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Pseudo-chordal mixed hypergraphs [1]

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Abstract

A mixed hypergraph contains two families of subsets: edges and co-edges. In every coloring any edge has at least two vertices of different colors, any co-edge has at least two vertices of the same color. The minimum (maximum) number of colors for which there exists a coloring of a mixed hypergraph H using all the colors is called lower (upper) chromatic number. A mixed hypergraph is called uniquely colorable if it has exactly one coloring apart from the permutation of colors. A vertex is called simplicial if its neighborhood induces a uniquely colorable mixed hypergraph. A mixed hypergraph is called pseudo-chordal if it can be decomposed by consecutive elimination of simplicial vertices. The main result of this paper is to provide a necessary and sufficient condition for a polynomial to be a chromatic polynomial of a pseudo-chordal mixed hypergraph. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

In the classical theory of coloring for hypergraphs [2,4], we ask for the coloring of vertices so that each edge requires at least two vertices in different colors, and ask for the minimum number of colors required. It is natural to ask the dual question to color vertices so that each edge admits at least two vertices in the same color, and ask for the maximum number of colors that can occur [1,5,10]. It is even more natural to ask the combination of the above two questions [8,10–12].

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In the present paper we deal with such a combination of constraints on colorings and use the terminology of [12]. The paper is organized as follows. The rest of the current section contains necessary definitions. Section 2 presents known or easy to prove results which are used in Section 3. The latter describes the main result and contains some corollaries and example as well.

Let X be a finite set, $\mathbf{S}$ a family of subsets of X. The pair $H=(X,\mathbf{S})$ is called a hypergraph on X. For any subset $Y \subseteq X$ we call the hypergraph $H/Y=(Y,\mathbf{S}')$ the *induced subhypergraph* of the hypergraph H if $\mathbf{S}'$ consists of all those subsets which are elements of $\mathbf{S}$, and which are contained in Y.

Let $H=(X,\mathbf{S})$, and $\mathbf{S}=\mathscr{A}\cup\mathscr{E}$, where both $\mathscr{A}$ and $\mathscr{E}$ are subfamilies of $\mathbf{S}$, in particular, $\mathscr{E}$ and/or $\mathscr{A}$ may be empty. We call every $E\in\mathscr{E}$ an 'edge', and every $A\in\mathscr{A}$ a 'co-edge'. In general we call $H=(X,\mathscr{A}\cup\mathscr{E})$ a *mixed hypergraph.*

We use the prefix 'co-' when a statement concerns the sets from $\mathscr{A}$.

A mixed hypergraph $H=(X,\mathscr{A}\cup\mathscr{E})$ is called connected if for any pair of vertices $x,y\in X$ there exists a sequence $xS_0z_1S_1z_2\dots z_tS_ty$ of alternating vertices and (co)-edges $xS_0z_1S_1z_2\dots z_tS_ty$ such that $z_1,z_2\dots z_t\in X$, $S_0,S_1,\dots S_t\in\mathscr{A}\cup\mathscr{E}$, $x\in S_0, y\in S_t$, and $z_i\in S_{i-1}\cap S_i$ for $i=1,2,\dots,t$. Throughout the paper we consider connected mixed hypergraphs.

Definition 1 (Tuza et al. [8]). In a connected mixed hypergraph $H=(X,\mathscr{A}\cup\mathscr{E})$ the nonempty subset $X_0\subset X$ (or the subhypergraph H/X_0) is called a *separator* if there exist two nonempty subsets X_1 and X_2 such that the following conditions hold:

(1) $X_1\cup X_2=X\backslash X_0$ and $X_1\cap X_2=\emptyset$;

(2) for any $A\in\mathscr{A}$ either $A\subseteq X_1\cup X_0$ or $A\subseteq X_0\cup X_2$; and

(3) for any $E\in\mathscr{E}$ either $E\subseteq X_1\cup X_0$ or $E\subseteq X_0\cup X_2$.

Induced subhypergraphs $H/(X_1\cup X_0)$ and $H/(X_2\cup X_0)$ are called *derived* subhypergraphs (with respect to the separator X_0).

