)}}\,dy\,dw$$

$$\leqslant CE_0(1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\big(1+|x|+\sqrt{t-s}+\sqrt{s}\big)^{-2r+1}$$

$$+CE_0\Big[(1+t-s)^{-1}(1+s)^{-r}e^{-\frac{|x|^2}{M'(1+t)}}+(1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}\Big]$$

$$\leqslant CE_0(1+s)^{-1}\Big[(1+t-s)^{-\frac{1}{2}}\big(1+|y|+\sqrt{t-s}+\sqrt{s}\big)^{-2r+1}+(1+t-s)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}\Big]. \qquad \square$$

**Theorem 6.23** *(Behavior). Suppose $u(x,t)$ satisfies $u_t=u_{xx}+u^q$ and $|u_0(y)|\leqslant E_0(1+|x|)^{-r}$, $r>2$, for $E_0>0$ sufficiently small, $M>1$ sufficiently large, and $q\geqslant 4$. Set*

$$U_*=\int_0^{\infty}U(s)\,ds+U_0=\int_0^{\infty}\int_{-\infty}^{\infty}u^q(y,s)\,dy\,ds+\int_{-\infty}^{\infty}u_0(y)\,dy.$$

*Then, $|U_*|<\infty$ and for some sufficiently large $M''>M'>M$,*

$$\big|u(x,t)-U_*k(x,t)\big|$$

$$\leqslant CE_0\Big[(1+t)^{-\frac{1}{2}}\big(1+|x|+\sqrt{t}\big)^{-r+1}+(1+t)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}\big(1+\ln(1+t)\big)\Big]. \qquad (6.32)$$

**Proof.** Recalling (6.5) and $q\geqslant 4$, $|U(s)|\leqslant CE_0(1+s)^{-\frac{3}{2}}$ and so

$$|U_*|\leqslant CE_0\int_0^{\infty}(1+s)^{-\frac{3}{2}}\,ds+|u_0|_{L^1}<\infty.$$

Now we break $|u(x,t)-U_*k(x,t)|$ into four parts like (6.14). Then we have

$$II\leqslant CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x|^2}{(1+t)}}\int_t^{\infty}(1+s)^{-\frac{3}{2}}\,ds\leqslant CE_0(1+t)^{-1}e^{-\frac{|x|^2}{(1+t)}}. \qquad (6.33)$$

By (6.31), we have

$$III\leqslant CE_0\int_0^t(1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\big(1+|x|+\sqrt{t-s}+\sqrt{s}\big)^{-2r+1}\,ds$$

$$+CE_0\int_0^t(1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x|^2}{M''(1+t)}}\,ds$$

$$\leqslant CE_0\big(1+|x|+\sqrt{t}\big)^{-2r+1}\int_0^t(1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\,ds$$

$$+ CE_0 e^{-\frac{|x|^2}{M''(1+t)}} \int_0^t (1+t-s)^{-1}(1+s)^{-1}\, ds$$

$$\leqslant CE_0\big[(1+t)^{-\frac{1}{2}}\big(1+|x|+\sqrt{t}\big)^{-r+1} + (1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}} \ln(1+t)\big].$$

Since $|U(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$, $IV$ is exactly the same as (6.23) which is

$$IV \leqslant CE_0(1+t)^{-1} e^{-\frac{|x|^2}{M''(1+t)}}. \tag{6.34}$$

By (6.30) and (6.33)–(6.34), we obtain (6.32). □

*6.4. Behavior of u for $u_t = u_{xx} + u_x^r$, $r \geqslant 2$*

A more exact analysis to our eventual equation (7.11) governing asymptotic behavior is the perturbed heat equation $u_t = u_{xx} + u_x^2$, where $|u_x|^2 \sim |u|^4$ by property of the heat equation. Indeed, the estimates in Sections 6.1–6.3 is a practice to estimates $\bar{u}'(x)\psi$ in Section 7. If you see (7.24), there are three terms and the last term is $|\bar{u}'(x)\psi - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)|$. The technique of estimates $|\bar{u}'(x)\psi - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)|$ is exactly same as Sections 6.1–6.3. Please see Lemmas 7.5 and 7.6 for linear and nonlinear estimates for $\bar{u}'(x)\psi$, respectively which is the same technique as Section 6. However, by (7.11), we can roughly say that

$$\psi = \int e v_0 + \iint e(t-s) O\big(|v|^2\big).$$

Since $|\psi_t, \psi_x| \sim |v|$ and $e$ is like a heat kernel, $\psi_t = \psi_{xx} + O(|\psi_x|^2)$. It is readily verified that all our above arguments go through in this case as well; see the more complicated analysis of Section 7 for the actual problem.

**Remark 6.24.** The relation between $u_t - u_{xx} = u^q$ and $u_t - u_{xx} = u_x^{q/2}$ was mentioned already by Schneider in [18,19].

## 7. Behavior of perturbations of (1.1)

In this section we prove the main theorem of the paper, Theorem 1.4 which is about the behavior of perturbation of (1.1). Let $\tilde{u}(x,t)$ be a solution of the system of reaction–diffusion equations

$$u_t = u_{xx} + f(u) + cu_x \tag{7.1}$$

and let $\bar{u}(x)$ be a stationary solution and define perturbations

$$u(x,t) = \tilde{u}(x,t) - \bar{u}(x) \quad \text{and} \quad v(x,t) = \tilde{u}\big(x+\psi(x,t),t\big) - \bar{u}(x), \tag{7.2}$$

for some unknown functions $\psi(x,t) : \mathbb{R}^2 \to \mathbb{R}$ to be determined later.

Plugging $\bar{u}(x,t) = u(x,t) - \bar{u}(x)$ in (7.1), we have

$$u_t = Lu + O\big(|u|^2\big), \tag{7.3}$$

where $L$ is the linear operator of (1.2).

In this section, using $v(x,t)$ and the linearized estimate of $L$ we have done in Section 4, we show the behavior of $u$ satisfying (7.3) similarly as in Section 6 for three cases of initial conditions:

(1) $|u_0|_{L^1 \cap H^2} \leqslant E_0$ and $|xu_0|_{L^1} \leqslant E_0$,
(2) $|u_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$ and $|u_0(x)|_{H^2} \leqslant E_0$,
(3) $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$ and $|u_0(x)|_{H^2} \leqslant E_0$,

where $E_0 > 0$ sufficiently small and $M > 0$ sufficiently large.

By Theorem 1.3, the Green function $G(x,t;y)$ for the linear equation $u_t = Lu$ satisfies the estimates:

$$G(x,t;y) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} \bar{u}'(x)\tilde{q}(y,0) + O\big((1+t)^{-1} + t^{-\frac{1}{2}} e^{-\eta t}\big) e^{-\frac{|x-y-at|^2}{Mt}},$$

for some sufficiently large constant $M > 0$ and $\eta > 0$. First off, let $\chi(t)$ be a smooth cut off function defined for $t \geqslant 0$ such that $\chi(t) = 0$ for $0 \leqslant t \leqslant 1$ and $\chi(t) = 1$ for $t \geqslant 2$ and define

$$E(x,t;y) := \bar{u}'(x) e(x,t;y), \tag{7.4}$$

where

$$e(x,t;y) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-y-at|^2}{4bt}} \tilde{q}(y,0)\chi(t).$$

Now we set

$$G(x,t;y) = E(x,t;y) + \tilde{G}(x,t;y) \quad \text{and} \quad G_y(x,t;y) = E(x,t;y) + \tilde{G}_y(x,t;y),$$

where

$$\big|\tilde{G}(x,t;y)\big| \leqslant C(1+t)^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{Mt}} \quad \text{and} \quad \big|\tilde{G}_y(x,t;y)\big| \leqslant Ct^{-1} e^{-\frac{|x-y-at|^2}{Mt}}.$$

Next, we now consider the nonlinear perturbation equations for $v$ defined in (7.2) which is already mentioned in [12]. This equation is used in the nonlinear iteration scheme which is the starting point for the perturbation behavior.

