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Determination of the source parameter in a heat equation with a non-local boundary condition

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Abstract

In this article we consider the inverse problem of identifying a time dependent unknown coefficient in a parabolic problem subject to initial and non-local boundary conditions along with an overspecified condition defined at a specific point in the spatial domain. Due to the non-local boundary condition, the system of linear equations resulting from the backward Euler approximation have a coefficient matrix that is a quasi-tridiagonal matrix. We consider an efficient method for solving the linear system and the predictor–corrector method for calculating the solution and updating the estimate of the unknown coefficient. Two model problems are solved to demonstrate the performance of the methods.

Keywords: Heat equation; Inverse problem; Non-local boundary condition

1. Introduction

In this article we shall study the numerical solution of an inverse problem of finding a source parameter $p = p(t)$ in the following diffusion equation:

$$u_t = u_{xx} + p(t)u + f(x,t) \quad x \in (0,1),\ 0 < t < T, \tag{1}$$

subject to the non-local boundary conditions

$$\begin{aligned} u(0,t) &= \int_0^1 K_0(x)u(x,t)\mathrm{d}x + g_0(t) \quad 0 < t < T, \\ u(1,t) &= \int_0^1 K_1(x)u(x,t)\mathrm{d}x + g_1(t) \quad 0 < t < T, \end{aligned} \tag{2}$$

with the initial condition

$$u(x,0) = u_0(t) \quad 0 \le x \le 1, \tag{3}$$

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and the overspecified condition at a point x_0 in the spatial domain

$$u(x_0, t) = E(t) \quad x_0 \in (0, 1) \text{ and } 0 \le t \le T. \tag{4}$$

The functions f, K_0, K_1, g_0, g_1, and $u_0(t)$ are sufficiently smooth functions.

Eq. (1) can be used to describe a heat transfer process with a source parameter $p(t)$, and (4) to represent the temperature $u(x, t)$ at a specific point x_0 in the spatial domain at any time t [8–12].

It is known that the class of problems for $p(t) = 0$ arises in the quasi-static theory of thermoelasticity [4,5,13,14]. For $p(t) \neq 0$ the problem represents the temperature distribution, and the model problem (1) with the given Dirichlet or Neumann boundary conditions can be regarded as a control problem with source control parameter [1–3]. Thus the purpose of solving (1) is to identify the source parameter $p(t)$ that will produce at each time t a desired temperature $u(x, t)$ at a given point x_0 in the spatial domain.

Existence and uniqueness and some properties of the solution to Eq. (1) with $p(t) = 0$ were established in [4,5] under the following assumptions:

$$\int_0^1 |K_0(x)| \mathrm{d}x < 1, \quad \text{and} \quad \int_0^1 |K_1(x)| \mathrm{d}x < 1. \tag{5}$$

The numerical solution of (1)($p(t) = 0$) and (2) and its variants has been considered in several articles. Ekolin [6] proved the convergence of the θ-method for $\theta = 0$ and 1 by assumption (5), and the convergence of the C–N method ($\theta = \frac{1}{2}$) was proved under the following assumption:

$$\left(\int_0^1 |K_0(x)|^2 \mathrm{d}x\right)^{1/2} + \left(\int_0^1 |K_1(x)|^2 \mathrm{d}x\right)^{1/2} \le \sqrt{3}/2. \tag{6}$$

Liu in [7] proved the convergence of a θ-method for $\theta \ge 0.5$ under the following assumption:

$$\int_0^1 |K_0(x)|^2 \mathrm{d}x + \int_0^1 |K_1(x)|^2 \mathrm{d}x \le 2. \tag{7}$$

The purpose of this article is to present the numerical solution algorithm of the model problem (1) with the given initial condition, the non-local boundary conditions (2) and the overspecified condition (23) to provide an accurate estimate for the solution $u(x, t)$ and $p(t)$ of the inverse problem.

The article is organized as follows. We first describe the finite difference approximation of the problem together with the non-local boundary conditions. In Section 3 we present the predictor–corrector method and the solution of the quasi-tridiagonal linear system. Finally in Section 4 we present the numerical results from the solution of two model problems.

2. Finite difference approximation

Let $x_m = m\delta x$ and $t_n = n\delta t$ for $m = 0, 1 \ldots, M$ and $n = 0, \ldots, N$, respectively, where $\delta x = \frac{1}{M}$ and $\delta t = \frac{1}{N}$ are the regular spatial and time step sizes respectively.

Let $u_m^n = u(x_m, t_n)$ be the approximations of $u(x, t)$ at (x_m, t_n). The backward Euler approximation of (1) is given by

$$u_m^{n+1} - u_m^n = r u_{m+1}^{n+1} - 2r u_m^{n+1} + r u_{m-1}^{n+1} + \delta t p(t_{n+1}) u_m^{n+1} \tag{8}$$

for $m = 1, \ldots, M - 1$, and $n = 0, 1, \ldots, N - 1$ with $r = \frac{\delta t}{\delta x^2}$ being the Courant number.