Definition 2 (Voloshin [12]). A free coloring of a mixed hypergraph $H=(X,\mathscr{A}\cup\mathscr{E})$ with λ colors is a mapping $c:X\to\{1,2,\dots,\lambda\}$ such that the following two conditions hold:

(1) any co-edge $A\in\mathscr{A}$ has at least two vertices of the same color;

(2) any edge $E\in\mathscr{E}$ has at least two vertices colored differently.

Moreover, c is called a strict coloring if c is onto.

A coloring using λ colors (no matter if it is free or strict) is sometimes called a λ-coloring. Two free colorings of a hypergraph H are said to be different if there exists at least one vertex that changes color when passing from one coloring to the other. Let $P(H,\lambda)$ $(\lambda\geqslant 0)$ be the chromatic polynomial of a hypergraph H, which gives the number of different free colorings of H with λ colors.

Definition 3 (Voloshin [12]). The maximum (minimum) i such that there exists a strict coloring of a mixed hypergraph H with i colors is called the upper (lower) chromatic number of H and is denoted by $\bar{\chi}(H)(\chi(H))$.

Two strict colorings of H are said to be different if there exist two vertices of H that have the same color for one of these colorings and different colors for the other. Notice the difference between the definitions of different coloring for free coloring and strict coloring. It is very important in describing their properties and formulas. Let $r_i = r_i(H)$ be the number of strict colorings of a hypergraph H with $i \geqslant 1$ colors. Call the vector $R(H) = (r_1, r_2, \ldots, r_n) \in \mathbf{Z}^n$ chromatic spectrum of H, hence $R(H) = (0, \ldots, 0, r_\chi, \ldots, r_{\bar{\chi}}, 0, \ldots, 0)$.

A mixed hypergraph H for which there exists no coloring is called uncolorable, in this case $R(H) = (0, 0, \ldots, 0)$. Throughout this paper we consider colorable mixed hypergraphs, i.e. there exists at least one coloring.

Definition 4 (Tuza et al. [8]). A mixed hypergraph H is called uniquely colorable (uc-mix-hypergraph) if there exists exactly one strict coloring.

Denote the class of all uc-mix-hypergraphs by $\mathscr{UC}$. For uc-mix-hypergraph H, $\chi(H) = \bar{\chi}(H) = \chi$, $r_\chi(H) = r_{\bar{\chi}}(H) = 1$, and $P(H, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - \chi + 1) = \lambda^{(\chi)}$.

Now let us consider the following situation, described in [8].

Let $H = (X, \mathscr{A} \cup \mathscr{E})$. For $x \in X$, let $\mathscr{A}_x$ be the subfamily of all co-edges from $\mathscr{A}$ containing x, and let $\mathscr{E}_x$ be the subfamily of all edges from $\mathscr{E}$ containing x. Assume that c is a strict t-coloring of the induced sub-mix-hypergraph H^* induced by the vertex set $X \backslash \{x\}$.

The co-edge $A \in \mathscr{A}_x$ is called *influencing* with respect to the coloring c of H^* if all its vertices except x are colored with different colors in the coloring c of H^*. Analogously, the edge $E \in \mathscr{E}_x$ is called *influencing* with respect to the coloring c of H^* if all its vertices except x are colored with the same color in the coloring c of H^*. Influencing co-edges and edges define all the possibilities in extending the coloring c of H^* to the vertex x.

Let $c(A)$ ($c(E)$) be the set of colors used by the vertices in the co-edge A (edge E) in the coloring c of H^*. Let $FS(x) = \cap \{c(A) : A \in \mathscr{A}_x, A$ is an influencing co-edge$\}$. This means that $FS(x)$ is the set of colors one of which must be used by x in extending the coloring c of H^* to the vertex x. We call $FS(x)$ the *Forcing Set* of x. Let $VS(x) = \cup \{c(E) : E \in \mathscr{E}_x, E$ is an influencing edge$\}$. It means that $VS(x)$ is the set of colors which must not be used by x in extending the coloring c of H^* to the vertex x. We call $VS(x)$ the Veto Set of x.

If there exist influencing co-edges but $FS(x) = \emptyset$, then the coloring c of the mixed hypergraph H^* is not extendable to the vertex x. Assume that $VS(x) = \{1, 2, \ldots, t\}$. Then c is extendable to x if and only if we are allowed to add a new color. In this paper we only consider colorable mixed hypergraphs, so we assume that any coloring in question of H^* is extendable to H.

