Total balancedness condition for Steiner tree games

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Abstract

In Steiner tree game associated with a graph \( G = (V, E) \), players consist of a subset \( N \subseteq V \) of nodes. The characteristic function value of a subset \( S \subseteq N \) of the players is the minimum weight of a Steiner tree that spans \( S \). We show that it is NP-hard to determine whether a Steiner tree game is totally balanced, i.e., cores for all its subgames are non-empty. In addition, the NP-hardness result is also proven for deciding whether the core is non-empty, or whether an imputation is a member of the core.

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1. Introduction

A cooperative game \( \Gamma = (N, c) \) consists of a player set \( N = \{1, 2, \ldots, n\} \) and a characteristic function \( c: 2^N \rightarrow R \), where, for each subset \( S \) of \( N \), \( c(S) \) represents the cost incurred by the coalition of players in \( S \) without participation of other players. The main issue is how to fairly distribute the total cost \( c(N) \) among all the players. A vector \( x = \{x_1, x_2, \ldots, x_n\} \) is called an imputation if and only if \( \sum_{i \in N} x_i = v(N) \).

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The core is defined by

\[
\text{Core}(\Gamma) = \{ x \in \mathbb{R}^n : x(N) = v(N) \text{ and } x(S) \leq v(S), \quad \forall S \subseteq N \},
\]

where \( x(S) = \sum_{i \in S} x_i \).

The study of the core is closely associated with another important concept, the balanced set. The collection \( \mathcal{B} \) of subsets \( S \) of \( N \) is balanced if there exists a set of positive numbers \( \gamma_S, S \in \mathcal{B} \), such that for each \( i \in N \), we have \( \sum_{i \in S} \gamma_S = 1 \). A game \((N,v)\) is called balanced if \( \sum_{S \in \mathcal{B}} \gamma_S v(S) \geq v(N) \) holds for every balanced collection \( \mathcal{B} \) with weights \( \{\gamma_S : S \in \mathcal{B}\} \). Bondareva [2] and Shapley [16] proved that a game has a non-empty core if and only if it is balanced.

For a subset \( S \subseteq N \), we define the induced subgame \((S,v_S)\) on \( S \): \( v_S(T) = v(T) \) for every subset \( T \) of \( S \). A cooperative game \( \Gamma \) is called totally balanced if all its subgames are balanced, i.e., all its subgames have non-empty cores. This concept was introduced by Shapley and Shubik [17] in the study of market games. They showed that the collection of market games are the same as the collection of totally balanced games. Several other classes of cooperative games were also shown to be equivalent to totally balanced games, such as, the maximum flow game introduced by Kalai and Zemel [13] and the linear production game discussed by Owen [15] and Curiel [3]. Recently, Deng et al. [6] considered total balancedness of some interesting combinatorial optimization games. They presented a complete characterization for a class of partition games to be totally balanced, and the relationship between some totally balanced games and their related combinatorial structures. Most were shown to be polynomially decidable. The worst complexity for totally balanced conditions is obtained for a coloring game, for which total balancedness was shown to be equivalent to a graph being perfect. The total balancedness condition is a very strong requirement (core being non-empty for all subgames), and it was challenged to find an example for which no polynomial time algorithm is known to decide total balancedness [4]. In this work we show this is true for Steiner tree games if NP is not the same as \( P \).

Steiner tree games were introduced by Megiddo [14]. Suppose that there is a central supplier offering service to a set of consumers through a given network. It is required to connect all the consumers to the central supplier. The connection is not limited to use direct links between two consumers or a consumer and the central supplier, it may pass through additional nodes, called switches, in the network. We wish to construct a cheapest connection and distribute the connection cost among the consumers fairly. Let \( G = (V,E;\omega) \) be a weighted graph with \( V = \{v_0\} \cup N \cup M \), where \( N,M \subseteq V \setminus \{v_0\} \) are disjoint. \( v_0 \) represents the central supplier, \( N \) represents the consumer set and \( M \) represents the switch set. The weight \( \omega(e) \) denotes the cost of connecting the two endpoints of edge \( e \) directly. Steiner tree game \( \Gamma(G) \equiv (N,v) \) is defined as follows:

1. The player set is \( N \);
2. For each coalition \( S \subseteq N \), \( v(S) \) is the weight of minimal Steiner tree on \( G \) w.r.t. the set \( S \cup \{v_0\} \), that is,

\[
v(S) = \min \{ \omega(T_S) : T_S = (V_S,E_S) \text{ is a subtree of } G \text{ such that } S \cup \{v_0\} \subseteq V_S \},
\]
where $\omega(T_S) = \sum_{e \in E_S} \omega(e)$. In a subgame $\Gamma(G_S) \equiv (S, v)$ $(S \subseteq N)$, for each coalition $S' \subseteq S$, $v(S')$ is the weight of minimum Steiner tree of $G$ w.r.t. the subset $S' \cup \{v_0\}$, where all vertices in $N \setminus S$ are treated as switches but not consumers.