**Lemma 7.1** *(Nonlinear perturbation equations). (See [12].) For $v$ defined in (7.2), we have*

$$(\partial_t - L)v = (\partial_t - L)\bar{u}'(x)\psi + Q + R_x - \big(\partial_x^2 + \partial_t\big)S + T, \tag{7.5}$$

*where*

$$Q := f\big(v(x,t) + \bar{u}(x)\big) - f\big(\bar{u}(x)\big) - df\big(\bar{u}(x)\big)v = \mathcal{O}\big(|v|^2\big), \tag{7.6}$$

$$R := v\psi_t - v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x}, \tag{7.7}$$

$$S := v\psi_x = O\big(|v||\psi_x|\big), \tag{7.8}$$

*and*

$$T := \big(f(v+\bar{u}) - f(\bar{u})\big)\psi_x = O\big(|v||\psi_x|\big). \tag{7.9}$$

**Proof.** Direct computation; see [12]. □

### 7.1. Integral representation and $\psi$-evolution scheme

We now recall the nonlinear iteration scheme of [12]. Using (7.5) and applying Duhamel's principle and setting

$$N(x,t) = \big(Q + R_x - \big(\partial_x^2 + \partial_t\big)S + T\big)(x,t), \tag{7.10}$$

we obtain the integral representation

$$v(x,t) = \bar{u}'(x)\psi(x,t) + \int_{-\infty}^{\infty} G(x,t;y)v_0(y)\,dy + \int_0^t \int_{-\infty}^{\infty} G(x,t-s;y)N(y,s)\,dy\,ds,$$

for the nonlinear perturbation $v$. Defining $\psi$ implicitly by

$$\psi(x,t) := -\int_{-\infty}^{\infty} e(x,t;y)v_0(y)\,dy - \int_0^t \int_{-\infty}^{\infty} e(x,t-s;y)N(y,s)\,dy\,ds, \tag{7.11}$$

we obtain the integral representation

$$v(x,t) = \int_{-\infty}^{\infty} \tilde{G}(x,t;y)v_0(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x,t-s;y)N(y,s)\,dy\,ds. \tag{7.12}$$

Differentiating and using $e(x,t;y) = 0$ for $0 < t \leqslant 1$ we obtain

$$\partial_t^k \partial_x^m \psi(x,t) = -\int_{-\infty}^{\infty} \partial_t^k \partial_x^m e(x,t;y)v_0\,dy - \int_0^t \int_{-\infty}^{\infty} \partial_t^k \partial_x^m e(x,t-s;y)N(y,s)\,dy\,ds. \tag{7.13}$$

Together, (7.12) and (7.13) form a complete system in $(v, \partial_t^k\psi, \partial_x^m\psi)$, $0 \leqslant k \leqslant 1$, $0 \leqslant m \leqslant 2$, that is, $v$ and derivatives of $\psi$, from solutions of which we may afterward recover the shift function $\psi$ by integration in $x$, completing the description of $\tilde{u}$.

**Plan of Sections 7.2, 7.3 and 7.4.** We describe here briefly how to get asymptotic behavior of the perturbation $u$ with respect to three initial data, respectively. Similarly as Section 6, the main purpose is to estimate $|u(x,t) - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)|_{L^p(x)}$ in Section 7.2 and estimate pointwise $|u(x,t) - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)|$ in Sections 7.3 and 7.4, where

$$\bar{U}_* = \int_0^{\infty}\int_{-\infty}^{\infty} N(y,s)\tilde{q}(y,0)\,dy\,ds + \int_{-\infty}^{\infty} u_0(y)\tilde{q}(y,0)\,dy,$$

and

$$\bar{k}(x,t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{4bt}}.$$

We show that this difference decays faster than a heat kernel with respect to $t$ so that the asymptotic behavior of $u$ converges to heat kernel. More precisely, setting

$$\bar{U}(s)=\int_{-\infty}^{\infty} N(y,s)\tilde{q}(y,0)\,dy \quad \text{and} \quad \bar{U}_0=\int_{-\infty}^{\infty} u_0(y)\tilde{q}(y,0)\,dy,$$

we separate $|u(x,t)-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)|$ into three parts (see (7.24) for detail),

$$\begin{aligned}\left|u(x,t)-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)\right| &\leqslant \left|v(x,t)\right|+\left|v_x\psi+O\left(\psi^2\right)\right|+\left|\bar{u}'(x)\psi-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)\right| \\ &= I+II+III.\end{aligned}$$

By the nonlinear iteration scheme, we first estimate $I$ and $II$. For $III$, by (7.11), we have

$$\begin{aligned}&\left|\bar{u}'(x)\psi-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)\right| \\ &=\left|\int_{-\infty}^{\infty} E(x,t;y)u_0(y)\,dy+\int_0^t\int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy\,ds-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)\right| \\ &\leqslant\left|\int_{-\infty}^{\infty} E(x,t;y)u_0(y)\,dy-\bar{U}_0\bar{u}'(x)\bar{k}(x,t)\right|+\int_t^{\infty}\left|\bar{u}'(x)\bar{k}(x,t)\bar{U}(s)\right|ds \\ &\quad+\int_0^t\left|\int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy-\bar{U}(s)\bar{u}'(x)\bar{k}(x-as,t-s)\right|ds \\ &\quad+\int_0^t\left|\bar{U}(s)\bar{u}'(x)\right|\left|\bar{k}(x-as,t-s)-\bar{k}(x,t)\right|ds. \end{aligned} \tag{7.14}$$

The estimate of $III$ is exactly same analysis as Section 6. The first and third terms are linear and nonlinear estimates of $\bar{u}'(x)\psi$.

### 7.2. Behavior for initial perturbation $|u_0|_{L^1\cap H^2}$, $|xu_0|_{L^1}$ sufficiently small

We start with the $L^p$ bounds of $v$, $u$ and $\psi$ from the iteration scheme. These are already proved in [12] as a nonlinear stability. As I mentioned above, $|v|_{L^p}$ is the fundamental decay estimates for the perturbation $u$.

**Theorem 7.2** *(Nonlinear stability). (See [12].) Let $v(x,t)$ and $u(x,t)$ be defined as in (7.2) and $|u_0(x)|=|v_0(x)|_{L^1\cap H^2(\mathbb{R})}<E_0$ sufficiently small. Then for all $t\geqslant 0$ and $p\geqslant 1$ we have the estimates*

$$\begin{aligned}&\left|v(\cdot,t)\right|_{L^p(\mathbb{R})}(t)\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \\ &\left|u(\cdot,t)\right|_{L^p(\mathbb{R})}(t),\left|\psi(\cdot,t)\right|_{L^p(\mathbb{R})}(t)\leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \\ &\left|v(\cdot,t)\right|_{H^2(\mathbb{R})}(t),\left|(\psi_t,\psi_x)(\cdot,t)\right|_{H^2(\mathbb{R})}(t)\leqslant CE_0(1+t)^{-\frac{3}{4}}.\end{aligned} \tag{7.15}$$

**Proof.** This is proved in [12] for $p\geqslant 2$. For $p=1$, we use integration by parts of (7.11) and (7.12) and use $|(Q,R,S,T)|_{L^1}\leqslant|(v,\psi_x,\psi_t)|_{H^1}^2\leqslant CE_0(1+t)^{-\frac{3}{2}}$ to prove $|v(\cdot,t)|_{L^1}\leqslant CE_0(1+t)^{-\frac{1}{2}}$ and $|\psi(\cdot,t)|_{L^1}\leqslant CE_0$. $\square$

We start with the following lemma which is used in the proof of a nonlinear estimate of $\bar{u}'(x)\psi$ (Lemma 7.6).