Simplifying (8) then

$$-\frac{1}{r} u_m^n = u_{m+1}^{n+1} - \left(2 + \frac{1}{r} - \delta x^2 p^{n+1}\right) u_m^{n+1} + u_{m-1}^{n+1}. \tag{9}$$

For $m = 0$ and $m = M$ we will consider the boundary condition given by (2). The non-local boundary conditions are discretized by the trapezoidal rule as follows;

$$u_0^{n+1} = \delta x \left(\frac{1}{2} K_0(x_0) u_0^{n+1} + \sum_{m=1}^{M-1} K_0(x_m) u_m^{n+1} + \frac{1}{2} K_0(x_M) u_M^{n+1} \right) + g_0(t_{n+1}),$$

and

$$u_M^{n+1} = \delta x \left(\frac{1}{2} K_1(x_0) u_0^{n+1} + \sum_{m=1}^{M-1} K_1(x_m) u_m^{n+1} + \frac{1}{2} K_1(x_M) u_M^{n+1} \right) + g_1(t_{n+1}), \tag{10}$$

for $n = 1, \ldots, N$.

Henceforth, by assembling the discretization for the points $m = 0, \ldots, M$, given by (9), together with (10), the following system of linear equations will be produced:

$$A u^{n+1} = w^{n+1}, \tag{11}$$

defined for each time interval $[t_n, t_{n+1}]$, $n = 0, 1, \ldots, N-1$, where A is given by

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{M-1} & a_M \\ 1 & \lambda & 1 & & & \\ & 1 & \lambda & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \lambda & 1 \\ b_0 & b_1 & b_2 & \dots & b_{M-1} & b_M \end{bmatrix},$$

with

$$\begin{aligned} a_0 &= 1 - \frac{\delta x}{2} K_0(x_0), & b_0 &= -\frac{\delta x}{2} K_1(x_0), \\ a_m &= -\delta x K_0(x_m), & b_m &= -\delta x K_1(x_m), \quad \text{for } m = 1, \ldots, M-1, \\ a_M &= -\frac{\delta x}{2} K_0(x_M), & b_M &= 1 - \frac{\delta x}{2} K_1(x_M), \end{aligned} \tag{12}$$

$\lambda = -(2 + \frac{1}{r} - \delta x^2 p^{n+1})$, and

$$w^{n+1} = (w_0^{n+1}, w_1^{n+1}, \ldots, w_M^{n+1}),$$

where

$$\begin{aligned} w_0^{n+1} &= g_0(t_{n+1}), \\ w_m^{n+1} &= \frac{-1}{r} u_m^n, \quad m = 1, 2, \ldots, M-1 \\ w_M^{n+1} &= g_1(t_{n+1}). \end{aligned}$$

If $p(t) \le 0$, $\delta x \sum_{m=0}^{M} |K_0(x_m)| < 1$, and $\delta x \sum_{m=0}^{M} |K_1(x_m)| < 1$ then the matrix A is diagonally dominant and it is non-singular.

The solution of the linear system (11) will provide an update for the solution u at the time step t_{n+1}, and it will be considered for the simulation of the given problem throughout the time interval $[0, T]$.

For the convergence of the numerical solution for the finite difference scheme (9) and (10), let $u(x, t)$ be the exact solution and the error be denoted by $e_m^n = u - u_m^n$; then the corresponding error equation for (9) is given by

$$-\frac{1}{r} e_m^n = e_{m+1}^{n+1} - \left(2 + \frac{1}{r} - \delta x^2 p^{n+1} \right) e_m^{n+1} + e_{m-1}^{n+1}. \tag{13}$$

and for the non-local boundary condition (10) is given by

$$\left(1 - \frac{\delta x}{2} K_0(x_0) \right) e_0^{n+1} - \delta x \sum_{m=1}^{M-1} K_0(x_m) e_m^{n+1} - \frac{\delta x}{2} K_0(x_M) e_M^{n+1} = g_0(t_{n+1}),$$

and

$$-\frac{\delta x}{2}K_0(x_0)e_0^{n+1} - \delta x\sum_{m=1}^{M-1}K_1(x_m)e_m^{n+1} + \left(1 - \frac{\delta x}{2}K_1(x_M)\right)e_M^{n+1} = g_1(t_{n+1}), \tag{14}$$

for $n = 1, \dots, N$.