There are two other related game theoretical formulations for this practical problem: minimum cost spanning tree (MCST) game and monotone MCST game. Let $G = (V, E; \omega)$ be a weighted graph with $V = N \cup \{v_0\}$. In MCST game $\Gamma_G \equiv (N, u)$, for each coalition $S \subseteq N$, $u(S) = \min\{\omega(T_S) : T_S = (V_S, E_S)$ is a subtree of $G$ with $V_S = S \cup \{v_0\}\}$. In monotone MCST game $\Gamma'_G \equiv (N, u')$, for each coalition $S \subseteq N$, $u'(S) = \min\{\omega(T_S) : T_S = (V_S, E_S)$ is a subtree of $G$ with $V_S \supseteq S \cup \{v_0\}\}$. The class of monotone MCST games is a subclass of Steiner tree games only under the restriction that the set of switches is empty. Bird [1], and independently, Granot and Huberman [11] presented a proof for the non-emptiness of the core for both MCST game and monotone MCST game: Find a minimum cost spanning tree $T$ of $G$ and allocate to player $i \in N$ the weight of the first edge that $i$ encounters on the path from $i$ to $v_0$ in $T$. Different from these two games, the core of Steiner tree game may in general be empty. An example with empty core was given in Megiddo [14]. On the other hand, Granot and Maschler [12] showed that, if there exist an optimal spanning tree $T_G$, which spans the nodes of all consumers and all switches, such that no two switches are adjacent in $T_G$, then the core of the corresponding Steiner tree game is not empty.

The computational complexity as a rationality measure for game theoretical concepts has attracted more and more attention recently. Various interesting complexity structures have started to emerge as a result, especially for the study of the core. Deng and Papadimitriou [7] found a game for which the core is non-empty if and only if a certain imputation (Shapley value in this case) is in the core. For the MCST game, the core is always non-empty and a member in the core can be found in polynomial time [11]; however, Faigle, et al. [8] show that membership testing is co-NP-complete. Deng et al. [5] discussed the complexity of the core for a class of combinatorial optimization games. Goemans and Skutella [10] recently showed that, for a facility location game, if the core is non-empty, a solution in the core can be found in polynomial time, and membership testing can be done in polynomial time. However, it is NP-complete to decide if a core is not empty.

In Section 2, we show it is NP-hard to decide whether a Steiner tree game is totally balanced, it is the first example of NP-hardness for the totally balanced condition. In Section 3, we prove that, given an imputation of a Steiner tree game, checking if it belongs to the core is also NP-hard.

2. Complexity of testing total balancedness

In this section, we prove that the problem of testing total balancedness of the Steiner tree game is NP-hard. And because of the specific reduction we constructed, the result also holds for the problem of testing non-emptiness of the core. In our proof, we will use 3-PARTITION, which is shown to be NP-complete in strong sense in the book of Garey and Johnson [9].
3-PARTITION is defined as follows:

\textbf{Instance}: A finite set \( A = \{a_1, a_2, \ldots, a_{3m}\} \), a bound \( B \in \mathbb{Z}^+ \), and a size \( s(a) \in \mathbb{Z}^+ \) for each \( a \in A \), such that

\[
\frac{B}{4} < s(a) < \frac{B}{2} \quad \forall a \in A,
\]

\[
\sum_{a \in A} s(a) = mb.
\]

\textbf{Question}: Can \( A \) be partitioned into \( m \) disjoint sets \( S_1, S_2, \ldots, S_m \), such that for \( 1 \leq i \leq m \)

\[
\sum_{a \in S_i} s(a) = B?
\]

\textbf{Theorem 2.1}. Testing total balancedness for Steiner tree games is NP-hard.

\textbf{Proof}. Given any instance of 3-PARTITION, we let \( \mathcal{R} = \{R : R \text{ is a 3-set of } A\} \), \( r = |\mathcal{R}| = \binom{3m}{3} \), and \( \mathcal{Q} = \{Q : Q \text{ is a } (3m - 1) \text{-set of } A\} \), \( q = |\mathcal{Q}| = 3m \).