**Lemma 7.3.** *Associated with the solution $(u, \psi_t, \psi_x, \psi_{xx})$ of integral system (7.12)–(7.13), we define*

$$\zeta(t) := \sup_{0\leqslant s\leqslant t} \left|(x-as)(v, \psi_t, \psi_x, \psi_{xx})\right|_{L^1(x)}(s). \tag{7.16}$$

*Then for all $t \geqslant 0$ for which $\zeta(t)$ is sufficiently small, we have the estimate*

$$\zeta(t) \leqslant C\big(E_0 + \zeta^2(t)\big) \tag{7.17}$$

*for some constant $C > 0$, as long as $|v_0|_{L^1\cap H^2}, |xv_0|_{L^1} < E_0$, for $E_0 > 0$ sufficiently small.*

**Proof.** To begin, notice first that

$$\begin{aligned}
&\left|(y-as)(Q+T+R+S)(y,s)\right|_{L^1(y)} \\
&\quad\leqslant \left|(y-as)\big(v^2+\psi_t^2+\psi_y^2+\psi_{yy}^2\big)\right|_{L^1(y)} \\
&\quad\leqslant \big(|v|_{L^\infty}+|\psi_t|_{L^\infty}+|\psi_y|_{L^\infty}+|\psi_{yy}|_{L^\infty}\big)\left|(y-as)(v,\psi_t,\psi_y,\psi_{yy})\right|_{L^1(y)} \\
&\quad\leqslant CE_0(1+t)^{-1}\zeta(t),
\end{aligned}$$

and

$$\left|(Q+T+R+S)(y,s)\right|_{L^1(y)} \leqslant \left|\big(v^2+\psi_t^2+\psi_y^2+\psi_{yy}^2\big)\right|_{L^1(y)} \leqslant (1+s)^{-\frac{3}{2}}. \tag{7.18}$$

By integration by parts, we have

$$\begin{aligned}
&\left|(x-at)v(x,t)\right|_{L^1(x)} \\
&\quad= \int_{-\infty}^{\infty} \left|(x-at-y)(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}}e^{-\frac{|x-at-y|^2}{Mt}}\right|_{L^1(x)}\left|v_0(y)\right|dy \\
&\qquad+ \int_{-\infty}^{\infty} \left|y(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}}e^{-\frac{|x-at-y|^2}{Mt}}\right|_{L^1(x)}\left|v_0(y)\right|dy \\
&\qquad+ \int_0^t\int_{-\infty}^{\infty} \left|\big(x-at-(y-as)\big)(1+t-s)^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}}\right|_{L^1(x)}|Q+T|\,dy\,ds \\
&\qquad+ \int_0^t\int_{-\infty}^{\infty} \left|(1+t-s)^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}}\right|_{L^1(x)}\left|(y-as)(Q+T)\right|dy\,ds \\
&\qquad+ \int_0^t\int_{-\infty}^{\infty} \left|\big(x-at-(y-as)\big)(t-s)^{-1}e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}}\right|_{L^1(x)}|R+S|\,dy\,ds
\end{aligned}$$

$$+\int_0^t \int_{-\infty}^{\infty} \left|(t-s)^{-1} e^{-\frac{|x-a(t-s)-y|^2}{M(t-s)}}\right|_{L^1(x)} \left|(y-as)(R+S)\right| dy\, ds$$

$$\leqslant |v_0|_{L^1} + (1+t)^{-\frac{1}{2}} |yv_0|_{L^1} + \int_0^t \left|(Q+R+S+T)\right|_{L^1} ds$$

$$+\int_0^t (1+t-s)^{-\frac{1}{2}} \left|(y-as)(Q+R+S+T)\right|_{L^1} ds$$

$$\leqslant CE_0 + C(1+t)^{-\frac{1}{2}} E_0 + CE_0 \int_0^t (1+s)^{-\frac{3}{2}} ds + CE_0 \zeta(t) \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-1} ds$$

$$\leqslant C\big(E_0 + \zeta^2(t)\big).$$

Similarly, we have

$$\left|(x-at)(\psi_t, \psi_x, \psi_{xx})\right|_{L^1(x)} \leqslant C\big(E_0 + \zeta^2(t)\big). \qquad \square$$

**Corollary 7.4.** *For $|v_0|_{L^1 \cap H^2}, |xv_0|_{L^1} < E_0$, and $E_0 > 0$ sufficiently small,*

$$\left|(y-as)(Q+T+R+S)(y,s)\right|_{L^1(y)} \leqslant CE_0(1+s)^{-1}. \tag{7.19}$$

**Proof.** Same proof as Corollary 6.4. $\square$

We now estimate linear and nonlinear parts of $\bar{u}'(x)\psi$.

**Lemma 7.5** *(Linear estimate). For $E$ defined as in (7.4) and $|u_0|_{L^1 \cap H^2}, |xu_0|_{L^1} < E_0$, we have*

$$\left| \int_{-\infty}^{\infty} E(x,t;y) u_0(y)\, dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x,t) \right|_{L^p(x)} \leqslant CE_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}, \tag{7.20}$$

*where $\bar{U}_0 = \int_{-\infty}^{\infty} u_0(y)\tilde{q}(y,0)\, dy$ and $\bar{k}(x,t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{4bt}}$.*

**Proof.** By the Mean Value Theorem,

$$\left| \int_{-\infty}^{\infty} E(x,t;y) u_0(y)\, dy - \bar{U}_0 \bar{u}'(x) \bar{k}(x,t) \right|_{L^p(x)} \leqslant C \int_{-\infty}^{\infty} \int_0^1 \left|\bar{k}_x(x-wy,t)\right|_{L^p(x)} \left|yu_0(y)\right| dw\, dy$$

$$\leqslant CE_0 (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \qquad \square$$

**Lemma 7.6** *(Nonlinear estimate). Recalling (7.4) and (7.10), we have*

$$\left| \int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy - \bar{U}(s)\bar{u}'(x)\bar{k}(x-as,t-s) \right|_{L^p(x)} \leqslant C(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+s)^{-1}, \tag{7.21}$$

*where* $\bar{U}(s) = \int_{-\infty}^{\infty} N(y,s)\tilde{q}(y,0)\,dy$.

**Proof.** By integration by parts, the Mean Value Theorem and (7.18)–(7.19), we have

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy - \bar{U}(s)\bar{u}'(x)\bar{k}(x-as,t-s) \right|_{L^p(x)} \\
&\leqslant \int_{-\infty}^{\infty} \left|\bar{u}'(x)\tilde{q}(y,0)\right| \left|\bar{k}(x-y,t-s) - \bar{k}(x-as,t-s)\right|_{L^p(x)} \left|(Q+T)(y,s)\right| dy \\
&\quad + \left| \int_{-\infty}^{\infty} \bar{u}'(x)\tilde{q}_y(y,0)\big(\bar{k}(x-y,t-s) - \bar{k}(x-as,t-s)\big)(R+S)(y,s)\,dy \right|_{L^p(x)} \\
&\quad + \left| \int_{-\infty}^{\infty} \bar{u}'(x)\tilde{q}(y,0)\partial_y\big(\bar{k}(x-y,t-s) - \bar{k}(x-as,t-s)\big)(R+S)(y,s)\,dy \right|_{L^p(x)} \\
&\leqslant CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \left|(y-as)(Q+T+R+S)(y,s)\right|_{L^1(y)} \\
&\quad + CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \left|(Q+T+R+S)(y,s)\right|_{L^1(y)} \\
&\leqslant CE_0(1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+s)^{-1}. \qquad \square
\end{aligned}$$

With these preparations, we are ready to get the asymptotic behavior of the perturbation $u$ of (1.1) in $L^p$.