For conventional notation in convergence analysis we present the following operators:

$$\mathcal{I}_0\omega^n = \omega_0^n - \delta x\left[\frac{1}{2}K_0(x_0)\omega_0^n + \sum_{k=1}^{M-1}K_0(x_m)\omega_m^n + \frac{1}{2}K_0(x_m)\omega_M^n\right] = g_0(t_n) \tag{15}$$

$$\mathcal{I}_1\omega^n = \omega_M^n - \delta x\left[\frac{1}{2}K_1(x_0)\omega_M^n + \sum_{k=1}^{M-1}K_1(x_m)\omega_m^n + \frac{1}{2}K_1(x_m)\omega_M^n\right] = g_1(t_n) \tag{16}$$

and

$$D_t\omega_m^n = \frac{\omega_m^{n+1} - \omega_m^n}{\delta t}, \quad \text{and} \quad D_x\omega_m^n = \frac{\omega_{m+1}^n - \omega_m^n}{\delta x} \tag{17}$$

for $n = 1, \dots, N-1$ and $m = 0, 1, \dots, M$.

For the error convergence analysis, we first present the following stability lemma.

Lemma 2.1 (*Stability Lemma*). *If the inequalities (5), (7) hold and $p(t_{n+1}) < 0$ then there exist positive constants δx_0 and $\mathcal{C}$, independent of δt and δx, such that*

$$\|u^n\|_\infty \le \mathcal{C}\{\|D_x u^0\| + |u_0^0| + |u_M^0| + |\mathcal{I}_0 u^n| + |\mathcal{I}_1 u^n|\} + \mathcal{C}\left\{\left[\delta t\sum_{n=0}^{N-1}\left(\|\partial_t u^{n+1} - \partial xx u^{n+1} - p(t_{n+1})u^{n+1}\|^2 + |\mathcal{I}_0 D_t u^n|^2 + |\mathcal{I}_1 D_t u^n|^2\right)\right]^{1/2}\right\}$$

for $n = 1, \dots, N-1$ and any $\delta x < \delta x_0$.

Proof. For the proof see [9]. ∎

Therefore, if we replace u_m^n in the inequality estimate, given by Lemma 2.1, by the error term e_m^n, we will obtain the following convergence result, given by the following theorem:

Theorem 2.2. *If the inequalities (5), (7) hold and $p(t_{n+1}) < 0$ then there exist positive constants δx_0 and $\mathcal{C}$, independent of δt and δx, such that the error e_m^n from the difference approximation (9) and the non-local boundary conditions (10) for the numerical solution of (1), (2) and (3), is bounded by*

$$|e_m^n| \le \mathcal{C}[\delta x^2 + \delta t]$$

for any $\delta x < \delta x_0$, and for each $m = 0, 1, \dots, M$, and $n = 1, \dots, N-1$.

3. Efficient methods of solution for $u(x, t)$ and $p(t)$

In this section we will discuss the methods for solving the generated linear system (11) which possesses two unknowns to estimate $u(x, t)$ and $p(t)$ at each time step t_{n+1}. The purpose of the solution of the inverse problem with the non-local boundary condition is to identify the source parameter that will produce at each time, at each time step t_n, a desired temperature $u(x, t)$ at a given point x_0 in the spatial domain.

As shown, in the diagonal entries of the matrix A, the discretization of (1) is defined by the unknown source function $p(t_{n+1})$ at each time step t_{n+1} and hence the solution $u(x_m, t_{n+1})$ requires an *accurate* estimate for the unknown function $p(t_{n+1})$.

The general solution methods possess two algorithms:

(1) Predictor–corrector method for $u(x, t)$ and $p(t)$ over $[t_n, t_{n+1}]$.

(2) Efficient solution method for the linear system (11) with quasi-tridiagonal matrix A.

3.1. Predictor–corrector method

The following predictor–corrector method is considered for solving the discretized equations (9) or solving the linear system (11) to update the solution $u(x_m, t_{n+1})$. For simplicity we will present the method for each equation of the linear system (11).

Consider the simplified discretization equation of (1) given by (9)

$$-\frac{1}{r}u_m^n = u_{m+1}^{n+1} - \left(2 + \frac{1}{r} - \delta x^2 p^{n+1}\right) u_m^{n+1} + u_{m-1}^{n+1}.$$

Then for each $m = 1, \dots, M-1$ and $n = 0, 1, \dots, N-1$ the solution for u_m^{n+1} requires the estimation $p^{n+1} = p(t_{n+1})$ at time t_{n+1}.

Consider (1) at $x = x_0$; then

$$E'(t) = u_{xx}(x_0, t) + p(t)E(t) + f(x_0, t),$$

with $u(x_0, t) = E(t)$ or we get an updating of $p(t)$ as follows:

$$p(t) = \frac{1}{E(t)}(E'(t) - u_{xx}(x_0, t) - f(x_0, t)). \tag{18}$$

Therefore at $t = t_{n+1}$

$$p^{n+1} = p(t_{n+1}) = \frac{1}{E(t_{n+1})}(E'(t_{n+1}) - u_{xx}(x_0, t_{n+1}) - f(x_0, t_{n+1})). \tag{19}$$

Thus p^0 ($p^0 = p(0)$), together with the initial condition values of $u(x, t)$ at $t = 0$, provides the starting value of p^{n+1} for our computation of $u(x, t)$ at $t = t_{n+1}$. Thus a good choice of the initial guess for p^{n+1} denoted by $p^{n+1(0)}$ can be considered as $p^{n+1}(t_n) = p^n,\ n = 0, 1, \dots, N$.

