J. Differential Equations 252 (2012) 745–767

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Global solution to the incompressible flow of liquid crystals

Xiaoli Li [a,b,*], Dehua Wang [b]

[a] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*
[b] *Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*

A R T I C L E   I N F O

Article history:
Received 28 December 2010
Revised 4 August 2011
Available online 8 September 2011

MSC:
35A05
76A10
76D03

Keywords:
Liquid crystals
Incompressible flow
Strong solution
Existence and uniqueness

A B S T R A C T

The initial–boundary value problem for the three-dimensional incompressible flow of liquid crystals is considered in a bounded smooth domain. The existence and uniqueness is established for both the local strong solution with large initial data and the global strong solution with small data. It is also proved that when the strong solution exists, a weak solution must be equal to the unique strong solution with the same data.

1. Introduction

Liquid crystals are a state of matter that have properties between those of a conventional liquid and those of a solid crystal that are optically anisotropic, even when they are at rest. In this work, we are interested in a Navier–Stokes type model for incompressible fluids that takes into account the crystallinity of the fluid molecules in the three-dimensional case, that is, a nematic liquid crystal model, which can be governed by the following nonlinear hydrodynamical system (see [5,13,14] and references therein):

* Corresponding author at: Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail addresses: xllithu@gmail.com (X. Li), dwang@math.pitt.edu (D. Wang).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.08.045

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla P = -\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \tag{1.1a}$$

$$\frac{\partial \mathbf{d}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{d} = \gamma \left(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\right), \tag{1.1b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.1c}$$

where $\mathbf{u} \in \mathbb{R}^3$ denotes the velocity, $\mathbf{d} \in \mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$) the unit-vector field that represents the macroscopic/continuum molecular orientations, $P \in \mathbb{R}$ is the pressure (including both the hydrostatic part and the induced elastic part from the orientation field) arising from the incompressibility $\nabla \cdot \mathbf{u} = 0$; and they all depend on the spatial variable $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the time variable $t > 0$. The term $\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})$ in the stress tensor represents the anisotropic feature of the system. The positive constants μ, λ, γ stand for viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Deborah number for the molecular orientation field, respectively. We set these three constants to be one since their exact values do not play any role in our analysis. The symbol $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ denotes a matrix whose (i, j)-th entry is $\partial_{x_i} \mathbf{d} \cdot \partial_{x_j} \mathbf{d}$ for $1 \leqslant i, j \leqslant 3$, and it is easy to see that $\nabla \mathbf{d} \odot \nabla \mathbf{d} = (\nabla \mathbf{d})^\top \nabla \mathbf{d}$, where $(\nabla \mathbf{d})^\top$ denotes the transpose of the 3×3 matrix $\nabla \mathbf{d}$.

System (1.1) is a simplified version, but still retains most of the interesting mathematical properties (without destroying the basic nonlinear structure) of the original Ericksen–Leslie model ([7,8,10,11,13, 14]) for the hydrodynamics of nematic liquid crystals; see [15,20,21] for more discussions on the relations of the two models. Both the Ericksen–Leslie system and the simplified one describe the macroscopic continuum time evolution of liquid crystal materials under the influence of both the velocity $\mathbf{u}$ and the orientation $\mathbf{d}$ which can be derived from the averaging/coarse graining of the directions of rod-like liquid crystal molecules. In particular, there is a force term in the $\mathbf{u}$-system depending on $\mathbf{d}$; the left-hand side of the $\mathbf{d}$-system stands for the kinematic transport by the flow field while the right-hand side represents the internal relaxation due to the elastic energy. In many situations, the flow velocity field does disturb the alignment of the molecule, and in turn, a change in the alignment will induce velocity.

We consider the initial–boundary value problem of system (1.1) in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ with the initial condition:

$$(\mathbf{u}, \mathbf{d})|_{t=0} = \big(\mathbf{u}_0(\mathbf{x}), \mathbf{d}_0(\mathbf{x})\big), \quad \mathbf{x} \in \Omega, \tag{1.2}$$

and the boundary condition:

$$(\mathbf{u}, \partial_\nu \mathbf{d})|_{\partial\Omega} = (0, 0), \quad t > 0, \tag{1.3}$$

where ν is the outer unit-normal vector field on $\partial\Omega$, $\mathbf{u}_0 : \Omega \to \mathbb{R}^3$, and $\mathbf{d}_0 : \Omega \to \mathbb{S}^2$ are given with compatibility; for the velocity $\mathbf{u}$ the non-slip boundary condition, i.e., homogeneous Dirichlet type, is considered, and for the orientation vector $\mathbf{d}$ the homogeneous Neumann boundary condition is posed here.

Roughly speaking, system (1.1) is a coupling between the incompressible Navier–Stokes equations and the transported flow of harmonic maps. There have been many studies on system (1.1), see [10, 11,14–17,19–22] and the references therein. Recently, in Lin, Lin, and Wang [17], they established both interior and boundary regularity theorem for such a system in dimension two under smallness conditions. And, they also established the existence of global weak solutions that are smooth away from at most finitely many singular times in any bounded smooth domain of $\mathbb{R}^2$. In Lin and Liu [15], they addressed both the regularity and existence of global weak solutions to the n-dimensional ($n = 2, 3$) Leslie system of variable length, i.e., the Dirichlet energy

$$\frac{1}{2} \int_\Omega |\nabla \mathbf{d}|^2 \, d\mathbf{x} \quad \text{for } \mathbf{d} : \Omega \to \mathbb{S}^{n-1}$$

is replaced by the Ginzburg–Landau energy

$$\int_{\Omega} \left(\frac{1}{2} |\nabla \mathbf{d}|^2 + \frac{(1 - |\mathbf{d}|^2)^2}{4\varepsilon^2} \right) d\mathbf{x} \quad \text{for } \mathbf{d} : \Omega \to \mathbb{R}^n.$$

More precisely, they proved the global existence of weak solutions with large initial data under the assumptions that $\mathbf{u}_0 \in L^2(\Omega), \mathbf{d}_0 \in H^1(\Omega)$ with $\mathbf{d}_0|_{\partial\Omega} \in H^{\frac{3}{2}}(\partial\Omega)$ in dimension two and three. The existence and uniqueness of global classical solution was also obtained if $\mathbf{u}_0 \in H^1(\Omega), \mathbf{d}_0 \in H^2(\Omega)$ in dimension two or dimension three when the fluid viscosity μ is large enough. The similar results were obtained also in [21] for a different but similar model. When weak solutions are discussed, the partial regularity theorem of the weak solution was investigated in [16] (and also [11]), similar to the classical theorem by Caffarelli, Kohn, and Nirenberg [3] for the Navier–Stokes equations.

In this paper, we are interested in the existence and uniqueness of global strong solution $(\mathbf{u}, \mathbf{d}, P)$ of (1.1) in $W^{2,q}(\Omega)^3 \times W^{3,q}(\Omega)^3 \times W^{1,q}(\Omega)$ with $q > 3$. By a *Strong Solution*, we mean a triplet $(\mathbf{u}, \mathbf{d}, P)$ satisfying (1.1) almost everywhere with the initial–boundary conditions (1.2)–(1.3). Our strategy to consider (1.1) is to linearize it as

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla P = -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla \cdot \big((\nabla \mathbf{f})^\top \nabla \mathbf{f} \big), \tag{1.4a}$$

$$\frac{\partial \mathbf{d}}{\partial t} - \Delta \mathbf{d} = -\mathbf{v} \cdot \nabla \mathbf{f} + |\nabla \mathbf{f}|^2 \mathbf{f}, \tag{1.4b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.4c}$$

for some given functions $\mathbf{v} \in \mathbb{R}^3$ and $\mathbf{f} \in \mathbb{R}^3$. One of the motivations of making such a linearization is that we can use the maximal regularity of Stokes equations (cf. Theorem 3.2) and the parabolic equation (cf. Theorem 3.1). We first use an iteration method to establish the local existence and uniqueness of strong solution with general large initial data. Then we prove the global existence by establishing some global estimates under the condition that the initial data are small in some norm. As system (1.1) contains the Navier–Stokes equations as a subsystem, one cannot expect generally better results than those for the Navier–Stokes equations. The uniqueness of global weak solution is still an open problem. We shall prove that when the strong solution exists, all the global weak solutions must be equal to the unique strong solution, which is called the weak–strong uniqueness. Similar results were obtained by Danchin [4] for the density-dependent incompressible Navier–Stokes equations. We shall establish our results in the spirit of [4], while developing new estimates for the orientation field $\mathbf{d}$. Due to the specific structure of the equations for $\mathbf{u}$, especially the strongly nonlinear term $(\nabla \mathbf{d})^\top \Delta \mathbf{d}$ in the $\mathbf{u}$-system, it will be necessary to obtain more regularity for $\mathbf{d}$.

The rest of the paper is organized as follows. In Section 2, we state our main results on local and global existence of strong solution, as well as the weak–strong uniqueness. In Section 3, we recall the maximal regularity for Stokes equations and the parabolic equation, and also some L^∞ estimates. In Section 4, we give the proof of the local existence. In Section 5, we prove the global existence. Finally in Section 6, we show the weak–strong uniqueness.

2. Main results

In this section, we state our main results. If $k > 0$ is an integer and $p \geqslant 1$, we denote by $W^{k,p}$ the set of functions in $L^p(\Omega)$ whose derivatives of up to order k belong to $L^p(\Omega)$. For $T > 0$ and a function space X, denote by $L^p(0, T; X)$ the set of Bochner measurable X-valued time dependent functions f such that $t \to \|f\|_X$ belongs to $L^p(0, T)$. Let us define the functional spaces in which the existence of solutions is going to be obtained:

Definition 2.1. For $T > 0$ and $1 < p, q < \infty$, we denote by $M_T^{p,q}$ the set of triplets $(\mathbf{u}, \mathbf{d}, P)$ such that

$$\mathbf{u} \in C\big([0,T]; D_{A_q}^{1-\frac{1}{p},p}\big) \cap L^p\big(0,T; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)\big), \qquad \frac{\partial \mathbf{u}}{\partial t} \in L^p\big(0,T; L^q(\Omega)\big), \qquad \nabla \cdot \mathbf{u} = 0,$$

$$\mathbf{d} \in C\big([0,T]; B_{q,p}^{3(1-\frac{1}{p})}\big) \cap L^p\big(0,T; W^{3,q}(\Omega)\big), \qquad \frac{\partial \mathbf{d}}{\partial t} \in L^p\big(0,T; L^q(\Omega)\big),$$

and

$$P \in L^p\big(0,T; W^{1,q}(\Omega)\big), \qquad \int_\Omega P \, d\mathbf{x} = 0.$$

The corresponding norm is denoted by $\|\cdot\|_{M_T^{p,q}}$.