**Theorem 7.7** *(Behavior). Suppose* $u(x,t)$ *is the perturbation of (1.1) with initial perturbation* $|u_0|_{L^1 \cap H^2}$, $|xu_0|_{L^1} < E_0$, $E_0 > 0$ *sufficiently small. Set*

$$\bar{U}_* = \int_0^{\infty} \bar{U}(s)\,ds + \bar{U}_0.$$

*Then* $|\bar{U}_*| < \infty$ *and*

$$\left|u(x,t) - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)\right|_{L^p(x)} \leqslant CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\big(1+\ln(1+t)\big). \tag{7.22}$$

**Proof.** Noting first, integration by parts and (7.15), that

$$\left|\bar U(s)\right| \leqslant C\left|(Q,R,S,T)(y,s)\right|_{L^1(y)} \leqslant \left|(v,\psi_t,\psi_x)\right|_{H^1}^2 \leqslant CE_0(1+s)^{-\frac{3}{2}}, \tag{7.23}$$

we have $|\bar U_*| \leqslant C\int_0^\infty (1+s)^{-\frac{3}{2}}\,ds + CE_0|u_0|_{L^1} < \infty$.

Now we break $|u(x,t)-\bar U_*\bar u'(x)\bar k(x,t)|_{L^p(x)}$ into three parts

$$\begin{aligned}
&\left|u(x,t)-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\quad=\left|\tilde u(x,t)-\bar u(x)-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\quad=\left|\tilde u(x+\psi,t)-\bar u(x)+\tilde u(x,t)-\tilde u(x+\psi,t)-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\quad\leqslant\left|v(x,t)\right|_{L^p(x)}+\left|\tilde u(x,t)-\tilde u(x+\psi,t)-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\quad=\left|v(x,t)\right|_{L^p(x)}+\left|\tilde u_x(x+\psi,t)(1+\psi_x)\psi+O\left(|\psi|^2\right)-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\quad=\left|v(x,t)\right|_{L^p(x)}+\left|\left(\bar u'(x)+v_x\right)\psi+O\left(|\psi|^2\right)-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\quad\leqslant\left|v(x,t)\right|_{L^p(x)}+\left(|v_x||\psi|+O\left(|\psi|^2\right)\right)_{L^p(x)}+\left|\bar u'(x)\psi-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}.
\end{aligned}\tag{7.24}$$

By (7.15), we easily see first two terms

$$\begin{aligned}
&\left|v(x,t)\right|_{L^p(x)}+\left(|v_x||\psi|+|\psi|^2\right)_{L^p(x)}\\
&\quad\leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}+|v_x|_{L^\infty}|\psi|_{L^p}+|\psi|_{L^{2p}}^2\\
&\quad\leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}+C(1+t)^{-\frac{3}{4}}(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}+C(1+t)^{-(1-\frac{1}{2p})}\\
&\quad\leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.
\end{aligned}\tag{7.25}$$

Now we estimate the last term

$$\begin{aligned}
&\left|\bar u'(x)\psi-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&=\left|\int_{-\infty}^{\infty}E(x,t;y)u_0(y)\,dy+\int_0^t\int_{-\infty}^{\infty}E(x,t-s;y)N(y,s)\,dy\,ds-\bar U_*\bar u'(x)\bar k(x,t)\right|_{L^p(x)}\\
&\leqslant\left|\int_{-\infty}^{\infty}E(x,t;y)u_0(y)\,dy-\bar U_0\bar u'(x)\bar k(x,t)\right|_{L^p(x)}+\int_t^{\infty}\left|\bar u'(x)\bar k(x,t)\bar U(s)\right|_{L^p(x)}ds\\
&\quad+\int_0^t\left|\int_{-\infty}^{\infty}E(x,t-s;y)N(y,s)\,dy-\bar U(s)\bar u'(x)\bar k(x-as,t-s)\right|_{L^p(x)}ds\\
&\quad+\int_0^t\left|\bar U(s)\bar u'(x)\right|\left|\bar k(x-as,t-s)-\bar k(x,t)\right|_{L^p(x)}ds\\
&=I+II+III+IV.
\end{aligned}\tag{7.26}$$

Since $|\bar{U}(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$,

$$II \leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \int_t^\infty (1+s)^{-\frac{3}{2}}\, ds \leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \tag{7.27}$$

By (7.21), we have

$$III \leqslant C \int_0^t (1+t-s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} (1+s)^{-1}\, ds \leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}} \big(1+\ln(1+t)\big). \tag{7.28}$$

By the Mean Value Theorem, for some $s^* \in (0, t/2)$, we have

$$\begin{aligned} IV &\leqslant C \int_{t/2}^t (1+s)^{-\frac{3}{2}} \big|\bar{k}(x-as, t-s) - \bar{k}(x,t)\big|_{L^p(x)}\, ds \\ &\quad + C \int_0^{t/2} (1+s)^{-\frac{3}{2}} s \big|\bar{k}_t\big(x-as, t-s^*\big)\big|_{L^p(x)}\, ds \\ &\leqslant C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \end{aligned} \tag{7.29}$$

By (7.20) and (7.25)–(7.29), we obtain the result (7.22). □

**Remark 7.8.** Untangling coordinate changes, we see that $\bar{U}_*\bar{k}(x,t)$ is an estimate for $\psi(x,t)$; that is, $|\bar{u}(x) - \bar{u}(x - \bar{U}_*\bar{k}(x,t))| \sim |\bar{U}_*\bar{u}'\bar{k}|$. This makes a connection between the analyses of [12] (where $v$ and $\psi$ but not $\bar{U}_*\bar{k}(x,t)$ appear) and [18,19] (where the equivalent of $\bar{U}_*\bar{k}(x,t)$ appears, but not $v$ or $\psi$).

### *7.3. Behavior for initial perturbation $|u_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$ and $|u_0(x)|_{H^2} \leqslant E_0$*

In this section, we take $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. We first consider pointwise bounds of $v$, $\psi_t$, $\psi_x$ and $\psi_{xx}$ like previous section.