In general we will use $p^{n+1(l)}$ to denote the l-th prediction for $p(t)$ at time $t = t_{n+1}$ and $u^{n+1(l)}$ to denote the corresponding approximation for $u(x, t)$ at time level t_{n+1} using the predicted value of $p^{n+1(l)}$. Also it should be noticed that we would expect a number of corrections, given as l corrections, for $p^{n+1(l)}$ and $u^{n+1(l)}$ to be made before $p^{n+1(l)}$ and $u^{n+1(l)}$ can be accepted as a *good* approximation at $t = t_{n+1}$.

The correction for $p^{n+1(l)}$ is given as follows:

$$p^{n+1(l)} = \frac{1}{E(t_{n+1})}(E'(t_{n+1}) - (u_{xx}\,|_{x_0}^{n+1(l-1)}) - f(x_0, t_{n+1})). \tag{20}$$

We will correct $p^{n+1(l)}$ and update the solution $u^{n+1(l)}$ until it converge using a prespecified tolerance; then the latest updated $u^{n+1(l)}$ is accepted.

3.2. The solution of the linear system with a quasi-tridiagonal matrix

To update the solution $u_m^{n+1(l)}$, $m = 1, 2, \dots$ and $n = 0, 1, \dots, N$, where (l) is the correction level after updating $p^{n+1(l)}$ using (20), it is required to solve, repeatedly, the system of linear equations (11) where

$$\begin{aligned} &\sum_{k=0}^{M} a_x u_k^{n+1} = w_0 \quad \text{for } m = 0,\\ &u_{m-1} + \lambda u_m^{n+1} + u_{m+1} = w_m \quad \text{for } m = 1, 2, \dots, M-1\\ &\sum_{k=0}^{M} b_x u_k^{n+1} = w_M \quad \text{for } m = M. \end{aligned} \tag{21}$$

The matrix A is a tridiagonal matrix except for the first and last rows (quasi-tridiagonal matrix). For the solution of such a linear system we will consider the algorithm by [7]. The algorithm outline is as follows;

Let

$$A_0 = \begin{bmatrix} \alpha & 1 & & & \\ 1 & \lambda & 1 & & \\ & 1 & \lambda & 1 & \\ & & & \ddots & \\ & & & 1 & \lambda \end{bmatrix}.$$

We consider the LU-matrix factorization of $A_0 = LU$ where

$$L = \begin{bmatrix} \alpha & & & & \\ 1 & \alpha & & & \\ & 1 & \alpha & & \\ & & & \ddots & \\ & & & 1 & \alpha \end{bmatrix}, \qquad U = \begin{bmatrix} \beta & 1 & & & \\ & \beta & 1 & & \\ & & \beta & 1 & \\ & & & \ddots & \\ & & & & \beta \end{bmatrix},$$

with

$$\alpha = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}, \quad \text{and} \quad \beta = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2},$$

are solutions of the quadratic equation

$$x^2 - \lambda x + 1 = 0.$$

The first step in the solution of the linear system (11) is to solve the following linear system:

$$A_0 y = w.$$

The solution is given as follows;

Solve, for v, the linear system

$$Lv = w$$

using the following recurrence relation:

$$v_0 = \frac{w_0}{\alpha},$$
$$v_m = \frac{w_m - v_{m-1}}{\alpha}, \quad m = 1, 2, \dots, M.$$

Then solve for y the linear system

$$Uy = v$$

using the following recurrence relation:

$$y_M = v_m,$$
$$y_m = v_m - \beta y_{m+1}, \quad m = 0, 1, \dots, M - 1.$$

Let $z = u^{n+1} - y$; then solve for z the linear system

$$Az = W, \quad \text{where } W = \begin{bmatrix} w_0 - \sum a_k y_k \\ \vdots \\ w_M - \sum b_k y_k \end{bmatrix}.$$

It is known that the general row entries of the matrix A, except for the first and last rows, are given by

$$z_{m-1} + \lambda z_m + z_{m+1} = 0, \quad \text{for } m = 0, 1, \dots, M - 1. \tag{22}$$

The solution of the multi-step equation (22) for z_m is given by

$$z_m = c_0\gamma^{M-m} + c_1\gamma^m, \quad \text{for } m = 0, 1, \ldots, M$$

where

$$\gamma = \frac{-\lambda - \sqrt{\lambda^2 - 4}}{2}$$

is a solution of $x^2 + \lambda x + 1 = 0$.

The arbitrary constants c_0 and c_1 are evaluated using the following system of equations:

$$c_0 \sum_{k=0}^{M} a_k \gamma^{M-k} + c_1 \sum_{k=0}^{M} a_k \gamma^k = w_0 - \sum_{k=0} a_k y_k,$$
$$c_0 \sum_{k=0}^{M} b_k \gamma^{M-k} + c_1 \sum_{k=0}^{M} b_k \gamma^k = w_M - \sum_{k=0} b_k y_k.$$

Then the final solution is $u^{n+1} = y + z$.