The following definitions are helpful in calculating the number of colorings of the mixed hypergraph H from the number of colorings of the mixed hypergraph H^*.

Definition 5. Call the set

$$\Gamma(x)=\{y\colon\ y\in X, y\neq x, \mathscr{A}_x\cap\mathscr{A}_y\neq\emptyset,\ \textit{or}\ \mathscr{E}_x\cap\mathscr{E}_y\neq\emptyset\}$$

the *neighborhood* of the vertex x in a mixed hypergraph H.

In other words, the neighborhood of a vertex x consists of those vertices which are contained in common edges or co-edges with x except x itself. It is clear that if $X\backslash(\{x\}\cup\Gamma(x))\neq\emptyset$, then $\Gamma(x)$ is a separator.

Definition 6. A vertex x is said to be *simplicial* in a mixed hypergraph if its neighborhood induces a uniquely colorable mixed subhypergraph.

Since in this case the coloring of neighborhood is unique, we may omit (unless a misunderstanding occurs) repeating 'with respect to a given strict t-coloring of H^*'.

Definition 7. A simplicial vertex x is called *free* if $\mathscr{A}_x$ does not contain influencing co-edges; otherwise it is called a *restricted* vertex.

Definition 8. A vertex x is called a *phantom* with respect to the given coloring of H^* if

$$|FS(x)\backslash VS(x)|=1.$$

We will show later that no coloring is changed after adding or removing simplicial phantom vertices.

Definition 9. A mixed hypergraph $H=(X,\mathscr{A}\cup\mathscr{E})$ is said to be *pseudo-chordal* if there exists an ordering σ for the vertex set X, $\sigma=(x_1,x_2,\dots,x_n)$, such that the vertex x_j is simplicial in the subhypergraph induced by the set $\{x_j,x_{j+1},\dots,x_n\}$ for each $j=1,2,\dots,n-1$.

2. Preliminary

The following results are either trivial or some simple consequences of old results [8,10–12].

Theorem 1. *Let $H^*\in\mathscr{U}\mathscr{C}$ and $\chi(H^*)=\bar{\chi}(H^*)=k$. If there is no influencing co-edge for x and $|VS(x)|=q$ where $0\leqslant q\leqslant k$, then*
(1) if $q=\chi(H^*)$, then $\chi(H)=\chi(H^*)+1$, otherwise, $\chi(H)=\chi(H^*)$;

(2) $\bar{\chi}(H)=\bar{\chi}(H^*)+1$;
(3) $r_i(H)=(i-q)r_i(H^*)+r_{i-1}(H^*)$ *for* $i=1,2,\ldots,n$, *specificly we have* $r_i(H)=0$ *for* $i\neq k, i\neq k+1$, $r_k(H)=(k-q)r_k(H^*)=k-q$, *and* $r_{k+1}(H)=r_k(H^*)=1$;
(4) $P(H,\lambda)=(\lambda-q)P(H^*,\lambda)$.

Proof. Evident. □

Corollary 2. *Under the conditions of the above theorem,* $H\in\mathscr{UC}$ *if and only if* $q=k$.

Note that in this case x might not be a simplicial vertex.

Theorem 3. *Let* $H^*\in\mathscr{UC}$, $\chi(H^*)=\bar{\chi}(H^*)=k$, *and* $|FS(x)-VS(x)|=t>0$. *Then*
(1) $\chi(H)=\chi(H^*)=\bar{\chi}(H)=\bar{\chi}(H^*)$,
(2) $r_i(H)=tr_i(H^*)$ *for* $i=1,2,\ldots,n$, *specificly we have* $r_i(H)=0$, $i\neq k$, $r_k(H)=tr_k(H^*)$,
(3) $P(H,\lambda)=tP(H^*,\lambda)$.

Proof. It is obvious. □

Corollary 4. *Under the conditions of the above theorem,* $H\in\mathscr{UC}$ *if and only if* $t=1$.

Remark. In the last case the vertex x is a phantom. Because the chromatic polynomial and the chromatic spectrum remain the same, no matter how many phantom vertices we have. This justifies the usage of the term 'phantom'.