We remark that the condition $\int_\Omega P \, d\mathbf{x} = 0$ in the definition (2.1) holds automatically if we replace P by

$$P - \frac{1}{|\Omega|} \int_\Omega P \, d\mathbf{x}$$

in (1.1). Also, in the above definition, the space $D_{A_q}^{1-\frac{1}{p},p}$ stands for some fractional domain of the Stokes operator in L^q (cf. Section 2.3 in [4]). Roughly, the vector-fields of $D_{A_q}^{1-\frac{1}{p},p}$ are vectors which have $2 - \frac{2}{p}$ derivatives in L^q, are divergence-free, and vanish on $\partial\Omega$. The Besov space (for definition, see [2]) $B_{q,p}^{3(1-\frac{1}{p})}$ can be regarded as the interpolation space between L^q and $W^{3,q}$, that is,

$$B_{q,p}^{3(1-\frac{1}{p})} = \big(L^q, W^{3,q}\big)_{1-\frac{1}{p},p}.$$

Moreover, we note that $B_{q,p}^{3(1-\frac{1}{p})} \hookrightarrow W^{1,q}$ if $p \geqslant \frac{3}{2}$. By the embedding $W^{1,q} \hookrightarrow L^\infty$ as $q > 3$, one has $B_{q,p}^{3(1-\frac{1}{p})} \hookrightarrow L^\infty$, which will be used repeatly in this paper.

The local existence will be shown by using an iterative method, and if the initial data are sufficiently small in some suitable function spaces, the solution is indeed global in time. More precisely, our existence result reads:

Theorem 2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Assume $\frac{3}{2} \leqslant p, q < \infty$ with $\frac{p}{2}(1 - \frac{3}{q}) \in (0,1)$ and $\mathbf{u}_0 \in D_{A_q}^{1-\frac{1}{p},p}$, $\mathbf{d}_0 \in B_{q,p}^{3(1-\frac{1}{p})}$. Then,*

(1) *There exists $T_0 > 0$, such that, system* (1.1) *with the initial–boundary conditions* (1.2)–(1.3) *has a unique strong solution $(\mathbf{u}, \mathbf{d}, P) \in M_{T_0}^{p,q}$ in $\Omega \times (0, T_0)$.*
(2) *Moreover, there exists $\delta_0 > 0$, such that, if the initial data satisfy*

$$\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} \leqslant \delta_0, \qquad \|\mathbf{d}_0\|_{B_{q,p}^{3(1-\frac{1}{p})}} \leqslant \delta_0,$$

then (1.1)–(1.3) *has a unique strong solution $(\mathbf{u}, \mathbf{d}, P) \in M_T^{p,q}$ in $\Omega \times (0, T)$ for all $T > 0$.*

According to [17], a *Weak Solution* to (1.1) with the initial–boundary conditions (1.2)–(1.3) means a triplet $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}}, \Pi)$ satisfying system (1.1) in $\Omega \times (0,T)$ for $0 < T \leqslant +\infty$ in the sense of distributions, i.e., for any smooth function $\psi(t)$ with $\psi(T) = 0$ and $\phi(\mathbf{x}) \in (H_0^1(\Omega))^3$ with $\nabla \cdot \phi = 0$, we have

$$-\int_0^T (\tilde{\mathbf{u}}, \psi'\phi)\,dt + \int_0^T (\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}, \psi\phi)\,dt + \mu \int_0^T (\nabla\tilde{\mathbf{u}}, \psi\nabla\phi)\,dt = \psi(0)(\mathbf{u}_0, \phi) + \lambda \int_0^T (\nabla\tilde{\mathbf{d}} \odot \nabla\tilde{\mathbf{d}}, \psi\nabla\phi)\,dt,$$

and

$$-\int_0^T (\tilde{\mathbf{d}}, \psi'\phi)\,dt + \int_0^T (\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{d}}, \psi\phi)\,dt + \gamma \int_0^T (\nabla\tilde{\mathbf{d}}, \psi\nabla\phi)\,dt = \psi(0)(\mathbf{d}_0, \phi) + \gamma \int_0^T |\nabla\tilde{\mathbf{d}}|^2(\tilde{\mathbf{d}}, \psi\phi)\,dt,$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)^3$. Moreover, $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}})$ satisfies (1.3) in the sense of trace.

Next, we will give a uniqueness result. For $0 < T < +\infty$, suppose $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}}, \Pi)$ with

$$\tilde{\mathbf{u}} \in L^{2,\infty}\big(\Omega \times [0,T]\big) \cap W_2^{1,0}(\Omega_T), \qquad \tilde{\mathbf{d}} \in L^\infty\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big),$$

and

$$\nabla\Pi \in L^{\frac{4}{3}}\big(0,T; L^{\frac{6}{5}}(\Omega)\big)$$

is a global weak solution to (1.1)–(1.3). Since $\nabla\tilde{\mathbf{d}} \in L^2(0,T;H^1(\Omega))$ and $|\tilde{\mathbf{d}}| = 1$, then

$$\Delta\tilde{\mathbf{d}} \cdot \tilde{\mathbf{d}} + |\nabla\tilde{\mathbf{d}}|^2 = 0.$$

Hence $|\nabla\tilde{\mathbf{d}}| \in L^4(\Omega \times [0,T])$. We have the following energy inequality (cf. [17], Section 5 for the two-dimensional case):

$$\frac{1}{2}\int_\Omega \big(|\tilde{\mathbf{u}}(t)|^2 + |\nabla\tilde{\mathbf{d}}(t)|^2\big)\,d\mathbf{x} + \int_0^t \int_\Omega \big(|\nabla\tilde{\mathbf{u}}|^2 + \big|\Delta\tilde{\mathbf{d}} + |\nabla\tilde{\mathbf{d}}|^2\tilde{\mathbf{d}}\big|^2\big)\,d\mathbf{x}\,ds \leqslant \frac{1}{2}\int_\Omega \big(|\mathbf{u}_0|^2 + |\nabla\mathbf{d}_0|^2\big)\,d\mathbf{x}, \tag{2.1}$$

for all $t \in (0,\infty)$. We remark that the assumption on pressure function holds since Π can be determined as in the Navier–Stokes equations (see [9]).

As for the standard Navier–Stokes equations, the question of uniqueness in the above class remains open. However, for the same initial–boundary conditions, a relation between the weak solution and the strong solution can be formulated as:

Theorem 2.2. *Assume that* $\mathbf{u}_0 \in D_{A_q}^{1-\frac{1}{p},p}$ *and* $\mathbf{d}_0 \in B_{q,p}^{3(1-\frac{1}{p})}$. *Then any weak solution to* (1.1)–(1.3) *in the above class is unique and indeed is equal to its unique strong solution.*

Usually, we call this kind of uniqueness as *Weak–Strong Uniqueness*. For the similar results on the compressible Navier–Stokes equations, we refer the readers to [6,18].

3. Maximal regularity

In this section, we recall the maximal regularities for the parabolic operator and the Stokes operator, as well as some L^∞ estimates.

For $T > 0$, $1 < p, q < \infty$, denote

$$\mathcal{W}(0,T) := \left(W^{1,p}\left(0,T;L^q(\Omega)\right)\right)^3 \cap \left(L^p\left(0,T;W^{3,q}(\Omega)\right)\right)^3.$$

Throughout this paper, C stands for a generic positive constant.

We first recall the maximal regularity for the parabolic operator (cf. Theorem 4.10.7 and Remark 4.10.9 in [1]):

Theorem 3.1. *Given* $1 < p, q < \infty$, $\omega_0 \in B_{q,p}^{3(1-\frac{1}{p})}$ *and* $f \in (L^p(0,T;L^q(\mathbb{R}^3)))^3$, *the Cauchy problem*

$$\begin{cases} \frac{\partial \omega}{\partial t} - \Delta \omega = f, \quad t > 0, \\ \omega(0) = \omega_0 \end{cases}$$

has a unique solution $\omega \in \mathcal{W}(0,T)$, *and*

$$\|\omega\|_{\mathcal{W}(0,T)} \leqslant C\left(\|f\|_{L^p(0,T;L^q(\mathbb{R}^3))} + \|\omega_0\|_{B_{q,p}^{3(1-\frac{1}{p})}}\right),$$

where C *is independent of* ω_0, f *and* T. *Moreover, there exists a positive constant* c_0 *independent of* f *and* T *such that*

$$\|\omega\|_{\mathcal{W}(0,T)} \geqslant c_0 \sup_{t\in(0,T)} \left\|\omega(t)\right\|_{B_{q,p}^{3(1-\frac{1}{p})}}.$$

Now we recall the maximal regularity for the Stokes equations (cf. Theorem 3.2 in [4]):

Theorem 3.2. *Let* Ω *be a bounded domain with* $C^{2+\varepsilon}$ *boundary in* $\mathbb{R}^3$ *and* $1 < p, q < \infty$. *Assume that* $\mathbf{u}_0 \in D_{A_q}^{1-\frac{1}{p},p}$ *and* $f \in L^p(\mathbb{R}^+;L^q(\Omega))$. *Then the system*

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla P = f, \qquad \displaystyle\int_\Omega P\,d\mathbf{x} = 0, \\ \nabla \cdot \mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Omega} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 \end{cases}$$

has a unique solution $(\mathbf{u}, P)$ *satisfying the following inequality for all* $T > 0$:

$$\left\|\mathbf{u}(T)\right\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \left\|\left(\nabla P, \Delta\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}\right)\right\|_{L^q}^p dt\right)^{\frac{1}{p}}$$

$$\leqslant C\left(\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \left\|f(t)\right\|_{L^q}^p dt\right)^{\frac{1}{p}}\right) \tag{3.1}$$

with $C = C(q, p, \sigma(\Omega))$ where $\sigma(\Omega)$ stands for the open set

$$\sigma(\Omega) = \left\{\frac{\mathbf{x}}{\delta(\Omega)}\,\middle|\,\mathbf{x} \in \Omega\right\}$$

with $\delta(\Omega)$ denoting the diameter of Ω.

Remark 3.1. We notice that (3.1) does not include the estimate for $\|\mathbf{u}\|_{L^p(0,T;L^q)}$. Indeed, thanks to $\mathbf{u}|_{\partial\Omega} = 0$, Poincare's inequality, and the fact $\int_\Omega \nabla\mathbf{u}\, d\mathbf{x} = 0$, we have

$$\|\mathbf{u}\|_{W^{2,q}} \leqslant C\|\Delta\mathbf{u}\|_{L^q},$$

and then (3.1) can be rewritten as

$$\left\|\mathbf{u}(T)\right\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \left\|(\nabla P, \mathbf{u}, \Delta\mathbf{u}, \partial_t\mathbf{u})\right\|_{L^q}^p dt\right)^{\frac{1}{p}}$$

$$\leqslant C\left(\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \left\|f(t)\right\|_{L^q}^p dt\right)^{\frac{1}{p}}\right).$$

We have the L^∞ estimate in the spatial variable as follows (cf. Lemma 4.1 in [4]).