**Lemma 7.9.** *Suppose $|v_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$ and $|v_0(x)|_{H^2} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. For $v$, $\psi_t$, $\psi_x$ and $\psi_{xx}$ defined in (7.12) and (7.13), define*

$$\zeta(t) := \sup_{0 \leqslant s \leqslant t,\, x \in \mathbb{R}} \big|(v, \psi_t, \psi_x, \psi_{xx})\big|(1+s) e^{\frac{|x-as|^2}{M(1+s)}}. \tag{7.30}$$

*Then, for all $t \geqslant 0$ for which $\zeta(t)$ defined in (7.30) is finite,*

$$\zeta(t) \leqslant C\big(E_0 + \zeta(t)^2\big) \tag{7.31}$$

*for some constant $C > 0$.*

**Proof.** Note first that by (7.15), we have $|v_x|_\infty \leqslant |v|_{H^2} \leqslant CE_0(1+t)^{-\frac{3}{4}} \leqslant C$ and so by (7.6)–(7.9) and (7.30) we have

$$\big|(Q,R,S,T)(x,t)\big| \leqslant \big|(v,\psi_t,\psi_x,\psi_{xx})(x,t)\big|^2 \leqslant \zeta(t)^2(1+t)^{-2}e^{-\frac{|x-at|^2}{M(1+t)}}.$$

Thus, from (7.12), we have

$$\begin{aligned}
\big|v(x,t)\big| &\leqslant \int_{-\infty}^{\infty}\big|\tilde{G}(x,t;y)\big|\big|v_0(y)\big|\,dy + \int_0^t\int_{-\infty}^{\infty}\big|\tilde{G}_y(x,t-s;y)\big|\big|(Q,R,S,T)(y,s)\big|\,ds\,dy \\
&\leqslant CE_0\int_{-\infty}^{\infty}(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}}e^{-\frac{|x-y-at|^2}{Mt}}e^{-\frac{|y|^2}{M}}\,dy \\
&\quad + C\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}(1+s)^{-2}e^{-\frac{|y-as|^2}{M(1+s)}}\,dy\,ds \\
&\leqslant C\big(E_0+\zeta^2(t)\big)(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}},
\end{aligned} \tag{7.32}$$

here we use integration by parts to exchange the $\partial_y$ and $(\partial_y^2+\partial_s)$ derivatives on $R$ and $S$ respectively for $-\partial_y$ and $(\partial_y^2-\partial_s)$ derivatives on $\tilde{G}$ and recall $|\tilde{G}_{yy}+\tilde{G}_t| \sim |\tilde{G}_y| \leqslant Ct^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{M(1+t)}}$.

Recalling $e(x,t;y)=0$ for $0<t\leqslant 1$ and from (7.13), we have

$$\begin{aligned}
\big|(\psi_t,\psi_x,\psi_{xx})(x,t)\big| &\leqslant \int_{-\infty}^{\infty}\big|e_x(x,t;y)\big|\big|v_0(y)\big|\,dy \\
&\quad + \int_0^t\int_{-\infty}^{\infty}\big|e_x(x,t-s;y)\big|\big|(Q,R,S,T)(y,s)\big|\,ds\,dy \\
&\leqslant E_0\int_{-\infty}^{\infty}(1+t)^{-1}e^{-\frac{|x-y-at|^2}{Mt}}e^{-\frac{|y|^2}{M}}\,dy \\
&\quad + \zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}(1+s)^{-2}e^{-\frac{|y-as|^2}{M(1+s)}}\,dy\,ds \\
&\leqslant C\big(E_0+\zeta^2(t)\big)(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}.
\end{aligned} \tag{7.33}$$

The (7.32) and (7.33) imply (7.31). □

**Corollary 7.10.** *For $v$ defined in (7.2) with $|v_0(x)| \leqslant E_0e^{-\frac{|x|^2}{M}}$ and $|v_0(x)|_{H^2} \leqslant E_0$, $E_0>0$ sufficiently small and $M>1$ sufficiently large,*

$$\big|v(x,t)\big| \leqslant CE_0(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}. \tag{7.34}$$

**Proof.** Same proof as Corollary 6.4. □

The following two lemmas are linear and nonlinear estimates of $\bar{u}'(x)\psi$.

**Lemma 7.11** *(Linear estimate). Let $E$ be defined as in (7.4) with $|u_0(x)| \leqslant E_0 e^{-\frac{|x|^2}{M}}$ and $|u_0(x)|_{H^2} < E_0$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Then, for some sufficiently large $M' > M$,*

$$\left| \int_{-\infty}^{\infty} E(x,t;y)u_0(y)\,dy - \bar{U}_0 \bar{u}'(x)\bar{k}(x,t) \right| \leqslant C E_0 (1+t)^{-1} e^{-\frac{|x-at|^2}{M'(1+t)}}, \tag{7.35}$$

*where $\bar{U}_0 = \int_{-\infty}^{\infty} u_0(y)\tilde{q}(y,0)\,dy$ and $\bar{k}(x,t) = \frac{1}{\sqrt{4\pi bt}} e^{-\frac{|x-at|^2}{(4bt)}}$.*

**Proof.** By the Mean Value Theorem,

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} E(x,t;y)u_0(y)\,dy - \bar{U}_0 \bar{u}'(x)\bar{k}(x,t) \right| &\leqslant C \int_{-\infty}^{\infty} \left|\bar{k}(x-y,t) - \bar{k}(x,t)\right| \left|u_0(y)\right| dy \\
&\leqslant C E_0 \int_{-\infty}^{\infty} \int_0^1 (1+t)^{-1} e^{-\frac{|x-wy-at|^2}{(1+t)}} e^{-\frac{|y|^2}{M}}\,dw\,dy \\
&\leqslant C E_0 (1+t)^{-1} e^{-\frac{|x-at|^2}{M'(1+t)}}. \qquad \square
\end{aligned}$$

**Lemma 7.12** *(Nonlinear estimate). Recalling (7.4) and (7.10), we have for some sufficiently large $M'' > M' > M$,*

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy - \bar{U}(s)\bar{u}'(x)\bar{k}(x-as,t-s) \right| \\
&\quad \leqslant C E_0 (1+t-s)^{-1}(1+s)^{-1} e^{-\frac{|x-at|^2}{M''(1+t)}},
\end{aligned} \tag{7.36}$$

*where $\bar{U}(s) = \int_{-\infty}^{\infty} N(y,s)\tilde{q}(y,0)\,dy$.*

**Proof.** Noting first that $|(Q,R,S,T)| \leqslant C E_0 (1+t)^{-2} e^{-\frac{|x-at|^2}{M(1+t)}}$, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy - \bar{U}(s)\bar{u}'(x)\bar{k}(x,t-s) \right| \\
&\quad \leqslant \int_{-\infty}^{\infty} \left|\bar{u}'(x)\tilde{q}(y,0)\right| \left|\bar{k}(x-y,t-s) - \bar{k}(x-as,t-s)\right| \left|(Q+T)(y,s)\right| dy
\end{aligned}$$

$$+\left|\int_{-\infty}^{\infty} \bar{u}'(x)\tilde{q}_y(y,0)\big(\bar{k}(x-y,t-s)-\bar{k}(x-as,t-s)\big)(R+S)(y,s)\,dy\right|$$

$$+\left|\int_{-\infty}^{\infty} \bar{u}'(x)\tilde{q}(y,0)\partial_y\big(\bar{k}(x-y,t-s)-\bar{k}(x-as,t-s)\big)(R+S)(y,s)\,dy\right|$$

$$\leqslant CE_0\int_{-\infty}^{\infty}\int_0^1 (1+t-s)^{-1}e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}}|y-as|(1+s)^{-2}e^{-\frac{|y-as|^2}{M(1+s)}}\,dw\,dy$$

$$+CE_0\int_{-\infty}^{\infty}(1+t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{(t-s)}}(1+s)^{-2}e^{-\frac{|y-as|^2}{M(1+s)}}\,dy$$

$$\leqslant CE_0\int_{-\infty}^{\infty}\int_0^1 (1+t-s)^{-1}e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}}(1+s)^{-\frac{3}{2}}e^{-\frac{|y-as|^2}{M'(1+s)}}\,dw\,dy$$

$$\leqslant CE_0(1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}. \qquad \square$$

We now prove the pointwise behavior of the perturbation $u$ with respect to $|u_0| \leqslant E_0 e^{-\frac{|x|^2}{M}}$.