The following separator theorem was proved in [8]:

Theorem 5. *If in a mixed hypergraph* $H=(X,\mathscr{A}\cup\mathscr{E})$, $H_0=H/X_0$ *is a uniquely colorable separator and* $H_1=H/(X_1\cup X_0)$ *and* $H_2=H/(X_2\cup X_0)$ *are the two derived subhypergraphs, then the following equalities hold*:
(1) $\chi(H)=max\{\chi(H_1),\chi(H_2)\}$;
(2) $\bar{\chi}(H)=\bar{\chi}(H_1)+\bar{\chi}(H_2)-\bar{\chi}(H_0)$;
(3) $P(H,\lambda)=P(H_1,\lambda)P(H_2,\lambda)/P(H_0,\lambda)$.

3. Main result

The main result of this paper is the following theorem.

Theorem 6. *A polynomial* $P(\lambda)$ *is a chromatic polynomial of a pseudo-chordal mixed hypergraph* H *with lower chromatic number* χ *if and only if it has the following form*:

$$P(\lambda)=N\lambda^{s_0}(\lambda-1)^{s_1}\ldots(\lambda-\chi+1)^{s_{\chi-1}},$$

where $s_i\geqslant 1$ $(i=0,1,\ldots,\chi-1)$ *and* $N=t_1t_2\ldots t_l$, (t_j *integer*, $1\leqslant t_j\leqslant\chi$ *for* $1\leqslant j\leqslant l$).

Proof. ($\Rightarrow$) Let $H=(X,\mathscr{A}\cup\mathscr{E})$ be a pseudo-chordal mixed hypergraph on n vertices and suppose the theorem is true for all pseudo-chordal mixed hypergraphs on fewer vertices (the theorem is trivial when $|X|=1,2$). Assume that there is an ordering $X=(x_1,x_2,\dots,x_n)$ such that the vertex x_i is simplicial in the subhypergraph induced by the set $x_i,x_{i+1},\dots x_n$ for all $i=1,2,\dots n-1$. Denote by $H_1=(X_1,\mathscr{A}_1\cup\mathscr{E}_1)$ the subhypergraph induced by the vertex set $X_1=X\backslash\{x_1\}$. Consider the following two possible cases:

Case 1. Neighborhood $\Gamma(x_1)$ is not a separator in H. Then $\Gamma(x_1)=X_1$. This implies that $H_1\in\mathscr{UC}$ since the vertex x_1 is simplicial. Hence $P(H_1,\lambda)=\lambda(\lambda-1)(\lambda-2)\dots(\lambda-\chi(H_1)+1)=\lambda^{(\chi(H_1))}$. If there is no influencing co-edge (x is free) and we let $|VS(x)|=q$, then $P(H,\lambda)=(\lambda-q)P(H_1,\lambda)$ by Theorem 1. Therefore, if $q=\chi(H_1)$, then

$$P(H,\lambda)=(\lambda-q)P(H_1,\lambda)=(\lambda-q)\lambda^{(q)}=\lambda^{(q+1)}=\lambda^{(\chi(H))},$$

where $\chi(H)=\chi(H_1)+1$; and, if $q<\chi(H_1)$, then

$$\begin{aligned}P(H,\lambda)&=(\lambda-q)P(H_1,\lambda)\\&=\lambda(\lambda-1)\dots(\lambda-q+1)(\lambda-q)^2(\lambda-q-1)\dots(\lambda-\chi(H)+1),\end{aligned}$$

where $\chi(H)=\chi(H_1)$.

If there exist influencing co-edges, i.e., x_1 is a restricted simplicial vertex, then by Theorem 3 we have

$$P(H,\lambda)=tP(H_1,\lambda)=t\lambda(\lambda-1)\dots(\lambda-\chi(H)+1),$$

and $\chi(H)=\chi(H_1)$, where $1\leqslant t=|FS(x_1)\backslash VS(x_1)|\leqslant\lambda(H)$.

Case 2. Neighborhood $\Gamma(x_1)$ is a separator in H.

Let $\Gamma(x_1)=X_0$ and denote by H_0 the subhypergraph induced by the vertex set X_0, and by H_2 the subhypergraph induced by the vertex set $X_0\cup\{x_1\}$. By the induction hypothesis we have that ($\chi=\chi(H_1)$)

$$P(H_1,\lambda)=N_1\lambda^{s'_0}(\lambda-1)^{s'_1}\dots(\lambda-\chi+1)^{s'_{\chi-1}}.$$

Since H_0 is a uniquely colorable separator, we can apply the separator theorem according to the following two subcases.