Lemma 3.1. *Let $1 < p, q, r, s < \infty$ satisfy $0 < \frac{p}{2} - \frac{3p}{2q} < 1$, then*

$$\|\nabla f\|_{L^p(0,T;L^\infty)} \leqslant CT^{\frac{1}{2}-\frac{3}{2q}}\|f\|^{1-\theta}_{L^\infty(0,T;D_{A_q}^{1-\frac{1}{p},p})}\|f\|^{\theta}_{L^p(0,T;W^{2,q})},$$

for some constant C depending only on Ω, p, q and

$$\frac{1-\theta}{p} = \frac{1}{2} - \frac{3}{2q}.$$

Similarly, we have,

Lemma 3.2. *Let $1 < p, q < \infty$ satisfy $0 < \frac{p}{2} - \frac{3p}{2q} < 1$, then*

$$\|\nabla f\|_{L^p(0,T;L^\infty)} \leqslant CT^{\frac{1}{2}-\frac{3}{2q}}\|f\|^{1-\theta}_{L^\infty(0,T;B_{q,p}^{2(1-\frac{1}{p})})}\|f\|^{\theta}_{L^p(0,T;W^{2,q})},$$

for some constant C depending only on Ω, p, q and

$$\frac{1-\theta}{p}=\frac{1}{2}-\frac{3}{2q}.$$

Proof. First, we notice that (cf. Theorem 6.4.5 in [2])

$$\left(B^{1-\frac{2}{p}-\frac{3}{q}}_{\infty,\infty}, B^{1-\frac{3}{q}}_{\infty,\infty}\right)_{\theta,1}=B^0_{\infty,1} \quad \text{with} \quad \frac{1-\theta}{p}=\frac{1}{2}-\frac{3}{2q}.$$

Also the embedding $B^0_{\infty,1}\hookrightarrow L^\infty$ holds due to Theorem 6.2.4 in [2]. Hence,

$$|\nabla f|_\infty \leqslant C\|\nabla f\|_{B^0_{\infty,1}} \leqslant C\|\nabla f\|^{\theta}_{B^{1-\frac{3}{q}}_{\infty,\infty}}\|\nabla f\|^{1-\theta}_{B^{1-\frac{2}{p}-\frac{3}{q}}_{\infty,\infty}}. \tag{3.2}$$

We remark that (cf. Theorems 6.2.4 and 6.5.1 in [2])

$$B^{2(1-\frac{1}{p})}_{q,p}\hookrightarrow B^{2-\frac{2}{p}-\frac{3}{q}}_{\infty,\infty}, \qquad W^{1,q}\hookrightarrow B^1_{q,\infty}\hookrightarrow B^{1-\frac{3}{q}}_{\infty,\infty}.$$

Thus, according to (3.2) and by applying Hölder's inequality, we deduce that

$$\begin{aligned}
\|\nabla f\|_{L^p(0,T;L^\infty)} &\leqslant C\left(\int_0^T \|\nabla f\|^{p\theta}_{B^{1-\frac{3}{q}}_{\infty,\infty}}\|\nabla f\|^{p(1-\theta)}_{B^{1-\frac{2}{p}-\frac{3}{q}}_{\infty,\infty}}\,dt\right)^{\frac{1}{p}}\\
&\leqslant C\left(\int_0^T \|f\|^{p\theta}_{W^{2,q}}\|f\|^{p(1-\theta)}_{B^{2-\frac{2}{p}-\frac{3}{q}}_{q,p}}\,dt\right)^{\frac{1}{p}}\\
&\leqslant C\left(\int_0^T \|f\|^{p\theta}_{W^{2,q}}\|f\|^{p(1-\theta)}_{B^{2(1-\frac{1}{p})}_{q,p}}\,dt\right)^{\frac{1}{p}}\\
&\leqslant CT^{\frac{1}{2}-\frac{3}{2q}}\|f\|^{1-\theta}_{L^\infty(0,T;B^{2(1-\frac{1}{p})}_{q,p})}\|f\|^{\theta}_{L^p(0,T;W^{2,q})}.
\end{aligned}$$

The proof is complete. □

Remark 3.2. In this paper, we will use the following weaker result:

$$\|\nabla f\|_{L^p(0,T;L^\infty)} \leqslant CT^{\frac{1}{2}-\frac{3}{2q}}\|f\|^{1-\theta}_{L^\infty(0,T;B^{3(1-\frac{1}{p})}_{q,p})}\|f\|^{\theta}_{L^p(0,T;W^{3,q})}.$$

Lemma 3.3. *Let* $1<p,q<\infty$ *satisfy* $0<\frac{p}{3}-\frac{p}{q}<1$, *then*

$$\|\Delta f\|_{L^p(0,T;L^\infty)} \leqslant CT^{\frac{1}{3}-\frac{1}{q}}\|f\|^{1-\theta}_{L^\infty(0,T;B^{3(1-\frac{1}{p})}_{q,p})}\|f\|^{\theta}_{L^p(0,T;W^{3,q})},$$

for some constant C depending only on Ω, p, q and

$$\frac{1-\theta}{p} = \frac{1}{3} - \frac{1}{q}.$$

Proof. First, we notice that (cf. Theorem 6.4.5 in [2])

$$\left(B_{\infty,\infty}^{2-\frac{3}{p}-\frac{3}{q}}, B_{\infty,\infty}^{2-\frac{3}{q}}\right)_{\theta,1} = B_{\infty,1}^{1} \quad \text{with} \quad \frac{1-\theta}{p} = \frac{1}{3} - \frac{1}{q}.$$

Also the embedding $B_{\infty,1}^{1} \hookrightarrow W^{1,\infty}$ is true due to Theorem 6.2.4 in [2]. Hence,

$$\|\Delta f\|_{L^\infty} \leqslant C\|\nabla f\|_{W^{1,\infty}} \leqslant C\|\nabla f\|_{B_{\infty,1}^{1}} \leqslant C\|\nabla f\|^{\theta}_{B_{\infty,\infty}^{2-\frac{3}{q}}} \|\nabla f\|^{1-\theta}_{B_{\infty,\infty}^{2-\frac{3}{p}-\frac{3}{q}}}. \tag{3.3}$$

We remark that (cf. Theorems 6.2.4 and 6.5.1 in [2])

$$B_{q,p}^{3(1-\frac{1}{p})} \hookrightarrow B_{\infty,\infty}^{3-\frac{3}{p}-\frac{3}{q}}, \qquad W^{2,q} \hookrightarrow B_{q,\infty}^{2} \hookrightarrow B_{\infty,\infty}^{2-\frac{3}{q}}.$$

Thus, according to (3.3) and by applying Hölder's inequality, we deduce that

$$\begin{aligned}
\|\Delta f\|_{L^p(0,T;L^\infty)} &\leqslant C\left(\int_0^T \|\nabla f\|^{p\theta}_{B_{\infty,\infty}^{2-\frac{3}{q}}} \|\nabla f\|^{p(1-\theta)}_{B_{\infty,\infty}^{2-\frac{3}{p}-\frac{3}{q}}}\, dt\right)^{\frac{1}{p}} \\
&\leqslant C\left(\int_0^T \|f\|^{p\theta}_{W^{3,q}} \|f\|^{p(1-\theta)}_{B_{q,p}^{3(1-\frac{1}{p})}}\, dt\right)^{\frac{1}{p}} \\
&\leqslant CT^{\frac{1}{3}-\frac{1}{q}} \|f\|^{1-\theta}_{L^\infty(0,T;B_{q,p}^{3(1-\frac{1}{p})})} \|f\|^{\theta}_{L^p(0,T;W^{3,q})}.
\end{aligned}$$

The proof is complete. □

4. Local existence

In this section, we prove the local existence and uniqueness of strong solution in Theorem 2.1. The proof will be divided into several steps, including constructing the approximate solutions by iteration, obtaining the uniform estimate, showing the convergence, consistency and uniqueness.

4.1. Construction of approximate solutions

We initialize the construction of approximate solutions by setting

$$\mathbf{d}^0 := \mathbf{d}_0, \qquad \mathbf{u}^0 := \mathbf{u}_0.$$

For given $(\mathbf{u}^n, \mathbf{d}^n, P^n)$, the Stokes equations (1.4a) and the parabolic equation (1.4b) enable us to define $(\mathbf{u}^{n+1}, \mathbf{d}^{n+1}, P^{n+1})$ as the global solution of

$$\begin{cases} \dfrac{\partial \mathbf{u}^{n+1}}{\partial t} - \Delta \mathbf{u}^{n+1} + \nabla P^{n+1} = -\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \nabla \cdot \big((\nabla \mathbf{d}^n)^\top \nabla \mathbf{d}^n \big), \\ \dfrac{\partial \mathbf{d}^{n+1}}{\partial t} - \Delta \mathbf{d}^{n+1} = -\mathbf{u}^n \cdot \nabla \mathbf{d}^n + |\nabla \mathbf{d}^n|^2 \mathbf{d}^n, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \qquad \displaystyle\int_\Omega P^{n+1}\, d\mathbf{x} = 0 \end{cases} \tag{4.1}$$

with the initial–boundary conditions:

$$\big(\mathbf{u}^{n+1}, \mathbf{d}^{n+1}\big)\big|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0), \qquad \big(\mathbf{u}^{n+1}, \partial_\nu \mathbf{d}^{n+1}\big)\big|_{\partial\Omega} = (0, 0).$$

According to Theorems 3.1–3.2, an argument by induction yields a sequence

$$\big\{ \big(\mathbf{u}^n, \mathbf{d}^n, P^n\big) \big\}_{n \in \mathbb{N}} \subset M_T^{p,q}$$

for all positive T.

4.2. Uniform estimate for some small fixed time T

We aim at finding a positive time T independent of n for which $\{(\mathbf{u}^n, \mathbf{d}^n, P^n)\}_{n\in\mathbb{N}}$ is uniformly bounded in the space $M_T^{p,q}$. Indeed, applying Theorem 3.2 to

$$\begin{cases} \dfrac{\partial \mathbf{u}^{n+1}}{\partial t} - \Delta \mathbf{u}^{n+1} + \nabla P^{n+1} = -\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \nabla \cdot \big((\nabla \mathbf{d}^n)^\top \nabla \mathbf{d}^n \big), \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \qquad \displaystyle\int_\Omega P^{n+1}\, d\mathbf{x} = 0, \\ \mathbf{u}^{n+1}\big|_{t=0} = \mathbf{u}_0, \qquad \mathbf{u}^{n+1}|_{\partial\Omega} = 0 \end{cases}$$

and Theorem 3.1 to

$$\begin{cases} \dfrac{\partial \mathbf{d}^{n+1}}{\partial t} - \Delta \mathbf{d}^{n+1} = -\mathbf{u}^n \cdot \nabla \mathbf{d}^n + |\nabla \mathbf{d}^n|^2 \mathbf{d}^n, \\ \mathbf{d}^{n+1}\big|_{t=0} = \mathbf{d}_0, \qquad \partial_\nu \mathbf{d}^{n+1}\big|_{\partial\Omega} = 0, \end{cases}$$

we obtain

$$\begin{aligned} &\big\| \mathbf{u}^{n+1}(T) \big\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \left\| \left(\nabla P^{n+1}, \mathbf{u}^{n+1}, \Delta \mathbf{u}^{n+1}, \frac{\partial \mathbf{u}^{n+1}}{\partial t} \right) \right\|_{L^q}^p dt \right)^{\frac{1}{p}} \\ &\quad \leqslant C \left(\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \left(\int_0^T \big\| \mathbf{u}^n \cdot \nabla \mathbf{u}^n + \nabla \cdot \big((\nabla \mathbf{d}^n)^\top \nabla \mathbf{d}^n \big) \big\|_{L^q}^p dt \right)^{\frac{1}{p}} \right), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\left\|\mathbf{d}^{n+1}(T)\right\|_{B_{q,p}^{3(1-\frac{1}{p})}}+\left\|\mathbf{d}^{n+1}\right\|_{\mathcal{W}(0,T)}\\&\quad\leqslant C\left(\|\mathbf{d}_0\|_{B_{q,p}^{3(1-\frac{1}{p})}}+\left\|-\mathbf{u}^n\cdot\nabla\mathbf{d}^n+\left|\nabla\mathbf{d}^n\right|^2\mathbf{d}^n\right\|_{L^p(0,T;L^q)}\right).\end{aligned}\tag{4.3}$$