1. First we construct a weighted graph \( G = (V, E; \omega) \).
   - The node set \( V = \{v_0\} \cup N \cup M \) is given as follows:
     - \( \{v_0\} \) is corresponding to the supplier;
     - \( N = A \) is corresponding to the set of consumers;
     - \( M = \mathcal{R} \cup \mathcal{Q} \cup \{u_0\} \) is corresponding to the set of switches.
   - The edge set \( E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \), and the edge weight function \( \omega: E \rightarrow \mathbb{R}^+ \) are defined as follows:
     - \( E_1 = \{(a, R) : a \in A, R \in \mathcal{R}\}, \forall e \in E_1, \omega(e) = 6mb; \)
     - \( E_2 = \{(a, Q) : a \in A, Q \in \mathcal{Q}\}, \forall e \in E_2, \omega(e) = 6mb + 5B/3; \)
     - \( E_3 = \{(R, u_0) : R \in \mathcal{R}\}, \forall e = (R, u_0) \in E_3, \omega(e) = \begin{cases} B & \text{if } \sum_{a \in R} s(a) \leq B, \\ \frac{3B}{2} & \text{if } \sum_{a \in R} s(a) > B, \end{cases} \)
     - \( E_4 = \{(Q, v_0) : Q \in \mathcal{Q}\}, \forall e \in E_4, \omega(e) = mb - B/3; \)
     - \( E_5 = \{(u_0, v_0)\}, \omega(u_0, v_0) = 5mb. \)

   Denote the corresponding Steiner tree game by \( I(G) \equiv (N, v) \). For convenience of comprehension, we give a sketch of the graph \( G = (V, E; \omega) \) in Fig. 1. Notice that

1. The weight of each edge \( e \in E_1 \cup E_2 \) is sufficiently large so that in any minimum Steiner tree of \( G \) w.r.t. \( S \cup \{v_0\} (S \subseteq N) \), there are exactly \( |S| \) edges in \( E_1 \cup E_2 \).

2. The construction of edge \( (u_0, v_0) \) is to ensure that for any proper subset of \( N \), the vertices in \( \mathcal{R} \) are not needed in the corresponding minimum Steiner tree.
(II) We show that the cores of all proper subgames of \( \Gamma(G) \) are non-empty. Let \( S \) be any proper subset of \( N \), \( |S| \leq 3m - 1 \). Assume that \( T_S = (V_S, E_S) \) is the minimum Steiner tree of \( G \) w.r.t. \( S \cup \{v_0\} \), then it must be the case

\[
V_S = \{v_0\} \cup S \cup Q,
\]

where the set \( Q \in \mathcal{Q} \) is an arbitrary set containing \( S \). That is, \( E_S \) consists of one edge in \( E_4 \) and \( |S| \) edges in \( E_2 \),

\[
v(S) = 6mB|S| + \frac{5}{2}B|S| + mB - \frac{1}{2}B = v^*_S.
\]

Let

\[
x(a) = \frac{v^*_S}{|S|} = 6mB + \frac{5B}{3} + \frac{(3m - 1)B}{3|S|} \quad \forall a \in S.
\]

It is easy to verify that it is an element in the core of \( \Gamma(G_S) \).

(III) We prove that the game \( \Gamma(G) \) has non-empty core if and only if \( A \) has a 3-partition.

Assume that \( T = (V_N, E_N) \) is the minimum Steiner tree of \( G \) w.r.t. \( N \cup \{v_0\} \), then we have

\[
V_N = \{v_0\} \cup N \cup \{R_1, \ldots, R_m\},
\]

where \( R_1, \ldots, R_m \in \mathcal{R} \) and \( \bigcup_{i=1}^m R_i = A \). That is, \( E_N \) consists of the edge \((u_0, v_0)\), \( m \) edges in \( E_3 \) and \( 3m \) edges in \( E_1 \), and

\[
v(N) \leq 6mB \cdot 3m + m\frac{3B}{2} + 5mB = 18m^2B + \frac{13}{2}mB = v^*_N.
\]

In fact, it is trivial for \( m = 1 \), and when \( m > 1 \) we have

(a) If \( E_N \) does not contain any edges in \( E_1 \cup E_3 \cup E_5 \), then it must consist of \( 3m \) edges in \( E_2 \) and two edges in \( E_4 \), and \( v(N) = 2(mB - B/3) + 3m(6mB + 5B/3) > v^*_N \), a contradiction.
(b) If \( E_N \) contains \( k \) \((0 < k < 3m)\) edges in \( E_2 \), then it must contain one edge in \( E_4 \), and \( v(N) > \left( mB - B/3 \right) + k(6mB + 5B/3) + 5mB + (3m - k)6mB + |(3m - k)/3|B > v_N^* \), also a contradiction.