Subcase 2.1. There is no influencing co-edge of x_1 in H_2 (x_1 is a free vertex). Then by the separator theorem and Theorem 1 we have

$$\begin{aligned}P(H,\lambda)&=P(H_1,\lambda)P(H_2,\lambda)/P(H_0,\lambda)\\&=P(H_1,\lambda)(\lambda-q)P(H_0,\lambda)/P(H_0,\lambda)\\&=(\lambda-q)P(H_1,\lambda),\end{aligned}$$

where $0\leqslant q=|VS(x_1)|\leqslant\chi(H_1)$.

Subcase 2.2. There exists an influencing co-edge of x_1 in H_2 (x_1 is restricted). Then by the separator theorem and by Theorem 3 we have

$$\begin{aligned}P(H,\lambda)&=P(H_1,\lambda)P(H_2,\lambda)/P(H_0,\lambda)\\&=P(H_1,\lambda)tP(H_0,\lambda)/P(H_0,\lambda)=tP(H_1,\lambda).\end{aligned}$$

If in subcase 2.1 we put $N=N_1$ and $s_i=s_i'+1$ provided that $i=q$, and put $s_i=s_i'$ otherwise, then we obtain that the theorem is true. Analogously, if in subcase 2.2 we put $s_i=s_i'$ and $N=tN_1$, then the theorem is true again.

$\Leftarrow$ Now let $P(\lambda)=N\lambda^{s_0}(\lambda-1)^{s_1}\dots(\lambda-k+1)^{s_{k-1}}$, where $s_i\geqslant 1, i=0,1,\dots,k-1$ and N,k are natural numbers such that $N=t_1t_2\dots t_l$, where $t_j\in\{1,2,\dots,k\}, l\geqslant 1$. We show how one can construct a pseudo-chordal mixed hypergraph $H=(X,\mathscr{A}\cup\mathscr{E})$ such that $P(H,\lambda)=P(\lambda)$ and $\chi(H)=k$. Take s_0 isolated vertices as $H_0=(X_0,\emptyset)$. $P(H_0,\lambda)=\lambda^{s_0}$. Add s_1 new isolated vertices to H_0 and join each of them by an edge of size 2 with the first vertex from X_0. Denote the hypergraph obtained by $H_1=(X_1,\mathscr{E}_1)$. $P(H_1,\lambda)=\lambda^{s_0}(\lambda-1)^{s_1}$. Let $(x_1,x_2)\in\mathscr{E}_1$.

Add s_2 new isolated vertices to H_1 and join each of them by an edge of size 2 with x_1 and by an edge of size 2 with x_2. Denote the hypergraph obtained by $H_2=(X_2,\mathscr{E}_2)$. $P(H_2,\lambda)=\lambda^{s_0}(\lambda-1)^{s_1}(\lambda-2)^{s_2}$. Let $(x_1,x_2),(x_1,x_3),(x_2,x_3)\in\mathscr{E}_2$.

Add s_3 new isolated vertices to H_2 and join each of them by an edge of size 2 with x_1, x_2 and with x_3. Denote the hypergraph obtained by $H_3=(X_3,\mathscr{E}_3)$. $P(H_3,\lambda)=\lambda^{s_0}(\lambda-1)^{s_1}(\lambda-2)^{s_2}(\lambda-3)^{s_3}$.

Continue this procedure until the hypergraph $H_{k-1}=(X_{k-1},\mathscr{E}_{k-1})$ is obtained such that $P(H_{k-1},\lambda)=\lambda^{s_0}(\lambda-1)^{s_1}\dots(\lambda-k+1)^{s_{k-1}}$. Observe that all its edges have the size 2 and it contains a complete graph on k vertices $x_1,x_2,\dots,x_k$.

Now, recalling that $t_j\leqslant k$, $j\in\{1,2,\dots l\}$, $l\geqslant 1$, choose t_1 vertices $x_1,x_2,\dots x_{t_1}$, add a new isolated vertex y_1 and form the first co-edge $A_1=(x_1,x_2,\dots x_{t_1},y_1)$. Denote the mixed hypergraph obtained by $H_1'=(X_1',\mathscr{A}_1\cup\mathscr{E}_{k-1})$. We have $P(H_1',\lambda)=t_1\lambda^{s_0}(\lambda-1)^{s_1}\dots(\lambda-k+1)^{s_{k-1}}$.