Now define

$$\begin{aligned}U^n(t):=&\left\|\mathbf{u}^n\right\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})}+\left\|\mathbf{u}^n\right\|_{L^p(0,t;W^{2,q})}+\left\|\frac{\partial\mathbf{u}^n}{\partial t}\right\|_{L^p(0,t;L^q)}\\&+\left\|\mathbf{d}^n\right\|_{L^\infty(0,t;B_{q,p}^{3(1-\frac{1}{p})})}+\left\|\mathbf{d}^n\right\|_{\mathcal{W}(0,t)},\end{aligned}$$

and

$$U^0:=\|\mathbf{u}_0\|_{D_{A_q}^{1-\frac{1}{p},p}}+\|\mathbf{d}_0\|_{B_{q,p}^{3(1-\frac{1}{p})}}.$$

Hence, from (4.2) and (4.3), we get, using Lemmas 3.1–3.3,

$$\begin{aligned}U^{n+1}(t)\leqslant\ &C\Big(U^0+\left\|\mathbf{u}^n\right\|_{L^\infty(0,t;L^q)}\left\|\nabla\mathbf{u}^n\right\|_{L^p(0,t;L^\infty)}\\&+2\left\|\nabla\mathbf{d}^n\right\|_{L^\infty(0,t;L^q)}\left\|\Delta\mathbf{d}^n\right\|_{L^p(0,t;L^\infty)}\\&+\left\|\mathbf{u}^n\right\|_{L^\infty(0,t;L^q)}\left\|\nabla\mathbf{d}^n\right\|_{L^p(0,t;L^\infty)}\\&+\left\|\mathbf{d}^n\right\|_{L^\infty(0,t;L^\infty)}\left\|\nabla\mathbf{d}^n\right\|_{L^\infty(0,t;L^q)}\left\|\nabla\mathbf{d}^n\right\|_{L^p(0,t;L^\infty)}\Big)\\\leqslant\ &C\left(U^0+2t^{\frac{1}{2}-\frac{3}{2q}}\left(U^n(t)\right)^2+2t^{\frac{1}{3}-\frac{1}{q}}\left(U^n(t)\right)^2+t^{\frac{1}{2}-\frac{3}{2q}}\left(U^n(t)\right)^3\right).\end{aligned}\tag{4.4}$$

Moreover, if we assume that $U^n(t)\leqslant 4CU^0$ on $[0,T]$ with

$$\begin{aligned}&0<T\leqslant\left(\frac{3}{64C^2U^0+64C^3(U^0)^2}\right)^{\frac{3q}{q-3}}\leqslant 1,\quad\text{or}\\&1<T\leqslant\left(\frac{3}{64C^2U^0+64C^3(U^0)^2}\right)^{\frac{2q}{q-3}},\end{aligned}\tag{4.5}$$

then a direct computation yields

$$U^{n+1}(t)\leqslant 4CU^0\quad\text{on }[0,T].$$

From (4.2)–(4.4), we conclude that the sequence $\{(\mathbf{u}^n,\mathbf{d}^n,P^n)\}_{n=1}^\infty$ is uniformly bounded in $M_T^{p,q}$. More precisely, we have:

Lemma 4.1. *For all $t\in[0,T]$ with T satisfying (4.5),*

$$U^n(t)\leqslant 4CU^0.$$

4.3. Convergence of the approximation sequence

Lemma 4.2. $\{(\mathbf{u}^n, \mathbf{d}^n, P^n)\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $M_{T_0}^{p,q}$ *and thus converges.*

Proof. Let

$$\bar{\mathbf{u}}^n := \mathbf{u}^{n+1} - \mathbf{u}^n, \qquad \bar{\mathbf{d}}^n := \mathbf{d}^{n+1} - \mathbf{d}^n, \qquad \bar{P}^n := P^{n+1} - P^n.$$

Then, the triplet $(\bar{\mathbf{u}}^n, \bar{\mathbf{d}}^n, \bar{P}^n)$ satisfies

$$\begin{cases} \dfrac{\partial \bar{\mathbf{u}}^n}{\partial t} - \Delta \bar{\mathbf{u}}^n + \nabla \bar{P}^n = -\bar{\mathbf{u}}^{n-1} \cdot \nabla \mathbf{u}^n - \mathbf{u}^{n-1} \cdot \nabla \bar{\mathbf{u}}^{n-1} - \nabla \cdot \big(\big(\nabla \bar{\mathbf{d}}^{n-1}\big)^\top \nabla \mathbf{d}^n \big) \\ \qquad\qquad - \nabla \cdot \big(\big(\nabla \mathbf{d}^{n-1}\big)^\top \nabla \bar{\mathbf{d}}^{n-1} \big), \\ \dfrac{\partial \bar{\mathbf{d}}^n}{\partial t} - \Delta \bar{\mathbf{d}}^n = -\bar{\mathbf{u}}^{n-1} \cdot \nabla \mathbf{d}^n - \mathbf{u}^{n-1} \cdot \nabla \bar{\mathbf{d}}^{n-1} + \big|\nabla \mathbf{d}^n\big|^2 \bar{\mathbf{d}}^{n-1} \\ \qquad\qquad + \big(\big(\nabla \mathbf{d}^n + \nabla \mathbf{d}^{n-1}\big) : \nabla \bar{\mathbf{d}}^{n-1} \big) \mathbf{d}^{n-1}, \\ \nabla \cdot \bar{\mathbf{u}}^n = 0, \qquad \displaystyle\int_\Omega \bar{P}^n d\mathbf{x} = 0 \end{cases} \tag{4.6}$$

with the initial–boundary conditions:

$$\big(\bar{\mathbf{u}}^n, \bar{\mathbf{d}}^n\big)\big|_{t=0} = (0,0), \qquad \big(\bar{\mathbf{u}}^n, \partial_\nu \bar{\mathbf{d}}^n\big)\big|_{\partial\Omega} = (0,0).$$

Define

$$\begin{aligned} \bar{U}^n(t) :=& \big\|\bar{\mathbf{u}}^n\big\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \big\|\bar{\mathbf{u}}^n\big\|_{L^p(0,t;W^{2,q})} + \left\|\frac{\partial \bar{\mathbf{u}}^n}{\partial t}\right\|_{L^p(0,t;L^q)} \\ &+ \big\|\nabla \bar{P}^n\big\|_{L^p(0,t;L^q)} + \big\|\bar{\mathbf{d}}^n\big\|_{L^\infty(0,t;B_{q,p}^{3(1-\frac{1}{p})})} + \big\|\bar{\mathbf{d}}^n\big\|_{\mathcal{W}(0,t)}. \end{aligned}$$

By using Lemmas 3.1–3.3, we obtain the following estimates:

$$\begin{aligned} &\big\|\bar{\mathbf{u}}^{n-1} \cdot \nabla \mathbf{u}^n + \mathbf{u}^{n-1} \cdot \nabla \bar{\mathbf{u}}^{n-1}\big\|_{L^p(0,t;L^q)} \\ &\quad \leqslant \big\|\bar{\mathbf{u}}^{n-1}\big\|_{L^\infty(0,t;L^q)} \big\|\nabla \mathbf{u}^n\big\|_{L^p(0,t;L^\infty)} + \big\|\mathbf{u}^{n-1}\big\|_{L^\infty(0,t;L^q)} \big\|\nabla \bar{\mathbf{u}}^{n-1}\big\|_{L^p(0,t;L^\infty)} \\ &\quad \leqslant 4CU^0 \big(t^{\frac{1}{2}-\frac{3}{2q}} \big\|\bar{\mathbf{u}}^{n-1}\big\|_{L^\infty(0,t;L^q)} + \big\|\nabla \bar{\mathbf{u}}^{n-1}\big\|_{L^p(0,t;L^\infty)}\big), \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\big\|\nabla \cdot \big(\big(\nabla \bar{\mathbf{d}}^{n-1}\big)^\top \nabla \mathbf{d}^n \big) + \nabla \cdot \big(\big(\nabla \mathbf{d}^{n-1}\big)^\top \nabla \bar{\mathbf{d}}^{n-1} \big)\big\|_{L^p(0,t;L^q)} \\ &\quad \leqslant \big\|\nabla \mathbf{d}^n\big\|_{L^\infty(0,t;L^q)} \big\|\Delta \bar{\mathbf{d}}^{n-1}\big\|_{L^p(0,t;L^\infty)} + \big\|\nabla \bar{\mathbf{d}}^{n-1}\big\|_{L^\infty(0,t;L^q)} \big\|\Delta \mathbf{d}^n\big\|_{L^p(0,t;L^\infty)} \\ &\qquad + \big\|\nabla \bar{\mathbf{d}}^{n-1}\big\|_{L^\infty(0,t;L^q)} \big\|\Delta \mathbf{d}^{n-1}\big\|_{L^p(0,t;L^\infty)} + \big\|\nabla \mathbf{d}^{n-1}\big\|_{L^\infty(0,t;L^q)} \big\|\Delta \bar{\mathbf{d}}^{n-1}\big\|_{L^p(0,t;L^\infty)} \\ &\quad \leqslant 8CU^0 \big(\big\|\Delta \bar{\mathbf{d}}^{n-1}\big\|_{L^p(0,t;L^\infty)} + t^{\frac{1}{3}-\frac{1}{q}} \big\|\nabla \bar{\mathbf{d}}^{n-1}\big\|_{L^\infty(0,t;L^q)}\big), \end{aligned} \tag{4.8}$$

$$\begin{aligned}&\left\|\bar{\mathbf{u}}^{n-1}\cdot\nabla\mathbf{d}^n+\mathbf{u}^{n-1}\cdot\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^q)}\\&\quad\leqslant\left\|\bar{\mathbf{u}}^{n-1}\right\|_{L^\infty(0,t;L^q)}\left\|\nabla\mathbf{d}^n\right\|_{L^p(0,t;L^\infty)}+\left\|\mathbf{u}^{n-1}\right\|_{L^\infty(0,t;L^q)}\left\|\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^\infty)}\\&\quad\leqslant 4CU^0\left(t^{\frac12-\frac{3}{2q}}\left\|\bar{\mathbf{u}}^{n-1}\right\|_{L^\infty(0,t;L^q)}+\left\|\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^\infty)}\right),\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\left\||\nabla\mathbf{d}^n|^2\bar{\mathbf{d}}^{n-1}+\left(\left(\nabla\mathbf{d}^n+\nabla\mathbf{d}^{n-1}\right):\nabla\bar{\mathbf{d}}^{n-1}\right)\mathbf{d}^{n-1}\right\|_{L^p(0,t;L^q)}\\&\quad\leqslant\left(4CU^0\right)^2\left(t^{\frac12-\frac{3}{2q}}\left\|\bar{\mathbf{d}}^{n-1}\right\|_{L^\infty(0,t;L^\infty)}+2\left\|\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^\infty)}\right).\end{aligned}\tag{4.10}$$