**Theorem 7.13** *(Behavior). Suppose $u(x,t)$ satisfies $u_t = Lu + O(|u|^2)$ and $|u_0| \leqslant E_0 e^{-\frac{|x|^2}{M}}$ and $|u_0(x)|_{H^2} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Set*

$$\bar{U}_* = \int_0^{\infty}\bar{U}(s)\,ds + \bar{U}_0.$$

*Then $|\bar{U}_*| < \infty$ and for some sufficiently large $M'' > M' > M$,*

$$\big|u(x,t) - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)\big| \leqslant C(1+t)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}\big(1+\ln(1+t)\big). \tag{7.37}$$

**Proof.** Recalling $|\bar{U}(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$, we have $|\bar{U}_*| < \infty$. We first break $|u(x,t) - \bar{U}_*\bar{u}'(x)\bar{k}(x,t)|$ into three parts exactly the same as (7.24). By (7.34) and (7.11), we easily see first two terms

$$\big|v(x,t)\big| + O\big(|v_x||\psi| + |\psi|^2\big) \leqslant C(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}. \tag{7.38}$$

Now we break the last term into four parts exactly the same as (7.26). Then

$$II \leqslant CE_0(1+t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{M(1+t)}}\int_t^{\infty}(1+s)^{-\frac{3}{2}}\,ds \leqslant CE_0(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}. \tag{7.39}$$

By (7.36), we have

$$III \leqslant CE_0 \int_0^t (1+t-s)^{-1}(1+s)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}\, ds \leqslant CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \ln(1+t). \tag{7.40}$$

By the Mean Value Theorem, for some $s^* \in (0, t/2)$, we have

$$\begin{aligned} IV &\leqslant CE_0\left[\int_{t/2}^t (1+s)^{-\frac{3}{2}} \left|\bar{k}(x-as, t-s) - \bar{k}(x,t)\right| ds + \int_0^{t/2} (1+s)^{-\frac{3}{2}} s \left|\bar{k}_t\left(x-as, t-s^*\right)\right| ds\right] \\ &\leqslant CE_0(1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}} \int_{t/s}^t (1+s)^{-\frac{3}{2}}\, ds + CE_0 e^{-\frac{|x-at|^2}{M(1+t)}} \int_0^{t/2} (1+s)^{-\frac{1}{2}} (1+t-s)^{-\frac{3}{2}}\, ds \\ &\leqslant CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}. \end{aligned} \tag{7.41}$$

By (7.35) and (7.38)–(7.41), we obtain the result (7.37). □

### 7.4. *Behavior for initial perturbation $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$ and $|u_0(x)|_{H^2} \leqslant E_0$*

In the last section, we consider the behavior of the perturbation $u$ for an algebraically decaying initial perturbation $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$. Like Section 6.3, we first some information about the effects on algebraically decaying data. The following lemma and corollary are exactly the same as (6.24) and (6.25) replacing $|x|$ by $|x-at|$. As usual, we take $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large.

**Lemma 7.14.** *For all $t \geqslant 0$ and $r > 1$, and any $x \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{t}} \left(1+|y|\right)^{-r} dy \leqslant C\left[t^{-\frac{1}{2}} \wedge \left(1+|x-at|\right)^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{Mt}}\right],$$

*for some sufficiently large $M > 0$ and $C > 0$.*

**Corollary 7.15.** *For all $t \geqslant 0$ and $r > 1$, and any $x \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{|x-y-at|^2}{t}} \left(1+|y|\right)^{-r} dy \leqslant C\left[\left(1+|x-at|+\sqrt{t}\right)^{-r} + (1+\sqrt{t})^{-1} e^{-\frac{|x-at|^2}{Mt}}\right], \tag{7.42}$$

*for some $M > 0$ sufficiently large and $C > 0$.*

With the above corollary, we first prove the pointwise bounds for $|v|$.

**Lemma 7.16.** *Suppose $|v_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 1$ and $|v_0(x)|_{H^2} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. For $v$, $\psi_t$, $\psi_x$ and $\psi_{xx}$ defined in (7.12) and (7.13), define*

$$\zeta(t) := \sup_{0\leqslant s\leqslant t,\, x\in\mathbb{R}} \big|(v,\psi_t,\psi_x,\psi_{xx})\big|(1+s)^{\frac{1}{2}}$$
$$\times\big[\big(1+|x-as|+\sqrt{s}\big)^{-r}+(1+\sqrt{s})^{-1}e^{\frac{-|x-as|^2}{M(1+s)}}\big]^{-1}. \tag{7.43}$$

*Then, for all $t \geqslant 0$ for which $\zeta(t)$ is finite, we have*

$$\zeta(t) \leqslant C\big(E_0+\zeta(t)^2\big) \tag{7.44}$$

*for some constant $C>0$.*

**Proof.** Note first that by (7.15), we have $|v_x|_\infty \leqslant |v|_{H^2} \leqslant CE_0(1+t)^{-\frac{3}{4}} \leqslant C$ and so by (7.6)–(7.9) and (7.43) we have

$$\big|(Q,R,S,T)(x,t)\big| \leqslant \big|(v,\psi_t,\psi_x,\psi_{xx})(x,t)\big|^2$$
$$\leqslant \zeta(t)^2(1+s)^{-1}\big[\big(1+|x-as|+\sqrt{s}\big)^{-r}+(1+\sqrt{s})^{-1}e^{\frac{-|x-as|^2}{M(1+s)}}\big]^2.$$

Then, from (7.12), we have

$$\big|v(x,t)\big| \leqslant \int_{-\infty}^{\infty}\big|\tilde G(x,t;y)\big|\big|v_0(y)\big|\,dy + \int_0^t\int_{-\infty}^{\infty}\big|\tilde G_y(x,t-s;y)\big|\big|(Q,R,S,T)(y,s)\big|\,ds\,dy$$
$$\leqslant CE_0\int_{-\infty}^{\infty}(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}}e^{-\frac{|x-y-at|^2}{Mt}}\big(1+|y|\big)^{-r}\,dy$$
$$+\,C\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}\big(1+|y-as|+\sqrt{s}\big)^{-2r}\,dy\,ds$$
$$+\,C\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}(1+\sqrt{s})^{-2}e^{-\frac{|y-as|^2}{M(1+s)}}\,dy\,ds$$
$$= I + II + III.$$

By (7.42), we have

$$I \leqslant CE_0(1+t)^{-\frac{1}{2}}\big[\big(1+|x-at|+\sqrt{t}\big)^{-r}+(1+\sqrt{t})^{-1}e^{-\frac{|x-at|^2}{Mt}}\big].$$

For $III$, we have

$$III \leqslant \zeta^2(t)\int_0^t(1+s)^{-2}\int_{-\infty}^{\infty}(t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}e^{-\frac{|y-as|^2}{M(1+s)}}\,dy\,ds$$
$$\leqslant \zeta^2(t)(1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}.$$

For $II$, by (7.42), we estimate

$$II \leqslant C\zeta^2(t)\int_0^t (1+s)^{-(1+\frac{r}{2})}(t-s)^{-\frac{1}{2}}\int_{-\infty}^{\infty}(t-s)^{-\frac{1}{2}}e^{-\frac{|x-(y-as)-at|^2}{M(t-s)}}\big(1+|y-as|\big)^{-r}\,dy\,ds$$

$$\leqslant C\zeta^2(t)\int_0^t (1+s)^{-\frac{3}{2}}(t-s)^{-\frac{1}{2}}\big(1+|x-at|+\sqrt{t-s}\big)^{-r}\,ds$$

$$+ C\zeta^2(t)\int_0^t (1+s)^{-(1+\frac{r}{2})}(t-s)^{-\frac{1}{2}}(1+\sqrt{t-s})^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\,ds$$

$$\leqslant C\zeta^2(t)(1+t)^{-\frac{1}{2}}\Big[\big(1+|x-at|+\sqrt{t}\big)^{-r}+(1+\sqrt{t})^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\Big].$$