Choose t_2 vertices $x_1,x_2,\dots,x_{t_2}$, add a new isolated vertex y_2 and form the second co-edge $A_2=(x_1,x_2,\dots x_{t_2},y_2)$. Denote the mixed hypergraph obtained by $H_2'=(X_2',\mathscr{A}_2\cup\mathscr{E}_{k-1})$. We have $P(H_2',\lambda)=t_1t_2\lambda^{s_0}(\lambda-1)^{s_1}\dots(\lambda-k+1)^{s_{k-1}}$. Continue this procedure until the mixed hypergraph $H_l'=(X_l',\mathscr{A}_l\cup\mathscr{E}_{k-1})$ is obtained. It is evident that $\chi(H_l')=k$ and $N=t_1t_2\dots t_l$, where $t_j\in\{1,2,\dots,k\}, l\geqslant 1$. Denote H_l' by $H=(X,\mathscr{A}\cup\mathscr{E})$. H is a pseudo-chordal mixed hypergraph and we have

$$P(H,\lambda)=N\lambda^{s_0}(\lambda-1)^{s_1}\dots(\lambda-\chi+1)^{s_{\chi-1}}. \qquad \square$$

Corollary 7. *Let $H=(X,\mathscr{A}\cup\mathscr{E})$ be a pseudo-chordal mixed hypergraph. Then*

$$\bar{\chi}(H)=|X|-l,$$

where l is the number of restricted simplicial vertices in the consecutive elimination procedure.

Proof. Since a simplicial vertex is either free or restricted, and when adding a free simplicial vertex the upper chromatic number increases by one, it follows that $\bar{\chi}(H)=|X|-l$. □

Corollary 8. *Let $H=(X,\mathscr{A}\cup\mathscr{E})$ be a pseudo-chordal mixed hypergraph and Σ be the set of all the orderings of a vertex set X which produce the consecutive eliminations of simplicial vertices. Then the numbers of free and restricted simplicial vertices are the same for all elements of Σ.*

Proof. We may have many simplicial vertices at any step of consecutive elimination. We may choose any of them. However, since the chromatic polynomial as a result of decomposition is unique, and the free and restricted simplicial vertices contribute to the chromatic polynomial in different ways, it follows that the number of free simplicial vertices is the same and the number of restricted simplicial vertices is the same. □

Corollary 9. *If the vertex x is simplicial in an arbitrary mixed hypergraph $H=(X,\mathscr{A}\cup\mathscr{E})$, and if $H_1=(X_1,\mathscr{A}_1\cup\mathscr{E}_1)$ is the subhypergraph induced by the vertex set $X_1=X\backslash x$ then*

$$P(H,\lambda)=(\lambda-q)P(H_1,\lambda)$$

or

$$P(H,\lambda)=tP(H_1,\lambda),$$

where $0\leqslant q\leqslant\chi(H_1)$ and $1\leqslant t\leqslant\chi(H_1)$.

A graph is called chordal if for any cycle of length $\geqslant 4$ there exists an edge connecting two non consecutive vertices (also called triangulated or rigid circuit graphs, introduced by Hajnal and Surányi [7] and characterized by Dirac [6], see also [2,3]).

In graph theory, a vertex is called simplicial (see for example [3]) if its neighborhood induces a complete subgraph (clique). Since the complete graphs are uniquely colorable and classic graphs do not contain co-edges, each simplicial vertex from graph theory is free simplicial by the definition of this paper.

Therefore, we have the following two theorems.