Applying Theorems 3.1–3.2 to (4.6), with the help of (4.7)–(4.10), we have

$$\begin{aligned}\bar{U}^n(t)\leqslant 4CU^0\Big(&2t^{\frac12-\frac{3}{2q}}\left\|\bar{\mathbf{u}}^{n-1}\right\|_{L^\infty(0,t;L^q)}+\left\|\nabla\bar{\mathbf{u}}^{n-1}\right\|_{L^p(0,t;L^\infty)}+2\left\|\Delta\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^\infty)}\\&+2t^{\frac13-\frac1q}\left\|\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^\infty(0,t;L^q)}+\left\|\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^\infty)}\\&+4CU^0\left(t^{\frac12-\frac{3}{2q}}\left\|\bar{\mathbf{d}}^{n-1}\right\|_{L^\infty(0,t;L^\infty)}+2\left\|\nabla\bar{\mathbf{d}}^{n-1}\right\|_{L^p(0,t;L^\infty)}\right)\Big).\end{aligned}\tag{4.11}$$

Combining (4.11) and Lemmas 3.1–3.3, one has

$$\bar{U}^n(t)\leqslant 16CU^0\left(\left(1+3CU^0\right)t^{\frac12-\frac{3}{2q}}+t^{\frac13-\frac1q}\right)\bar{U}^{n-1}(t).$$

Thus, if we choose T_0 satisfying (4.5) such that the condition

$$16CU^0\left(2+3CU^0\right)T_0^{\frac13-\frac1q}\leqslant\frac12,\quad\text{or}\quad 16CU^0\left(2+3CU^0\right)T_0^{\frac12-\frac{3}{2q}}\leqslant\frac12$$

is fulfilled, it is clear that $\{(\mathbf{u}^n,\mathbf{d}^n,P^n)\}_{n=1}^\infty$ is a Cauchy sequence in $M_{T_0}^{p,q}$ and thus converges in $M_{T_0}^{p,q}$. □

4.4. The limit is a solution

Since $\{(\mathbf{u}^n,\mathbf{d}^n,P^n)\}_{n=1}^\infty$ is a Cauchy sequence in $M_{T_0}^{p,q}$, then it converges. Let $(\mathbf{u},\mathbf{d},P)\in M_{T_0}^{p,q}$ be the limit of the sequence $\{(\mathbf{u}^n,\mathbf{d}^n,P^n)\}_{n=1}^\infty$ in $M_{T_0}^{p,q}$. We claim all those nonlinear terms in (4.1) converge to their corresponding terms in (1.1) in $(L^p(0,T_0;L^q(\Omega)))^3$. Indeed, due to the convergence of $\mathbf{u}^n$ to $\mathbf{u}$ in $M_{T_0}^{p,q}$, we have

$$\begin{aligned}&\left\|\mathbf{u}^n\cdot\nabla\mathbf{u}^n-\mathbf{u}\cdot\nabla\mathbf{u}\right\|_{L^p(0,T_0;L^q)}\\&\quad=\left\|\left(\mathbf{u}^n-\mathbf{u}\right)\cdot\nabla\mathbf{u}^n+\mathbf{u}\cdot\nabla\left(\mathbf{u}^n-\mathbf{u}\right)\right\|_{L^p(0,T_0;L^q)}\\&\quad\leqslant\left\|\mathbf{u}^n-\mathbf{u}\right\|_{L^\infty(0,T_0;L^q)}\left\|\nabla\mathbf{u}^n\right\|_{L^p(0,T_0;L^\infty)}+\|\mathbf{u}\|_{L^\infty(0,T_0;L^q)}\left\|\nabla\mathbf{u}^n-\nabla\mathbf{u}\right\|_{L^p(0,T_0;L^\infty)}\\&\quad\leqslant 4CU^0T_0^{\frac12-\frac{3}{2q}}\left\|\mathbf{u}^n-\mathbf{u}\right\|_{M_{T_0}^{p,q}}+CT_0^{\frac12-\frac{3}{2q}}\|\mathbf{u}\|_{L^\infty(0,T_0;L^q)}\left\|\mathbf{u}^n-\mathbf{u}\right\|_{M_{T_0}^{p,q}}\\&\quad\to 0.\end{aligned}$$

Hence,

$$\mathbf{u}^n \cdot \nabla \mathbf{u}^n \to \mathbf{u} \cdot \nabla \mathbf{u} \quad \text{in } \big(L^p\big(0, T_0; L^q(\Omega)\big)\big)^3.$$

Similarly, we have

$$\nabla \cdot \big(\big(\nabla \mathbf{d}^n\big)^\top \nabla \mathbf{d}^n\big) \to \nabla \cdot \big((\nabla \mathbf{d})^\top \nabla \mathbf{d}\big) \quad \text{in } \big(L^p\big(0, T_0; L^q(\Omega)\big)\big)^3,$$
$$\mathbf{u}^n \cdot \nabla \mathbf{d}^n \to \mathbf{u} \cdot \nabla \mathbf{d} \quad \text{in } \big(L^p\big(0, T_0; L^q(\Omega)\big)\big)^3.$$

Since

$$\begin{aligned}
&\big\| \big|\nabla \mathbf{d}^n\big|^2 \mathbf{d}^n - |\nabla \mathbf{d}|^2 \mathbf{d} \big\|_{L^p(0,T_0;L^q)} \\
&\quad \leqslant \big\| \big|\nabla \mathbf{d}^n\big|^2 \mathbf{d}^n - \big|\nabla \mathbf{d}^n\big|^2 \mathbf{d} \big\|_{L^p(0,T_0;L^q)} + \big\| \big|\nabla \mathbf{d}^n\big|^2 \mathbf{d} - |\nabla \mathbf{d}|^2 \mathbf{d} \big\|_{L^p(0,T_0;L^q)} \\
&\quad \leqslant \big\| \mathbf{d}^n - \mathbf{d} \big\|_{L^\infty(0,T_0;L^\infty)} \big\| \nabla \mathbf{d}^n \big\|_{L^\infty(0,T_0;L^q)} \big\| \nabla \mathbf{d}^n \big\|_{L^p(0,T_0;L^\infty)} \\
&\qquad + \|\mathbf{d}\|_{L^\infty(0,T_0;L^\infty)} \big\| \nabla \mathbf{d}^n + \nabla \mathbf{d} \big\|_{L^\infty(0,T_0;L^q)} \big\| \nabla \mathbf{d}^n - \nabla \mathbf{d} \big\|_{L^p(0,T_0;L^\infty)} \\
&\quad \leqslant \Big(\big(4CU^0\big)^2 + \|\mathbf{d}\|_{L^\infty(0,T_0;L^\infty)} \big(4CU^0 + \|\nabla \mathbf{d}\|_{L^\infty(0,T_0;L^q)}\big)\Big) T_0^{\frac{1}{2} - \frac{3}{2q}} \big\| \mathbf{d}^n - \mathbf{d} \big\|_{M_{T_0}^{p,q}} \\
&\quad \to 0,
\end{aligned}$$

then

$$\big|\nabla \mathbf{d}^n\big|^2 \mathbf{d}^n \to |\nabla \mathbf{d}|^2 \mathbf{d} \quad \text{in } \big(L^p\big(0, T_0; L^q(\Omega)\big)\big)^3.$$

Thus, taking the limit as $n \to \infty$ in (4.1), we conclude that (1.1) holds in $L^p(0, T_0; L^q(\Omega))$, and hence almost everywhere in $\Omega \times (0, T_0)$.

Multiply the $\mathbf{d}$-system, i.e., (1.1b) by $\mathbf{d}$, we obtain

$$\frac{1}{2} \frac{\partial |\mathbf{d}|^2}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \nabla\big(|\mathbf{d}|^2\big) = \Delta \mathbf{d} \cdot \mathbf{d} + |\nabla \mathbf{d}|^2 |\mathbf{d}|^2.$$

Since

$$\Delta\big(|\mathbf{d}|^2\big) = 2|\nabla \mathbf{d}|^2 + 2\mathbf{d} \cdot (\Delta \mathbf{d}),$$

then it follows that

$$\frac{1}{2} \frac{\partial |\mathbf{d}|^2}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \nabla\big(|\mathbf{d}|^2\big) = \frac{1}{2} \Delta\big(|\mathbf{d}|^2\big) - |\nabla \mathbf{d}|^2 + |\nabla \mathbf{d}|^2 |\mathbf{d}|^2.$$

Therefore, it is easy to deduce that

$$\frac{\partial (|\mathbf{d}|^2 - 1)}{\partial t} - \Delta\big(|\mathbf{d}|^2 - 1\big) + \mathbf{u} \cdot \nabla\big(|\mathbf{d}|^2 - 1\big) - 2|\nabla \mathbf{d}|^2 \big(|\mathbf{d}|^2 - 1\big) = 0. \tag{4.12}$$

Multiplying (4.12) by $(|\mathbf{d}|^2 - 1)$ and then integrating over Ω, using (1.1c) and (1.3), we get the following inequality:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \left(|\mathbf{d}|^2-1\right)^2 d\mathbf{x} &\leqslant 4\int_\Omega |\nabla \mathbf{d}|^2\left(|\mathbf{d}|^2-1\right)^2 d\mathbf{x}\\ &\leqslant 4\|\nabla \mathbf{d}\|_{L^\infty}^2 \int_\Omega \left(|\mathbf{d}|^2-1\right)^2 d\mathbf{x}.\end{aligned} \tag{4.13}$$

Remark that interpolating between $L^\infty(0,T_0;W^{1,q})$ and $L^p(0,T_0;W^{3,q})$ shows that for some positive $\beta>\frac{1}{2}$, $\mathbf{d}$ belongs to $L^2(0,T_0;H^{2+\beta})$ and that $\|\nabla\mathbf{d}\|_{L^\infty}^2\in L^1(0,T_0)$. Notice that

$$\int_\Omega \left(|\mathbf{d}|^2-1\right)^2 d\mathbf{x}=0, \quad \text{at time } t=0.$$

Thus, using (4.13) together with Grönwall's inequality, it yields $|\mathbf{d}|=1$ in $\Omega\times(0,T_0)$.