4. Numerical results

In this section we will present the numerical results from the solution of two model problems for testing the performance of the presented algorithms. The first model problem, model problem 1, has been designed using the basic model problem defined by Day [4], and considered by Ekolin [6] as well, with some alterations.

The standard model problem definition is given by

$$u_t = u_{xx} + p(t)u + f(x, t).$$

For the model problem 1, we have $f(x, t) = -2\mathrm{e}^{-t^2}$, with the following initial and boundary conditions:

$$u(x, 0) = x(x - 1) + \frac{\delta}{6(1 + \delta)} \quad 0 < x < 1$$
$$u(0, t) = -\delta \int_0^1 u(x, t)\mathrm{d}x, \quad t > 0$$
$$u(1, t) = -\delta \int_0^1 u(x, t)\mathrm{d}x, \quad t > 0$$

and where the overspecified condition is given by

$$u(x_0, t) = \mathrm{e}^{-t^2}\left[x_0(x_0 - 1) + \frac{\delta}{6(1 + \delta)}\right] = E_0(t) \quad t > 0 \tag{23}$$

with $\delta = 0.0144$. For the point x_0 in (23) we selected two points, $x_0 = 0.3$ and $x_0 = 0.6$, in the spatial domain (0, 1).

It is easy to check that the exact solution $u(x, t)$ is given by

$$u(x, t) = \mathrm{e}^{-t^2}\left\{x(x - 1) + \frac{\delta}{6(1 + \delta)}\right\},$$

and for $p(t) = -2t$.

We also considered the solution of $u(x, t)$ and $p(t)$ for the second model problem, model problem 2, with the following specific definition for $f(x, t)$ and the initial and boundary conditions:

$$f(x, t) = ((t - 1)^2 - \pi^2)\mathrm{e}^{-t^2}(\cos(\pi x) + \sin(\pi x))$$
$$u(x, 0) = \cos(\pi x) + \sin(\pi x)$$
$$u(1, t) = -\frac{\pi}{2}\int_0^1 u(x, t)\mathrm{d}t$$

Table 1
The error of the solution $u(x,t)$ and the estimated value of $p(t)$ at time $T=1$ for model problem 1, with $p(t)=-2t$

$\delta x = 0.01$	$r = 0.4$		$r = 0.8$		$r = 1$	
	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$
u	3.3183e−6	3.2382e−6	4.8914e−6	5.001e−6	6.24937e−6	6.11358e−6
p	−1.999743	−1.999735	−1.99948	−1.99947	−1.99933	−1.99935
$\delta x = 0.005$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$
u	6.11703e−7	6.253e−7	1.2505e−6	1.2233e−6	1.56310e−6	1.52911e−6
p	−1.999935	−1.999937	−1.99986	−1.99987	−1.99983	−1.99983

Table 2
The error of the solution $u(x,t)$ and the estimated value of $p(t)$ at time $T=1$ for model problem 2, with $p(t)=-(1+t^2)$

$\delta x = 0.01$	$r = 0.4$		$r = 0.8$		$r = 1$	
	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$
u	6.1941e−5	3.4493e−5	6.901555e−5	1.23973e−5	1.5502e−4	8.6287e−5
p	−1.999212	−1.99997	−1.9976	−1.99913	−1.9968	−1.9987
$\delta x = 0.005$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$	$x_0 = 0.3$	$x_0 = 0.6$
u	1.548199e−5	8.6235e−6	3.09697e−5	1.72488e−5	3.87157e−5	2.15621e−5
p	−1.999803	−1.99999	−1.9999403	−1.999783	−1.9992	−1.99967

$$u(0,t) = \frac{\pi}{2}\int_0^1 u(x,t)\mathrm{d}t.$$

The overspecified condition defined at the point x_0 is given by

$$u(x,t) = \mathrm{e}^{-t^2}(\cos(\pi x_0) + \sin(\pi x_0)) = E_0(t) \tag{24}$$

with $x_0 = 0.3$ and $x_0 = 0.6$.

The exact solution $u(x,t)$ of the model problem 2 with the specified conditions is given by

$$u(x,t) = \mathrm{e}^{-t^2}(\cos(\pi x) + \sin(\pi x))$$

and with $p(t) = -(t^2+1)$.

To demonstrate the performance of the algorithm presented we considered the backward Euler and the central difference approximation for the time and Laplace operators, respectively. The discretization of the space variable is performed for $\delta x = 0.01, 0.005$. The time spacing is selected using different values of r ($\delta t = r\delta x^2$), where r is the Courant number, with the given values $r = 0.4$ $r = 0.8$ and $r = 1$ to simulate over the time interval [0, 1].