Theorem 10. *If $G=(X,V)$ is a chordal graph, then it is a pseudo-chordal mixed hypergraph.*

Proof. Follows immediately from the well known fact that G is chordal if and only if it has a simplicial elimination ordering [6,3]. □

Theorem 11. *The problem to recognize if a mixed hypergraph is pseudo-chordal is* NP-*complete.*

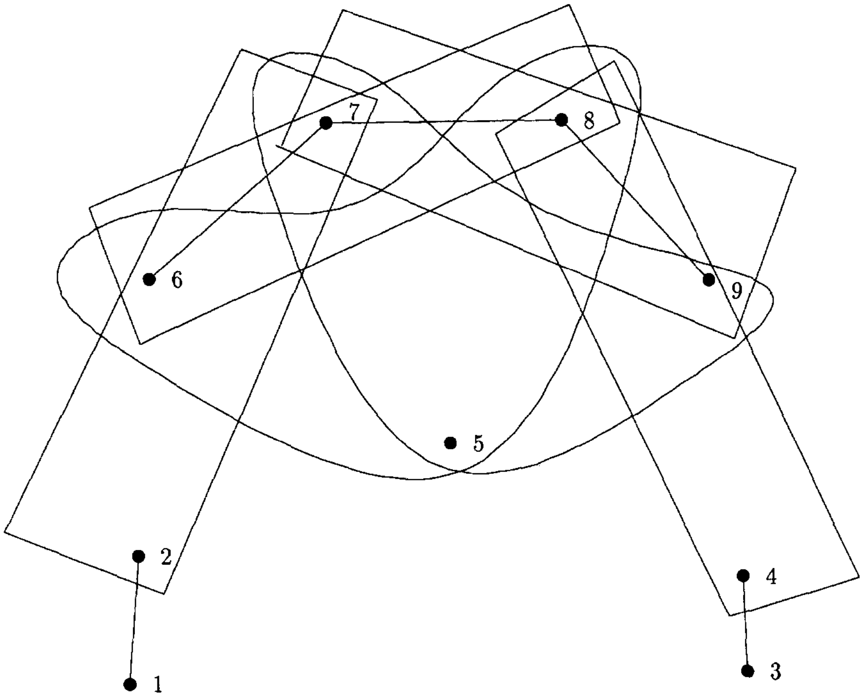

Fig. 1. Example of pseudo-chordal mixed hypergraph.

Proof. To determine if a vertex is simplicial it is necessary to find out whether its neighborhood induces a uniquely colorable mixed subhypergraph. However, as it was shown in [8], this problem is NP-complete. □

Example. Let $H=(X,\mathscr{A}\cup\mathscr{E})$, where $X=\{1,2,3,4,5,6,7,8,9\}$, $\mathscr{A}=\{(2,6,7),(4,8,9),(6,7,8),(7,8,9)\}$, $\mathscr{E}=\{(1,2),(3,4),(5,6,8),(5,7,9),(6,7),(7,8),(8,9)\}$. The mixed hypergraph H is shown on the Fig. 1 above, the co-edges are drawn as rectangles, the edges as usually.

We have that $\sigma_1=(1,2,3,4,5,6,7,8,9)$, and in this ordering the vertices 1, 3, 5, 8 and 9 are free simplicial, the vertices 2, 4, 6 and 7 are the restricted simplicial vertices, and the vertices 6 and 7 are phantom. It is easy to see another ordering $\sigma_2=(5,3,4,1,2,9,8,7,6)$ and in this ordering the vertices 5, 3, 1, 7 and 6 are free simplicial, the vertices 4, 2, 9 and 8 are the restricted simplicial vertices, and the vertices 9 and 8 are phantom.

Obviously, in both cases

$$P(H,\lambda)=4\lambda(\lambda-1)^3(\lambda-2),$$

$$R(H)=(0,0,16,20,4,0,0,0,0),$$

$$\chi(H)=3 \text{ and } \bar{\chi}(H)=5,$$

and the maximal uniquely colorable mixed subhypergraph is induced by the subset $Y = \{5,6,7,8,9\}$.

From the point of view of coloring, the uniquely colorable mixed hypergraphs are generalizations of complete graphs. In a similar way, the pseudo-chordal mixed hypergraphs are generalizations of chordal graphs. Comparing to the simple structure of a complete graph, the structure of a uniquely colorable mixed hypergraph is quite complicated [8]. Therefore, the structure of a pseudo-chordal mixed hypergraphs is not expected to be simple. In addition, there is a notion of a chordal hypergraph. Hypergraph is called chordal [9] if each cycle of length at least 4 has two non consecutive vertices with common hyperedge. The conformal [4, p. 30] chordal hypergraphs coincide with dual to the arboreal hypergraphs [9; 4, p. 186]. Coloring properties of chordal hypergraphs, however, are far to be as nice as of chordal graphs. This justifies the use of the prefix 'pseudo-'.

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