4.5. *Uniqueness*

Let $(\mathbf{u}_1,\mathbf{d}_1,P_1)$ and $(\mathbf{u}_2,\mathbf{d}_2,P_2)$ be two solutions to (1.1) with the initial–boundary conditions (1.2)–(1.3). Denote

$$\bar{\mathbf{u}}=\mathbf{u}_1-\mathbf{u}_2,\qquad \bar{\mathbf{d}}=\mathbf{d}_1-\mathbf{d}_2,\qquad \bar{P}=P_1-P_2.$$

Note that the triplet $(\bar{\mathbf{u}},\bar{\mathbf{d}},\bar{P})$ satisfies the following system:

$$\begin{cases}\dfrac{\partial\bar{\mathbf{u}}}{\partial t}-\Delta\bar{\mathbf{u}}+\nabla\bar{P}=-\bar{\mathbf{u}}\cdot\nabla\mathbf{u}_1-\mathbf{u}_2\cdot\nabla\bar{\mathbf{u}}-\nabla\cdot\big((\nabla\mathbf{d}_1)^\top\nabla\bar{\mathbf{d}}\big)-\nabla\cdot\big((\nabla\bar{\mathbf{d}})^\top\nabla\mathbf{d}_2\big),\\ \dfrac{\partial\bar{\mathbf{d}}}{\partial t}-\Delta\bar{\mathbf{d}}=-\mathbf{u}_1\cdot\nabla\bar{\mathbf{d}}-\bar{\mathbf{u}}\cdot\nabla\mathbf{d}_2+|\nabla\mathbf{d}_1|^2\bar{\mathbf{d}}+\big((\nabla\mathbf{d}_1+\nabla\mathbf{d}_2):\nabla\bar{\mathbf{d}}\big)\mathbf{d}_2,\\ \nabla\cdot\bar{\mathbf{u}}=0,\qquad \displaystyle\int_\Omega \bar{P}\,d\mathbf{x}=0\end{cases}$$

with the initial–boundary conditions:

$$(\bar{\mathbf{u}},\bar{\mathbf{d}})|_{t=0}=(0,0),\qquad (\bar{\mathbf{u}},\partial_\nu\bar{\mathbf{d}})|_{\partial\Omega}=(0,0).$$

Define

$$\begin{aligned}X(t):=\|\bar{\mathbf{u}}\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})}+\|\bar{\mathbf{u}}\|_{L^p(0,t;W^{2,q})}+\left\|\frac{\partial\bar{\mathbf{u}}}{\partial t}\right\|_{L^p(0,t;L^q)}\\ +\|\nabla\bar{P}\|_{L^p(0,t;L^q)}+\|\bar{\mathbf{d}}\|_{L^\infty(0,t;B_{q,p}^{3(1-\frac{1}{p})})}+\|\bar{\mathbf{d}}\|_{\mathcal{W}(0,t)}.\end{aligned}$$

Thus, repeating the arguments in (4.7)–(4.10), we have

$$\begin{aligned}X(t)&\leqslant 16CU^0\big((1+3CU^0)t^{\frac{1}{2}-\frac{3}{2q}}+t^{\frac{1}{3}-\frac{1}{q}}\big)X(t)\\ &\leqslant\frac{1}{2}X(t).\end{aligned}$$

Hence, $X(t)=0$ for all $t\in[0,T_0]$, which guarantees the uniqueness on the time interval $[0,T_0]$.

5. Global existence

In this section, we prove that, if the initial data is sufficiently small, the local solution established in the previous section is indeed global in time. To this end, we first denote by T^* the maximal time of existence for $(\mathbf{u}, \mathbf{d}, P)$. Define the function $H(t)$ as

$$H(t) := \|\mathbf{u}\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \|\mathbf{u}\|_{L^p(0,t;W^{2,q})} + \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^p(0,t;L^q)} + \|\nabla P\|_{L^p(0,t;L^q)} + \|\mathbf{d}\|_{L^\infty(0,t;B_{q,p}^{3(1-\frac{1}{p})})} + \|\mathbf{d}\|_{\mathcal{W}(0,t)}.$$

To extend the local solution, we need to control the maximal time T^* only in terms of the initial data. For this purpose, it is obvious to observe that $H(t)$ is an increasing and continuous function in $[0, T^*)$, and for all $t \in [0, T^*)$, we have, using Theorems 3.1–3.2,

$$H(t) \leqslant C\big(U^0 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,t;L^q)} + \big\|\nabla\cdot\big((\nabla\mathbf{d})^\top\nabla\mathbf{d}\big)\big\|_{L^p(0,t;L^q)} + \|\mathbf{u}\cdot\nabla\mathbf{d}\|_{L^p(0,t;L^q)} + \big\||\nabla\mathbf{d}|^2\mathbf{d}\big\|_{L^p(0,t;L^q)}\big). \tag{5.1}$$

On the other hand, Lemmas 3.1–3.3 imply that

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,t;L^q)} \leqslant \|\mathbf{u}\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{u}\|_{L^p(0,t;L^\infty)} \leqslant Ct^{\frac{1}{2}-\frac{3}{2q}}H^2(t), \tag{5.2}$$

$$\big\|\nabla\cdot\big((\nabla\mathbf{d})^\top\nabla\mathbf{d}\big)\big\|_{L^p(0,t;L^q)} \leqslant C\|\nabla\mathbf{d}\|_{L^\infty(0,t;L^q)}\|\Delta\mathbf{d}\|_{L^p(0,t;L^\infty)} \leqslant Ct^{\frac{1}{3}-\frac{1}{q}}H^2(t), \tag{5.3}$$

$$\|\mathbf{u}\cdot\nabla\mathbf{d}\|_{L^p(0,t;L^q)} \leqslant \|\mathbf{u}\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{d}\|_{L^p(0,t;L^\infty)} \leqslant Ct^{\frac{1}{2}-\frac{3}{2q}}H^2(t), \tag{5.4}$$

and for the fact that $|\mathbf{d}| = 1$, we have

$$\big\||\nabla\mathbf{d}|^2\mathbf{d}\big\|_{L^p(0,t;L^q)} \leqslant \|\nabla\mathbf{d}\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{d}\|_{L^p(0,t;L^\infty)} \leqslant Ct^{\frac{1}{2}-\frac{3}{2q}}H^2(t). \tag{5.5}$$

Substituting (5.2)–(5.5) into (5.1), we get

$$H(t) \leqslant C\big(U^0 + \big(3t^{\frac{1}{2}-\frac{3}{2q}} + t^{\frac{1}{3}-\frac{1}{q}}\big)H^2(t)\big). \tag{5.6}$$

Assume that T is the smallest number such that

$$H(T) = 4CU^0.$$

This is possible because $H(t)$ is an increasing and continuous function in time. Then,

$$H(t) < H(T) = 4CU^0, \quad \text{for all } t \in [0, T),$$

and from (5.6), we deduce that

$$16C^2U^0\big(3T^{\frac{1}{2}-\frac{3}{2q}}+T^{\frac{1}{3}-\frac{1}{q}}\big)\geqslant 3.$$

This implies that the maximal time of existence T^* will go to infinity when the initial data approaches zero. More precisely, we can show that, if the initial data is sufficiently small, the solution exists globally in time. To this end, we need some other estimates for the terms on the right side of (5.1). Indeed, by $W^{1,q}\hookrightarrow L^\infty$ as $q>3$, we have

$$\begin{aligned}\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p(0,t;L^q)}&\leqslant\|\mathbf{u}\|_{L^\infty(0,t;L^q)}\|\nabla\mathbf{u}\|_{L^p(0,t;L^\infty)}\\&\leqslant CH^2(t).\end{aligned}$$

Similarly,

$$\big\|\nabla\cdot\big((\nabla\mathbf{d})^\top\nabla\mathbf{d}\big)\big\|_{L^p(0,t;L^q)},\ \|\mathbf{u}\cdot\nabla\mathbf{d}\|_{L^p(0,t;L^q)},\ \big\||\nabla\mathbf{d}|^2\mathbf{d}\big\|_{L^p(0,t;L^q)}\leqslant CH^2(t).$$

Thus, (5.1) turns out to be

$$H(t)\leqslant C\big(U^0+H^2(t)\big). \tag{5.7}$$

Now we take U^0 sufficiently small such that

$$U^0\leqslant\delta_0:=\frac{1}{4C^2}.$$

Then, we compute directly from (5.7) and the continuity of $H(t)$ that

$$H(t)\leqslant\frac{1-\sqrt{1-4C^2U^0}}{2C}\leqslant\frac{1}{2C}$$

for all $t\in[0,T^*)$, which implies $\|(\mathbf{u},\mathbf{d},P)\|_{M^{p,q}_{T^*}}$ bounded. Hence, according to the local existence in the previous section, we can extend the solution on $[0,T^*)$ to some larger interval $[0,T^*+\eta)$ with $\eta>0$. This is impossible since T^* is already the maximal time of existence. Hence, when the initial data are sufficiently small, the strong solution is indeed global in time.

The proof of Theorem 2.1 is complete.

6. Weak–strong uniqueness

The purpose of this section is to show *Weak–Strong Uniqueness* in Theorem 2.2. To this end, we first formally deduce and obtain an energy estimate for the strong solution to (1.1)–(1.3).

Lemma 6.1. *Let p,q satisfy the same conditions as Theorem* 2.1 *and $(\mathbf{u},\mathbf{d},P)\in M^{p,q}_T$ be the unique solution to* (1.1) *on $\Omega\times[0,T]$. Then for any $0<t\leqslant T$, we have*

$$\begin{aligned}&\int_\Omega\big(|\mathbf{u}(t)|^2+|\nabla\mathbf{d}(t)|^2\big)\,d\mathbf{x}+2\int_0^t\int_\Omega\big(|\nabla\mathbf{u}|^2+\big|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\big|^2\big)\,d\mathbf{x}\,ds\\&\quad=\int_\Omega\big(|\mathbf{u}_0|^2+|\nabla\mathbf{d}_0|^2\big)\,d\mathbf{x}.\end{aligned}$$

Proof. Note that

$$\mathbf{u} \in C\big([0,T]; D_{A_q}^{1-\frac{1}{p},p}\big) \cap L^p\big(0,T; W^{2,q} \cap W_0^{1,q}\big),$$
$$\mathbf{d} \in C\big([0,T]; B_{q,p}^{3(1-\frac{1}{p})}\big) \cap L^p\big(0,T; W^{3,q}\big),$$
$$D_{A_q}^{1-\frac{1}{p},p} \hookrightarrow B_{q,p}^{2(1-\frac{1}{p})} \cap X^q \quad \text{(see Proposition 2.5 in [4])},$$

where

$$X^q = \big\{\mathbf{z} \in L^q(\Omega)^3 \;\big|\; \nabla\cdot\mathbf{z} = 0 \text{ in } \Omega \text{ and } \mathbf{z}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\big\},$$

for some positive $\beta > \frac{1}{2}$, it follows from the standard interpolation inequalities that

$$\mathbf{u} \in C\big([0,T]; H^\beta\big) \cap L^2\big(0,T; H^{1+\beta}\big), \qquad \mathbf{u} \in L^4\big(\Omega \times [0,T]\big),$$
$$\mathbf{d} \in C\big([0,T]; H^{1+\beta}\big) \cap L^2\big(0,T; H^{2+\beta}\big), \qquad \nabla\mathbf{d} \in L^4\big(\Omega \times [0,T]\big).$$

This enables us to justify the following computations.