Now we consider $|(\psi_t,\psi_x,\psi_{xx})|$. Recalling $e(x,t;y)=0$ for $0<t\leqslant 1$, similarly we have

$$\big|(\psi_t,\psi_x,\psi_{xx})(x,t)\big|$$

$$\leqslant E_0\int_{-\infty}^{\infty}(1+t)^{-1}e^{-\frac{|x-y-at|^2}{Mt}}\big(1+|y|\big)^{-r}\,dy$$

$$+ C\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(1+t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}\big(1+|y-as|+\sqrt{s}\big)^{-2r}\,dy\,ds$$

$$+ C\zeta^2(t)\int_0^t\int_{-\infty}^{\infty}(1+s)^{-1}(1+t-s)^{-1}e^{-\frac{|x-y-a(t-s)|^2}{M(t-s)}}(1+\sqrt{s})^{-2}e^{-\frac{|y-as|^2}{M(1+s)}}\,dy\,ds$$

$$\leqslant C\zeta^2(t)(1+t)^{-\frac{1}{2}}\Big[\big(1+|x-at|+\sqrt{t}\big)^{-r}+(1+\sqrt{t})^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\Big]. \qquad \square$$

**Corollary 7.17.** *For $v$ defined in (7.2) with $|v_0(x)|\leqslant E_0(1+|x|)^{-r}$, $r>1$ and $|v_0(x)|_{H^2}\leqslant E_0$, $E_0>0$ sufficiently small and $M>1$ sufficiently large,*

$$\big|v(x,t)\big|\leqslant CE_0(1+t)^{-\frac{1}{2}}\Big[\big(1+|x-at|+\sqrt{t}\big)^{-r}+(1+\sqrt{t})^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\Big]. \tag{7.45}$$

**Proof.** Same proof as Corollary 6.4. $\square$

The proofs of following two lemmas are the same proofs of Lemmas 6.19 and 6.20, respectively. These are needed when we use Mean Value Theorem in estimating linear and nonlinear parts of $\bar{u}'\psi$.

**Lemma 7.18.** *For all $t>0$, $x\in\mathbb{R}$, $r>2$ and all $0<w<1$,*

$$\int_{-\infty}^{\infty}(1+t)^{-\frac{1}{2}}e^{-\frac{|x-wy-at|^2}{M(1+t)}}\big(1+|y|\big)^{-r}\,dy$$

$$\leqslant CE_0\Big[\big(1+|x-at|+\sqrt{t}\big)^{-r}+(1+t)^{-\frac{1}{2}}e^{-\frac{|x-at|^2}{M'(1+t)}}\Big], \tag{7.46}$$

*for some sufficiently large $M'>M$.*

**Lemma 7.19.** *For all $t > s > 0$, $x \in \mathbb{R}$, $r > 2$ and all $0 < w < 1$,*

$$\int_{-\infty}^{\infty} (1+t-s)^{-\frac{1}{2}} e^{-\frac{|x-wy-at|^2}{M(1+t-s)}} \left(1+|y|+\sqrt{s}\right)^{-r} dy$$

$$\leqslant CE_0\left[\left(1+|x-at|+\sqrt{t-s}+\sqrt{t}\right)^{-r} + (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{r}{2}} e^{-\frac{|x-at|^2}{M'(1+t)}}\right], \tag{7.47}$$

*for some sufficiently large $M' > M$.*

**Lemma 7.20** *(Linear estimate). Suppose $u(x,t)$ satisfies $u_t = Lu$ and $|u_0(x)| \leqslant E_0(1+|x|)^{-r}$, $r > 2$ and $|u_0(x)|_{H^2} \leqslant E_0$, for $E_0 > 0$ sufficiently small and $M > 1$ sufficiently large. Then for some sufficiently large $M' > M$,*

$$\left|\int_{-\infty}^{\infty} E(x,t;y)u_0(y)\,dy - \bar{U}_0\bar{u}'(x)\bar{k}(x,t)\right|$$

$$\leqslant CE_0\left[(1+t)^{-\frac{1}{2}}\left(1+|x-at|+\sqrt{t}\right)^{-r+1} + (1+t)^{-1}e^{-|x-at|^2/M'(1+t)}\right], \tag{7.48}$$

*where $\bar{U}_0 = \int_{-\infty}^{\infty} u_0(y)\tilde{q}(y,0)\,dy$ and $\bar{k}(x,t) = \frac{1}{\sqrt{4\pi bt}}e^{-\frac{|x-at|^2}{(4bt)}}$.*

**Proof.** By (7.42) and (6.28), we have

$$\left|\int_{-\infty}^{\infty} E(x,t;y)u_0(y)\,dy - \bar{U}_0\bar{u}'(x)\bar{k}(x,t)\right|$$

$$\leqslant CE_0 \int_{-\infty}^{\infty}\int_0^1 (1+t)^{-1} e^{-\frac{|x-wy-at|^2}{(1+t)}} \left(1+|y|\right)^{-r+1} dw\,dy$$

$$\leqslant CE_0\left[(1+t)^{-\frac{1}{2}}\left(1+|x-at|+\sqrt{t}\right)^{-r+1} + (1+t)^{-1}e^{-\frac{|x-at|^2}{M'(1+t)}}\right]. \qquad \square$$

**Lemma 7.21** *(Nonlinear estimate). Recalling (7.4) and (7.10), we have for some sufficiently large $M' > M$*

$$\left|\int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy - \bar{U}(s)\bar{u}'(x)\bar{k}(x-as,t-s)\right|$$

$$\leqslant CE_0(1+s)^{-1}\left[(1+t-s)^{-\frac{1}{2}}\left(1+|x-at|+\sqrt{t-s}+\sqrt{s}\right)^{-2r+1} + (1+t-s)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\right], \tag{7.49}$$

*where $\bar{U}(s) = \int_{-\infty}^{\infty} N(y,s)\tilde{q}(y,0)\,dy$.*

**Proof.** Noting first that

$$\left|(Q,R,S,T)\right| \leqslant CE_0(1+t)^{-1}\left[\left(1+|x-at|+\sqrt{t}\right)^{-2r} + (1+t)^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\right],$$

we have, from (7.47) and by the Mean Value Theorem,

$$\left|\int_{-\infty}^{\infty} E(x,t-s;y)N(y,s)\,dy - \bar{U}(s)\bar{u}'(x)\bar{k}(x,t-s)\right|$$

$$\leqslant C\int_{-\infty}^{\infty}\left|\bar{k}(x-y,t-s)-\bar{k}(x,t-s)\right|\left|(Q,R,S,T)(y,s)\right|dy$$

$$\leqslant CE_0\int_{-\infty}^{\infty}\int_0^1(1+t-s)^{-1}e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}}(1+s)^{-1}\left(1+|y-as|+\sqrt{s}\right)^{-2r+1}dw\,dy$$

$$+CE_0\int_{-\infty}^{\infty}\int_0^1(1+t-s)^{-1}e^{-\frac{|x-w(y-as)-at|^2}{(t-s)}}(1+s)^{-\frac{3}{2}}e^{-\frac{|y-as|^2}{M'(1+s)}}dw\,dy$$

$$\leqslant CE_0(1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\left(1+|x-at|+\sqrt{t-s}+\sqrt{t}\right)^{-2r+1}$$

$$+CE_0(1+t-s)^{-1}(1+s)^{-r}e^{-\frac{|x-at|^2}{M(1+t)}}+CE_0(1+t-s)^{-1}(1+s)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}$$

$$\leqslant CE_0(1+s)^{-1}\Big[(1+t-s)^{-\frac{1}{2}}\left(1+|x-at|+\sqrt{t-s}+\sqrt{t}\right)^{-2r+1}$$

$$+(1+t-s)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}\Big]. \qquad \square$$

We now prove the final asymptotic behavior of $u$ with respect to $|u_0|\leqslant E_0(1+|x|)^{-r}$, $r>2$.