The integrals of the boundary conditions are approximated using the trapezoidal integration method. The resulting linear system from the discretization of the boundary conditions and the model problem is solved using the methods presented in Section 3.

The numerical results for the solution $u(x,t)$ of problem 1 and problem 2 using $r = 0.4$ and $r = 1$ are graphed and compared in Figs. 1 and 2 with the respective exact solutions, while the estimated values of the unknown $p(t)$ at time $T = 1$ for problem 1 and 2 are graphed in Figs. 3 and 4, respectively.

In Tables 1 and 2 we tabulate the error of the numerical solution $u(x,t)$ together with the estimated $p(t)$ at time $T = 1$, for different values of r.

As can be seen in Figs. 1 and 2 there is close agreement between the numerical value and the exact value of $u(x,t)$, for different points $x_0 = 0.3$ and 0.6. That close accurate estimate is also confirmed from the tabulated values of the error for the solution $u(x,t)$, in Tables 1 and 2 respectively. Furthermore we also refer to the maximum error, of the solution $u(x,t)$, calculated at each time step and graphed in Figs. 5 and 6.

The estimated values of $p(t)$ for problems 1 and 2 are presented in Figs. 7 and 8 respectively. The figures show very accurate significant estimates of $p(t) = -2t$ and $p(t) = -(1+t^2)$ for problems 1 and 2 respectively.

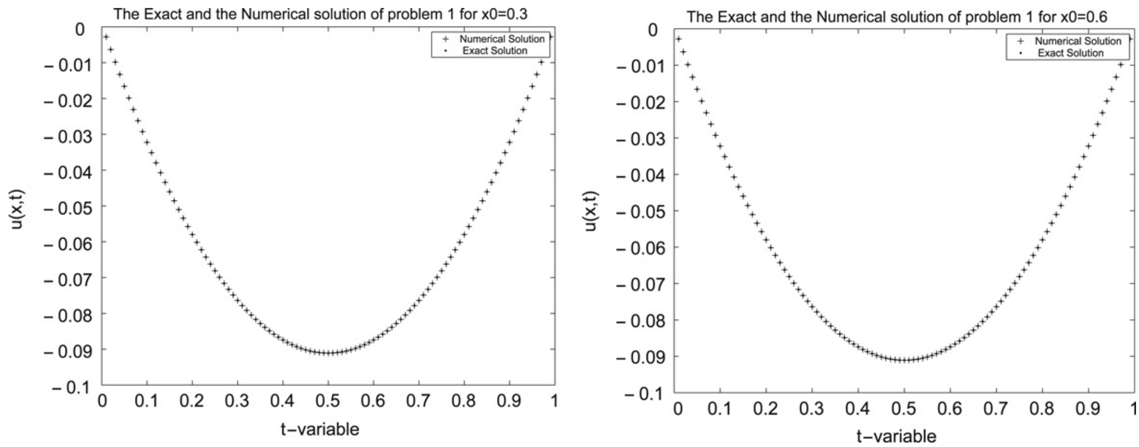

Fig. 1. The numerical and exact solutions of problem 1 at time $T = 1$, $x_0 = 0.3$ using $h = 0.01$ for $r = 1$ (on the left) and $r = 0.4$ (on the right).

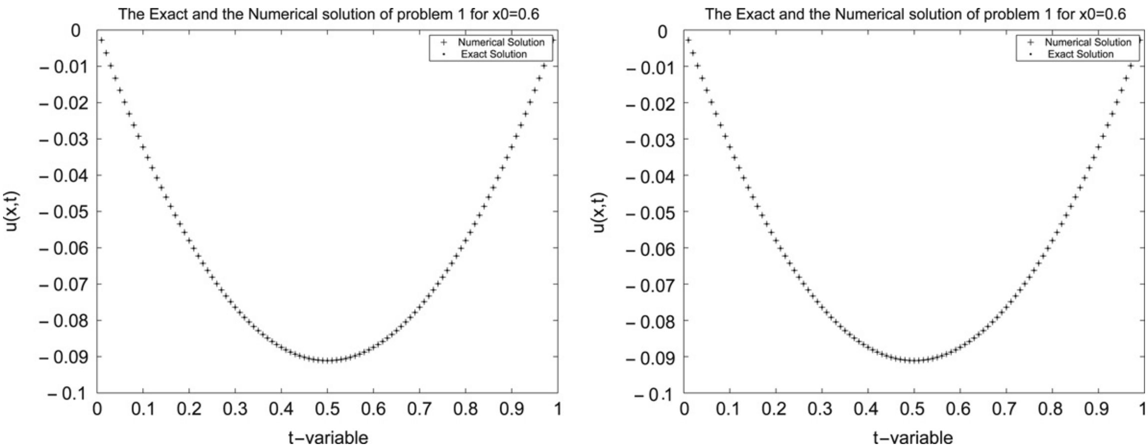

Fig. 2. The numerical and exact solutions of problem 1 at time $T = 1$, $x_0 = 0.6$ using $h = 0.01$ for $r = 1$ (on the left) and $r = 0.4$ (on the right).