Multiplying (1.1a) by $\mathbf{u}$, integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2\,d\mathbf{x} + \int_\Omega |\nabla\mathbf{u}|^2\,d\mathbf{x} = -\int_\Omega \mathbf{u}\cdot\big((\nabla\mathbf{d})^\top\Delta\mathbf{d}\big)\,d\mathbf{x}. \tag{6.1}$$

Here we have used the facts

$$\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) = \nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right) + (\nabla\mathbf{d})^\top\Delta\mathbf{d},$$

and $\nabla\cdot\mathbf{u} = 0$ in Ω, $\mathbf{u} = 0$ on $\partial\Omega$, as well as

$$\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega \nabla P\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega \nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right)\cdot\mathbf{u}\,d\mathbf{x} = 0.$$

Multiplying (1.1b) by $-(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d})$ and integrating over Ω, we obtain

$$-\int_\Omega \frac{\partial\mathbf{d}}{\partial t}\cdot\Delta\mathbf{d}\,d\mathbf{x} - \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{d})\cdot\Delta\mathbf{d}\,d\mathbf{x} = -\int_\Omega \big|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\big|^2\,d\mathbf{x}.$$

Here we have used the fact that $|\mathbf{d}| = 1$ to get

$$\left(\frac{\partial\mathbf{d}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{d}\right)\cdot|\nabla\mathbf{d}|^2\mathbf{d} = \frac{1}{2}\left(|\nabla\mathbf{d}|^2\frac{\partial|\mathbf{d}|^2}{\partial t} + \mathbf{u}\cdot\nabla|\mathbf{d}|^2|\nabla\mathbf{d}|^2\right) = 0.$$

Since $\partial_\nu\mathbf{d} = 0$ on $\partial\Omega$, integrating by parts, we have

$$\int_\Omega \frac{\partial\mathbf{d}}{\partial t}\cdot\Delta\mathbf{d}\,d\mathbf{x} = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\mathbf{d}|^2\,d\mathbf{x}.$$

Hence we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |\nabla \mathbf{d}|^2\,d\mathbf{x} + \int_{\Omega} \left|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\right|^2 d\mathbf{x} = \int_{\Omega} (\mathbf{u}\cdot\nabla \mathbf{d})\cdot \Delta \mathbf{d}\,d\mathbf{x}. \tag{6.2}$$

By adding (6.1) and (6.2), we eventually get the identity:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\right) d\mathbf{x} + \int_{\Omega} \left(|\nabla \mathbf{u}|^2 + \left|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\right|^2\right) d\mathbf{x} = 0, \tag{6.3}$$

for all $t \in (0, T]$.

Integrating (6.3) over the time interval $[0, t]$, we obtain the energy equality of this lemma. □

We remark that (6.3) is usually called the basic energy law governing the system (1.1)–(1.3). It reflects the energy dissipation property of the flow of liquid crystals.

Now we proceed to prove *Weak–Strong Uniqueness*. In view of the regularity of $\mathbf{u}$, we deduce from the weak formulation of (1.1a) that

$$\begin{aligned} &\int_{\Omega} \tilde{\mathbf{u}}\cdot \mathbf{u}\,d\mathbf{x} + \int_0^t \int_{\Omega} \nabla \tilde{\mathbf{u}} : \nabla \mathbf{u}\,d\mathbf{x}\,ds \\ &\quad = \int_{\Omega} |\mathbf{u}_0|^2\,d\mathbf{x} - \int_0^t \int_{\Omega} (\nabla \tilde{\mathbf{d}})^\top \Delta \tilde{\mathbf{d}}\cdot \mathbf{u}\,d\mathbf{x}\,ds + \int_0^t \int_{\Omega} \tilde{\mathbf{u}}\cdot\left(\frac{\partial \mathbf{u}}{\partial s} + \tilde{\mathbf{u}}\cdot\nabla \mathbf{u}\right) d\mathbf{x}\,ds. \end{aligned} \tag{6.4}$$

On the other hand, since $\mathbf{u}$ satisfies (1.1a), i.e.,

$$\frac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u} - \mathbf{u}\cdot\nabla \mathbf{u} - \nabla P - \nabla\left(\frac{|\nabla \mathbf{d}|^2}{2}\right) - (\nabla \mathbf{d})^\top \Delta \mathbf{d},$$

then we have, from (6.4),

$$\begin{aligned} \int_{\Omega} \tilde{\mathbf{u}}\cdot \mathbf{u}\,d\mathbf{x} - \int_{\Omega} |\mathbf{u}_0|^2\,d\mathbf{x} &= -2\int_0^t \int_{\Omega} \nabla \tilde{\mathbf{u}} : \nabla \mathbf{u}\,d\mathbf{x}\,ds - \int_0^t \int_{\Omega} (\nabla \tilde{\mathbf{d}})^\top \Delta \tilde{\mathbf{d}}\cdot \mathbf{u}\,d\mathbf{x}\,ds \\ &\quad - \int_0^t \int_{\Omega} \tilde{\mathbf{u}}\cdot\left(\mathbf{u}\cdot\nabla \mathbf{u} + (\nabla \mathbf{d})^\top \Delta \mathbf{d} - \tilde{\mathbf{u}}\cdot\nabla \mathbf{u}\right) d\mathbf{x}\,ds. \end{aligned} \tag{6.5}$$

Similarly, in view of the regularity of $\mathbf{d}$, we have

$$\begin{aligned} &\int_{\Omega} \nabla \tilde{\mathbf{d}} : \nabla \mathbf{d}\,d\mathbf{x} - \int_{\Omega} |\nabla \mathbf{d}_0|^2\,d\mathbf{x} \\ &\quad = \int_0^t \int_{\Omega} \left(-\tilde{\mathbf{d}}\cdot \Delta \mathbf{d}_s + \tilde{\mathbf{u}}\cdot\nabla \tilde{\mathbf{d}}\cdot \Delta \mathbf{d} - \Delta \tilde{\mathbf{d}}\cdot \Delta \mathbf{d} - |\nabla \tilde{\mathbf{d}}|^2 \tilde{\mathbf{d}}\cdot \Delta \mathbf{d}\right) d\mathbf{x}\,ds. \end{aligned} \tag{6.6}$$

Taking advantage of (1.1b), we obtain, from (6.6),

$$\int_\Omega \nabla\tilde{\mathbf{d}} : \nabla\mathbf{d}\,d\mathbf{x} - \int_\Omega |\nabla\mathbf{d}_0|^2\,d\mathbf{x} = \int_0^t\int_\Omega \big(-2\Delta\tilde{\mathbf{d}}\cdot\Delta\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d}\cdot\Delta\tilde{\mathbf{d}} + \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{d}}\cdot\Delta\mathbf{d} - |\nabla\mathbf{d}|^2\mathbf{d}\cdot\Delta\tilde{\mathbf{d}} - |\nabla\tilde{\mathbf{d}}|^2\tilde{\mathbf{d}}\cdot\Delta\mathbf{d}\big)\,d\mathbf{x}\,ds. \tag{6.7}$$

From (2.1), (6.5), (6.7) and the fact that $(\mathbf{u},\mathbf{d},P)$ (resp. $(\tilde{\mathbf{u}},\tilde{\mathbf{d}},\Pi)$) is a strong solution (resp. weak solution) to (1.1) with the initial–boundary conditions (1.2)–(1.3), we have the following energy estimate of $(\mathbf{u}-\tilde{\mathbf{u}},\mathbf{d}-\tilde{\mathbf{d}})$:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \big(|\mathbf{u}-\tilde{\mathbf{u}}|^2 + |\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\big)\,d\mathbf{x}\\
&\quad\leqslant -\int_0^t\int_\Omega \big(|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}|^2 + |\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}|^2\big)\,d\mathbf{x}\,ds\\
&\qquad -\int_0^t\int_\Omega \big((\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla\mathbf{u}\cdot(\mathbf{u}-\tilde{\mathbf{u}}) + (\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\cdot\Delta\mathbf{d}\cdot(\mathbf{u}-\tilde{\mathbf{u}})\\
&\qquad - \mathbf{u}\cdot(\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\cdot(\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}})\\
&\qquad + \big(|\nabla\mathbf{d}|^2\mathbf{d} - |\nabla\tilde{\mathbf{d}}|^2\tilde{\mathbf{d}}\big)\cdot(\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}})\big)\,d\mathbf{x}\,ds\\
&\quad = -\int_0^t\int_\Omega \big(|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}|^2 + |\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}|^2\big)\,d\mathbf{x}\,ds + I,
\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}
I = &-\int_0^t\int_\Omega \big((\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla\mathbf{u}\cdot(\mathbf{u}-\tilde{\mathbf{u}}) + (\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\cdot\Delta\mathbf{d}\cdot(\mathbf{u}-\tilde{\mathbf{u}})\\
&- \mathbf{u}\cdot(\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\cdot(\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}})\\
&+ \big(|\nabla\mathbf{d}|^2\mathbf{d} - |\nabla\tilde{\mathbf{d}}|^2\tilde{\mathbf{d}}\big)\cdot(\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}})\big)\,d\mathbf{x}\,ds.
\end{aligned}$$

Next, we will estimate I term by term. By the zero boundary condition, we have

$$\begin{aligned}
-\int_\Omega (\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla\mathbf{u}\cdot(\mathbf{u}-\tilde{\mathbf{u}})\,d\mathbf{x} &= \int_\Omega (\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{u}}\cdot\mathbf{u}\,d\mathbf{x}\\
&= \int_\Omega (\mathbf{u}-\tilde{\mathbf{u}})\cdot(\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}})\cdot\mathbf{u}\,d\mathbf{x}\\
&\leqslant \|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}\|_{L^2}\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^2}\\
&\leqslant \frac{1}{2}\|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}\|_{L^2}^2 + \frac{\|\mathbf{u}\|_{L^\infty}^2}{2}\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^2}^2,
\end{aligned} \tag{6.9}$$

$$-\int_\Omega (\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\cdot\Delta\mathbf{d}\cdot(\mathbf{u}-\tilde{\mathbf{u}})\,d\mathbf{x}\leqslant\|\Delta\mathbf{d}\|_{L^\infty}\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}$$
$$\leqslant\frac{\|\Delta\mathbf{d}\|_{L^\infty}}{2}\big(\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}^2\big), \tag{6.10}$$

$$\int_\Omega \mathbf{u}\cdot(\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\cdot(\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}})\,d\mathbf{x}\leqslant\|\mathbf{u}\|_{L^\infty}\|\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}\|_{L^2}\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}$$
$$\leqslant\frac{1}{2}\|\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}\|_{L^2}^2+\frac{\|\mathbf{u}\|_{L^\infty}^2}{2}\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}^2, \tag{6.11}$$

$$-\int_\Omega \big(|\nabla\mathbf{d}|^2\mathbf{d}-|\nabla\tilde{\mathbf{d}}|^2\tilde{\mathbf{d}}\big)\cdot(\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}})\,d\mathbf{x}$$
$$\leqslant\|\nabla\mathbf{d}\|_{L^\infty}^2\|\mathbf{d}-\tilde{\mathbf{d}}\|_{L^2}\|\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}\|_{L^2}$$
$$+\|\nabla\mathbf{d}+\nabla\tilde{\mathbf{d}}\|_{L^\infty}\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}\|\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}\|_{L^2}$$
$$\leqslant\frac{1}{2}\|\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^\infty}^4\|\mathbf{d}-\tilde{\mathbf{d}}\|_{L^2}^2+\|\nabla\mathbf{d}+\nabla\tilde{\mathbf{d}}\|_{L^\infty}^2\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}^2. \tag{6.12}$$