If \( A \) has a 3-PARTITION, then

\[
v(N) = 6mB \cdot 3m + mB + 5mB = 18m^2B + 6mB.
\]

Let \( x(a) = 6mB + 2B, \forall a \in N \). By formula (2.1), it is easy to verify that it is a core member of \( \Gamma(G) \).

If \( A \) has no 3-partition, suppose the core of \( \Gamma(G) \) is not empty and \( x \in \text{Core}(\Gamma(G)) \).

Since \( x(N) = v(N) > 18m^2B + 6mB \), there must be a \((3m - 1)\)-element subset \( Q^* \) of \( N \) such that

\[
x(Q^*) = \frac{3m - 1}{3m}(18m^2B + 6mB) = (3m - 1)(6mB + 2B).
\]

By formula (2.1), we have

\[
v(Q^*) = 6mB(3m - 1) + \frac{5}{3}B(3m - 1) + mB - \frac{B}{3} = (3m - 1)(6mB + 2B) < x(Q^*),
\]

which is contrary to our hypothesis that \( x \in \text{Core}(\Gamma(G)) \).

Therefore, the \( \Gamma(G) \) is totally balanced if and only if \( A \) has 3-PARTITION. Also the construction of \( \Gamma(G) \) can be carried out in polynomial time, so testing total balancedness of Steiner tree games is NP-hard.

From the proof of Theorem 2.1, we can get the complexity on testing balancedness obviously.

**Corollary 2.2.** Testing balancedness for Steiner tree games is NP-hard.

### 3. Complexity of checking membership

Faigle et al. [8] proved that the problem of checking membership in the core for MCST games is co-NP-complete. Much in the spirit of their proof, we will show that checking membership in the core for Steiner tree games is co-NP-hard, even for a balanced game.

**Theorem 3.1.** The following problem is NP-hard: Given a balanced Steiner tree game \( \Gamma(G) = (N, v) \) on weighted graph \( G = (V, E; \omega) \) and a vector \( x \in \mathbb{R}^n (n = |N|) \) with \( x(N) = v(N) \), does there exist a coalition \( S \subseteq N \) such that \( x(S) > v(S) \)?

**Proof.** Given an arbitrary instance of EXACT COVER BY 3-SETS (X3C): a finite set \( X = \{x_1, x_2, \ldots, x_q\} \) and a collection \( F = \{f_1, f_2, \ldots, f_{|F|}\} \) of 3-element subsets of \( X \) \((q \geq 2, |F| \geq 2q)\). We first construct a Steiner tree game \( \Gamma(G) \) on weighted graph \( G = (V, E; \omega) \) and a candidate vector \( x \) for a core member.
The vertex set $V$ of $G$ consists of four parts:
- $N_X = \{v_1, v_2, \ldots, v_{3q}\}$, the vertex $v_i \in N_X$ is corresponding to the element $x_i \in X$, $i = 1, 2, \ldots, 3q$, $N_X$ represents a subset of consumers;
- $N_F = \{u_1, u_2, \ldots, u_{|F|}\}$, the vertex $u_j \in N_F$ is corresponding to the subset $f_j \in F$, $j = 1, 2, \ldots, |F|$, $N_F$ also represents a subset of consumers;
- $g_1$ and $g_2$ are two additional vertices, they represent switches;
- $v_0$ represents central supplier.

Thus, $V = N_X \cup N_F \cup \{g_1, g_2, v_0\}$. Let $N = N_X \cup N_F$, $n = |N| = 3q + |F|$ (See Fig. 2).

The edge set $E$ of $G$ and edge weight function $\omega : E \to \mathbb{R}^+$ are defined as follows:
- $a_0 = (v_0, g_1)$, $\omega(a_0) = 2q - 1$;
- $b_0 = (g_1, g_2)$, $\omega(b_0) = q + 1$;
- $E_1 = \{(g_1, u_i) : i = 1, 2, \ldots, |F|\}$, $\forall e \in E_1$: $\omega(e) = q + 1$;
- $E_2 = \{(g_2, u_i) : i = 1, 2, \ldots, |F|\}$, $\forall e \in E_2$: $\omega(e) = q$;
- $E_3 = \{(u_i, v_j) : x_j \in f_i, i = 1, 2, \ldots, |F|, j = 1, 2, \ldots, 3q\}$, $\forall e \in E_3$: $\omega(e) = 2q + 1$.