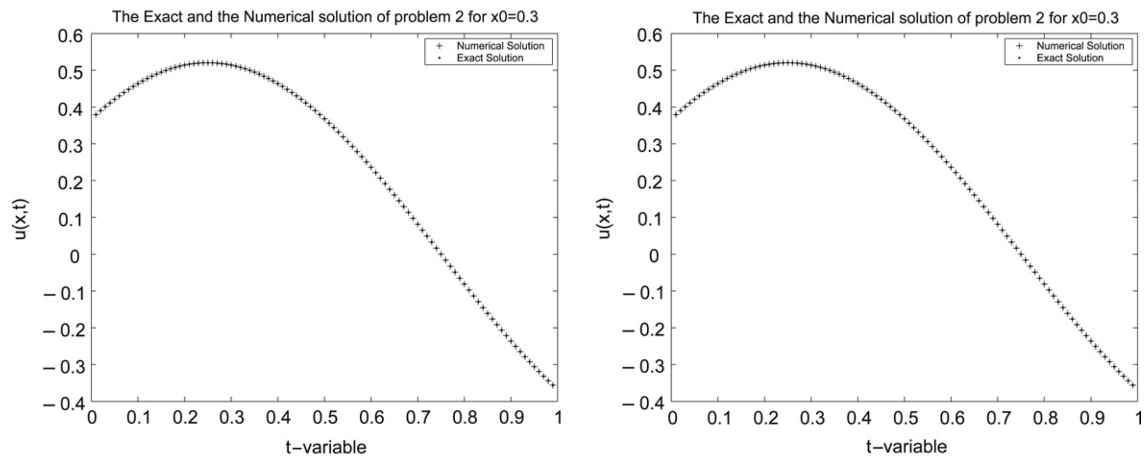

Fig. 3. The numerical and exact solutions of problem 2 at time $T = 1$, $x_0 = 0.3$ using $h = 0.01$ for $r = 1$ (on the left) and $r = 0.4$ (on the right).

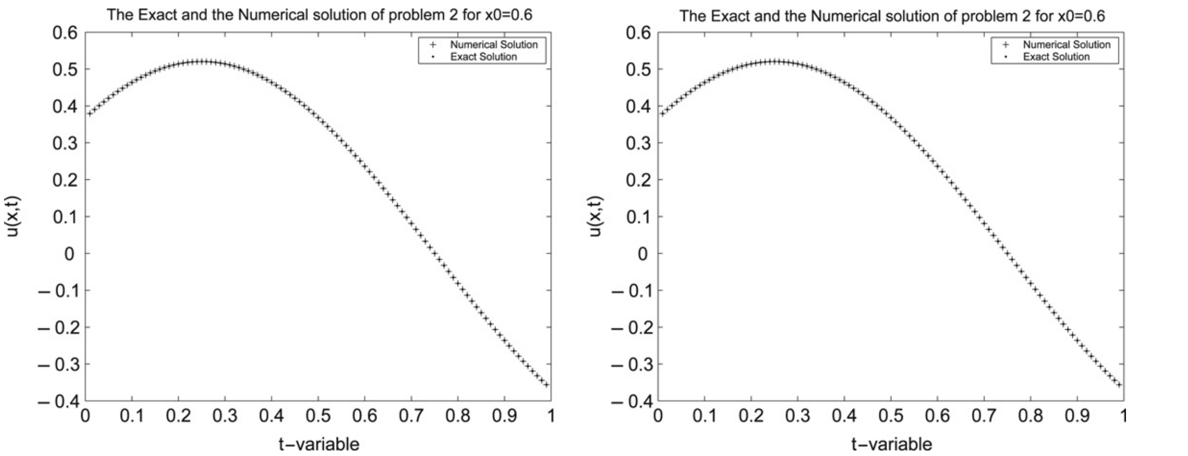

Fig. 4. The numerical and exact solutions of problem 2 at time $T = 1$, $x_0 = 0.6$ using $h = 0.01$ for $r = 1$ (on the left) and $r = 0.4$ (on the right).