**Theorem 7.22** *(Behavior). Suppose $u(x,t)$ satisfies $u_t = Lu + O(|u|^2)$ and $|u_0|\leqslant E_0(1+|x|)^{-r}$, $r>2$ and $|u_0|_{H^2}\leqslant E_0$, for $E_0>0$ sufficiently small and $M>1$ sufficiently large. Set*

$$\bar{U}_* = \int_0^{\infty}\bar{U}(s)\,ds + \bar{U}_0.$$

*Then $|\bar{U}_*|<\infty$ and for some sufficiently large $M''>M'>M$,*

$$\left|u(x,t)-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)\right|$$

$$\leqslant CE_0\Big[(1+t)^{-\frac{1}{2}}\left(1+|x-at|+\sqrt{t}\right)^{-r+1}+(1+t)^{-1}e^{-\frac{|x-at|^2}{M''(1+t)}}\big(1+\ln(1+t)\big)\Big]. \tag{7.50}$$

**Proof.** Recalling $|\bar{U}(s)| = |N(y,s)|_{L^1(y)}\leqslant CE_0(1+s)^{-\frac{3}{2}}$, we have $|\bar{U}_*|<\infty$. Now we break $|u(x,t)-\bar{U}_*\bar{u}'(x)\bar{k}(x,t)|$ into three parts exactly the same as (7.24). By (7.45) and (7.11), first two terms are trivial

$$\left|v(x,t)\right|+O\big(|v_x||\psi|+|\psi|^2\big)$$

$$\leqslant CE_0(1+t)^{-\frac{1}{2}}\Big[\left(1+|x-at|+\sqrt{t}\right)^{-r}+(1+\sqrt{t})^{-1}e^{-\frac{|x-at|^2}{M(1+t)}}\Big]. \tag{7.51}$$

Like (7.26), we break the last term into four parts. Since $|\bar{U}(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$,

$$II \leqslant CE_0(1+t)^{-\frac{1}{2}} e^{-\frac{|x-at|^2}{M(1+t)}} \int_t^\infty (1+s)^{-\frac{3}{2}}\, ds \leqslant CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}. \tag{7.52}$$

By (7.49), we have

$$\begin{aligned} III &\leqslant CE_0 \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\bigl(1+|x-at|+\sqrt{t-s}+\sqrt{s}\bigr)^{-2r+1}\, ds \\ &\quad + CE_0 \int_0^t (1+t-s)^{-1}(1+s)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}\, ds \\ &\leqslant CE_0\bigl(1+|x-at|+\sqrt{t}\bigr)^{-2r+1} \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-1}\, ds \\ &\quad + CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \left[\int_0^{t/2} (1+s)^{-1}\, ds + \int_{t/2}^t (1+t-s)^{-1}\, ds\right] \\ &\leqslant CE_0\Bigl[(1+t)^{-\frac{1}{2}}\bigl(1+|x-at|+\sqrt{t}\bigr)^{-r+1} + (1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}} \ln(1+t)\Bigr]. \end{aligned} \tag{7.53}$$

Since $|\bar{U}(s)| \leqslant CE_0(1+s)^{-\frac{3}{2}}$, the estimate of $IV$ is exactly the same as (7.41) which is

$$IV \leqslant CE_0(1+t)^{-1} e^{-\frac{|x-at|^2}{M(1+t)}}. \tag{7.54}$$

By (7.48) and (7.51)–(7.54), we obtain the result (7.50). □

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## References

[1] B. Barker, M. Johnson, P. Noble, M. Rodrigues, K. Zumbrun, Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto–Sivashinsky equation, preprint, 2012.
[2] B. Barker, M. Johnson, P. Noble, M. Rodrigues, K. Zumbrun, Efficient numerical evaluation of the periodic Evans function of Gardner and spectral stability of periodic viscous roll waves, in preparation.
[3] B.I. Cohen, J.A. Krommes, W.M. Tang, M.N. Rosenbluth, Non-linear saturation of the dissipative trapped-ion mode by mode coupling, Nucl. Fusion 16 (1976) 971.
[4] Arjen Doelman, Björn Sandstede, Arnd Scheel, Guido Schneider, The dynamics of modulated wave trains, Mem. Amer. Math. Soc. 199 (934) (2009), viii+105.
[5] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice–Hall, Englewood Cliffs, NY, 1964, reprint ed., 1983.
[6] U. Frisch, Z.S. She, O. Thual, Viscoelastic behaviour of cellular solutions to the Kuramoto–Sivashinsky model, J. Fluid Mech. 168 (1986) 221–240.
[7] R. Gardner, On the structure of the spectra of periodic traveling waves, J. Math. Pures Appl. 72 (1993) 415–439.
[8] R.A. Gardner, Spectral analysis of long wavelength periodic waves and applications, J. Reine Angew. Math. 491 (1997) 149–181.

[9] P. Howard, K. Zumbrun, Stability of undercompressive viscous shock waves, J. Differential Equations 225 (1) (2006) 308–360.
[10] M. Johnson, P. Noble, M. Rodrigues, K. Zumbrun, Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto–Sivashinsky equation, preprint, 2011.
[11] M. Johnson, K. Zumbrun, Nonlinear stability of periodic traveling waves of viscous conservation laws in the generic case, J. Differential Equations 249 (5) (2010) 1213–1240.
[12] M. Johnson, K. Zumbrun, Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction diffusion equations, Ann. Inst. H. Poincare Anal. Non Lineaire 28 (4) (2011) 471–483.
[13] M. Johnson, K. Zumbrun, P. Noble, Nonlinear stability of viscous roll waves, SIAM J. Math. Anal. 43 (2) (2011) 577–611.
[14] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin/Heidelberg, 1985.
[15] C. Mascia, K. Zumbrun, Pointwise green's function bounds for shock profiles of systems with real viscosity, Arch. Ration. Mech. Anal. 169 (3) (2003) 177–263.
[16] M. Oh, K. Zumbrun, Stability of periodic solutions of conservation laws with viscosity: Pointwise bounds on the Green function, Arch. Ration. Mech. Anal. 166 (2) (2003) 167–196.
[17] M. Raoofi, K. Zumbrun, Stability of undercompressive viscous shock profiles of hyperbolic–parabolic systems, J. Differential Equations (2009) 1539–1567.
[18] G. Schneider, Nonlinear diffusive stability of spatially periodic solutions – abstract theorem and higher space dimensions, in: Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems, Sendai, 1997, in: Tohoku Math. Publ., vol. 8, Tohoku Univ., Sendai, 1998, pp. 159–167.
[19] G. Schneider, Diffusive stability of spatial periodic solutions of the Swift–Hohenberg equation, Comm. Math. Phys. 178 (3) (1996) 679–702 (English summary).
[20] K. Zumbrun, P. Howard, Pointwise semigroup methods and stability of viscous shock waves, Indiana Math. J. 47 (1998) 741–871.