The edge set $E = \{a_0, b_0\} \cup E_1 \cup E_2 \cup E_3$ (See Fig. 2). Denote the Steiner tree game on the weighted graph $G$ by $\Gamma(G) = (N, v)$.

Our construction of graph $G$ is much similar to that in Faigle et al. [8]. The only difference is the weights of edges in $E_3$. The weights of edges in $E_3$ are sufficiently large so that each vertex in $N_X$ does not serve as a switch in the minimum Steiner tree of $G$. For any coalition $S \subseteq N$, denote the minimum Steiner tree w.r.t. $S \cup \{v_0\}$ by $T_S = (V_S, E_S)$. We have

1. If $|S \cap N_F| < q + 1$, then $E_S \cap E_2 = \emptyset$, and
   \[ v(S) = \omega(E_S) = 2q - 1 + (q + 1)|S \cap N_F| + (2q + 1)|S \cap N_X|. \quad (3.1) \]
2. If $|S \cap N_F| > q + 1$, then $E_S \cap E_1 = \emptyset$, and
   \[ v(S) = \omega(E_S) = 2q - 1 + (q + 1) + q|S \cap N_F| + (2q + 1)|S \cap N_X|. \quad (3.2) \]
(3) If $|S \cap \mathcal{F}| = q + 1$, then either $E_S \cap E_1 = \emptyset$ or $E_S \cap E_2 = \emptyset$, and

$$v(S) = o(E_S) = (2q - 1) + (q + 1)^2 + (2q + 1)|S \cap \mathcal{X}|.$$  

Therefore, for the grand coalition $\mathcal{N}$, $v(\mathcal{N}) = 6q(q + 1) + q|\mathcal{F}|$. Let

$$y(u_i) = q + \frac{3q}{3q + |\mathcal{F}|} \quad \forall i = 1, 2, \ldots, |\mathcal{F}|,$$

$$y(v_j) = 2q + 1 + \frac{3q}{3q + |\mathcal{F}|} \quad \forall j = 1, 2, \ldots, 3q,$$

it is easy to verify that $y \in \text{Core}(\Gamma(G))$, this game is balanced.

Define $x \in \mathbb{R}^n$ be a candidate for a core member

$$x(u_i) = q \quad \forall i = 1, 2, \ldots, |\mathcal{F}|,$$

$$x(v_j) = 2q + 2 \quad \forall j = 1, 2, \ldots, 3q.$$

We shall claim that $x$ is not in $\text{Core}(\Gamma(G))$ if and only if $\mathcal{F}$ contains an exact cover for $\mathcal{X}$.

Assume $x$ is not in $\text{Core}(\Gamma(G))$, and let $S \subseteq \mathcal{N}$ be a coalition satisfying $x(S) > v(S)$. Then $S$ must possess the following properties:

1. $|S \cap \mathcal{F}| = q$. If $|S \cap \mathcal{F}| \geq q + 1$, then by formulas (3.2) and (3.3) we have $0 < x(S) - v(S) \leq |S \cap \mathcal{X}| - 3q$, which is contrary to the fact that $|S \cap \mathcal{X}| \leq 3q$.

2. $|S \cap \mathcal{X}| = 3q$. Suppose that $|S \cap \mathcal{X}| = k$, by the analysis of (2) and formula (3.1), we have $0 < x(S) - v(S) = k - 3q + 1$, it implies that $k = 3q$.

Therefore, the assumption of $x \in \text{Core}(\Gamma(G))$ implies $\mathcal{F}$ contains an exact cover of $\mathcal{X}$.

On the other hand, if $\mathcal{F}$ admits an exact 3-cover $\mathcal{F}' = \{f_{i_1}, f_{i_2}, \ldots, f_{i_q}\}$, let $S = \{u_{i_k}: k = 1, 2, \ldots, q\} \cup \mathcal{X}$, then $v(S) = 7q^2 + 6q - 1 < 7q^2 + 6q = x(S)$, which implies that $x$ is not in $\text{Core}(\Gamma(G))$.

Our construction of game $\Gamma(G)$ and the candidate core member $x$ can be carried out in polynomial time, the proof is finished. □

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