Then, we eventually get from (6.9)–(6.12) that

$$I\leqslant\int_0^t\bigg(\frac{1}{2}\|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}\|_{L^2}^2+\|\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}\|_{L^2}^2+\frac{\|\mathbf{u}\|_{L^\infty}^2+\|\Delta\mathbf{d}\|_{L^\infty}}{2}\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^2}^2$$
$$+\bigg(\|\nabla\mathbf{d}+\nabla\tilde{\mathbf{d}}\|_{L^\infty}^2+\frac{\|\mathbf{u}\|_{L^\infty}^2+\|\Delta\mathbf{d}\|_{L^\infty}}{2}\bigg)\|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^\infty}^4\|\mathbf{d}-\tilde{\mathbf{d}}\|_{L^2}^2\bigg)\,ds. \tag{6.13}$$

Now, we wish to estimate $\|\mathbf{d}-\tilde{\mathbf{d}}\|_{L^2}$. We write

$$\partial_t(\mathbf{d}-\tilde{\mathbf{d}})+\mathbf{u}\cdot\nabla(\mathbf{d}-\tilde{\mathbf{d}})+(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{d}}=\Delta\mathbf{d}-\Delta\tilde{\mathbf{d}}+|\nabla\mathbf{d}|^2\mathbf{d}-|\nabla\tilde{\mathbf{d}}|^2\tilde{\mathbf{d}}. \tag{6.14}$$

Multiply (6.14) by $\mathbf{d}-\tilde{\mathbf{d}}$ and integrate over $\Omega\times(0,t)$, we have

$$\frac{1}{2}\int_\Omega|\mathbf{d}-\tilde{\mathbf{d}}|^2\,d\mathbf{x}$$
$$=-\int_0^t\int_\Omega(\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla\tilde{\mathbf{d}}\cdot(\mathbf{d}-\tilde{\mathbf{d}})\,d\mathbf{x}\,ds-\int_0^t\int_\Omega|\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\,d\mathbf{x}\,ds$$
$$+\int_0^t\int_\Omega|\nabla\mathbf{d}|^2|\mathbf{d}-\tilde{\mathbf{d}}|^2\,d\mathbf{x}\,ds+\int_0^t\int_\Omega(\nabla\mathbf{d}+\nabla\tilde{\mathbf{d}}):(\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}})\,\tilde{\mathbf{d}}\cdot(\mathbf{d}-\tilde{\mathbf{d}})\,d\mathbf{x}\,ds.$$

Using Sobolev's inequality $\|\mathbf{u}-\tilde{\mathbf{u}}\|_{L^6}\leqslant C\|\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}\|_{L^2}$ and for some $\varepsilon>0$ small enough, it is easy to get

$$\frac{1}{2}\int_\Omega |\mathbf{d}-\tilde{\mathbf{d}}|^2\,d\mathbf{x} \leqslant \int_0^t \left(C_\varepsilon\|\nabla\tilde{\mathbf{d}}\|_{L^3}^2 + \|\nabla\mathbf{d}\|_{L^\infty}^2 + \frac{1}{2}\right)\int_\Omega |\mathbf{d}-\tilde{\mathbf{d}}|^2\,d\mathbf{x}\,ds + \varepsilon\int_0^t\int_\Omega |\nabla\mathbf{u}-\nabla\tilde{\mathbf{u}}|^2\ d\mathbf{x}\,ds$$
$$+\int_0^t \left(1+\frac{\|\nabla\mathbf{d}+\nabla\tilde{\mathbf{d}}\|_{L^\infty}^2}{2}\right)\int_\Omega |\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\,d\mathbf{x}\,ds. \tag{6.15}$$

Now we have from (6.8), (6.13) and (6.15) that

$$\frac{1}{2}\int_\Omega \left(|\mathbf{u}-\tilde{\mathbf{u}}|^2 + |\mathbf{d}-\tilde{\mathbf{d}}|^2 + |\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\right)d\mathbf{x}$$
$$\leqslant \int_0^t \frac{\|\mathbf{u}\|_{L^\infty}^2 + \|\Delta\mathbf{d}\|_{L^\infty}}{2}\int_\Omega |\mathbf{u}-\tilde{\mathbf{u}}|^2\,d\mathbf{x}\,ds$$
$$+\int_0^t \left(\frac{1}{2} + \|\nabla\mathbf{d}\|_{L^\infty}^4 + C\|\nabla\tilde{\mathbf{d}}\|_{L^3}^2 + \|\nabla\mathbf{d}\|_{L^\infty}^2\right)\int_\Omega |\mathbf{d}-\tilde{\mathbf{d}}|^2\,d\mathbf{x}\,ds$$
$$+\int_0^t \left(1 + \frac{\|\mathbf{u}\|_{L^\infty}^2 + \|\Delta\mathbf{d}\|_{L^\infty}}{2} + \frac{3\|\nabla\mathbf{d}+\nabla\tilde{\mathbf{d}}\|_{L^\infty}^2}{2}\right)\int_\Omega |\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\,d\mathbf{x}\,ds$$
$$\leqslant C\int_0^t \varphi(s)\int_\Omega \left(|\mathbf{u}-\tilde{\mathbf{u}}|^2 + |\mathbf{d}-\tilde{\mathbf{d}}|^2 + |\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\right)d\mathbf{x}\,ds, \tag{6.16}$$

where

$$\varphi(s) = 1 + \|\nabla\mathbf{d}\|_{L^\infty}^4 + \|\nabla\mathbf{d}\|_{L^\infty}^2 + \|\nabla\tilde{\mathbf{d}}\|_{L^\infty}^2 + \|\mathbf{u}\|_{L^\infty}^2 + \|\Delta\mathbf{d}\|_{L^\infty}.$$

Notice that $\|\nabla\mathbf{d}\|_{L^\infty}^2, \|\mathbf{u}\|_{L^\infty}^2, \|\Delta\mathbf{d}\|_{L^\infty} \in L^1(0,t)$. Moreover, by applying the quasi-linear equations of parabolic type estimates (cf. [12], Chapter VI, Section 2) to (1.1b), we see $\mathbf{d}(\cdot,t)$, $\tilde{\mathbf{d}}(\cdot,t) \in C^{1,\alpha}$ with respect to the space variables, for some $\alpha > 0$, and its $C^{1,\alpha}$ norm is independent of t. Then we have $\varphi(s) \in L^1(0,t)$. Applying Grönwall's inequality to (6.16), we obtain

$$\int_\Omega \left(|\mathbf{u}-\tilde{\mathbf{u}}|^2 + |\mathbf{d}-\tilde{\mathbf{d}}|^2 + |\nabla\mathbf{d}-\nabla\tilde{\mathbf{d}}|^2\right)d\mathbf{x} = 0$$

for all t. Thus, $\mathbf{u} = \tilde{\mathbf{u}}$, $\mathbf{d} = \tilde{\mathbf{d}}$ *a.e.* and $P = \Pi$ up to a constant in $\Omega \times (0,T)$.

The proof of Theorem 2.2 is now complete.

Acknowledgments

X. Li's research was supported in part by a joint project from the NSAF of China (China Scholarship Council), and by Doctoral Innovation Fund of Tsinghua University. D. Wang's research was supported in part by the National Science Foundation under Grant DMS-0906160 and by the Office of Naval Research under Grant N00014-07-1-0668.

References

[1] H. Amann, Linear and Quasilinear Parabolic Problems, vol. I. Abstract Linear Theory, Birkhäuser Boston, Inc., Boston, 1995.
[2] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Grundlehren Math. Wiss., Springer-Verlag, Berlin, New York, 1976.
[3] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of Navier–Stokes equations, Comm. Pure Appl. Math. 35 (1982) 771–831.
[4] R. Danchin, Density-dependent incompressible fluids in bounded domains, J. Math. Fluid Mech. 8 (2006) 333–381.
[5] J.L. Ericksen, Hydrostatic theory of liquid crystal, Arch. Ration. Mech. Anal. 9 (1962) 371–378.
[6] B. Desjardins, Regularity of weak solutions of the compressible isentropic Navier–Stokes equations, Comm. Partial Differential Equations 22 (1997) 977–1008.
[7] J.L. Ericksen, Conservation laws for liquid crystals, Trans. Soc. Rheology 5 (1961) 23–34.
[8] J.L. Ericksen, Continuum theory of nematic liquid crystals, Res. Mechanica 21 (1987) 381–392.
[9] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vol. I. Linearized Steady Problems, Springer-Verlag, New York, 1994.
[10] R. Hardt, D. Kinderlehrer, Mathematical Questions of Liquid Crystal Theory, IMA Vol. Math. Appl., vol. 5, Springer-Verlag, New York, 1987.
[11] R. Hardt, D. Kinderlehrer, F. Lin, Existence and partial regularity of static liquid crystal configurations, Comm. Math. Phys. 105 (1986) 547–570.
[12] O.A. Ladyzhenskaya, N.A. Solonnikov, N.N. Uraltseva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr., vol. 23, American Mathematical Society, 1968.
[13] F.M. Leslie, Some constitutive equations for liquid crystals, Arch. Ration. Mech. Anal. 28 (1968) 265–283.
[14] F.H. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, Comm. Pure Appl. Math. 42 (1989) 789–814.
[15] F.H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Comm. Pure Appl. Math. 48 (1995) 501–537.
[16] F.H. Lin, C. Liu, Partial regularity of the dynamic system modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst. 2 (1996) 1–22.
[17] F.H. Lin, J. Lin, C. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal. 197 (2010) 297–336.
[18] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1. Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 3, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.
[19] C. Liu, J. Shen, On liquid crystal flows with free-slip boundary conditions, Discrete Contin. Dyn. Syst. 7 (2001) 307–318.
[20] C. Liu, N.J. Walkington, Approximation of liquid crystal flow, SIAM J. Numer. Anal. 37 (2000) 725–741.
[21] H. Sun, C. Liu, On energetic variational approaches in modeling the nematic liquid crystal flows, Discrete Contin. Dyn. Syst. 23 (2009) 455–475.
[22] X. Xu, L.Y. Zhao, C. Liu, Axisymmetric solutions to coupled Navier–Stokes/Allen–Cahn equations, SIAM J. Math. Anal. 41 (6) (2010) 2246–2282.