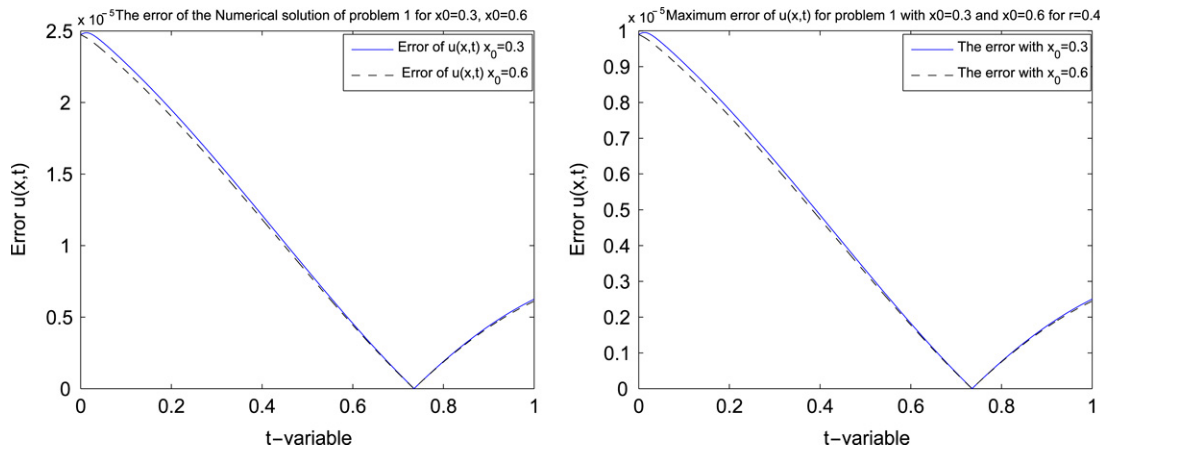

Fig. 5. The error for the solution of problem 1 at each time step through [0, 1] for $x_0 = 0.3$ and $x_0 = 0.6$, on the left for $r = 1$ and on the right for $r = 0.4$.

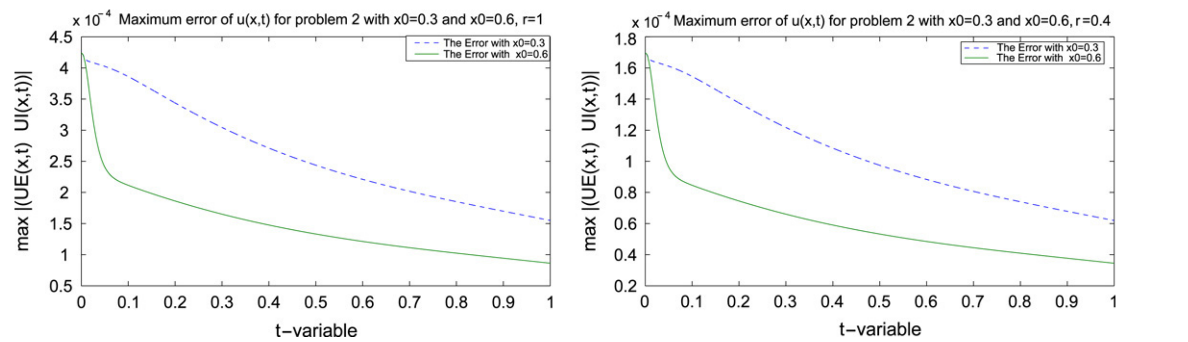

Fig. 6. The error for the solution of problem 2 at each time step through [0, 1] for $x_0 = 0.3$ and $x_0 = 0.6$, on the left for $r = 1$ and on the right for $r = 0.4$.

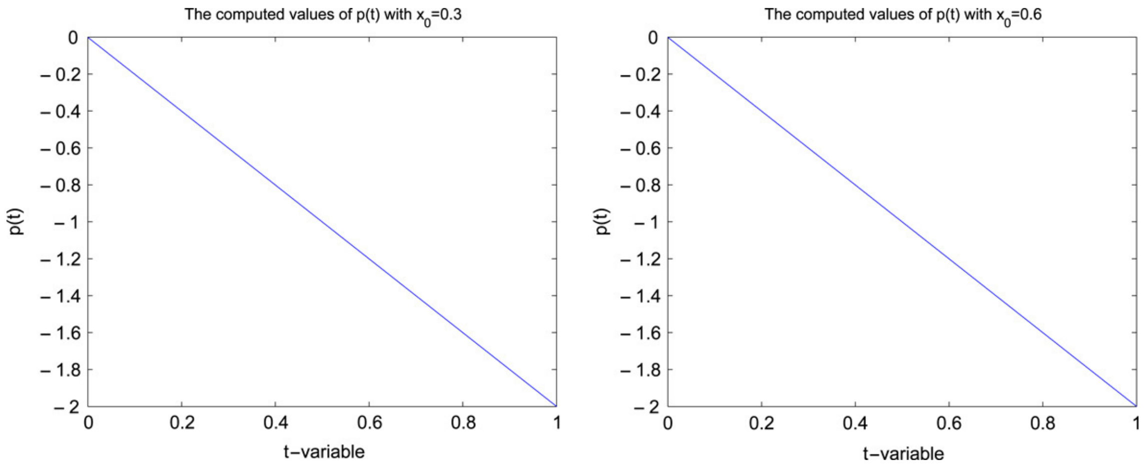

Fig. 7. The estimated value of $p(t)$ of problem 1 at each time step through [0, 1] for $x_0 = 0.3$ (on the left) and $x_0 = 0.6$ (on the right).

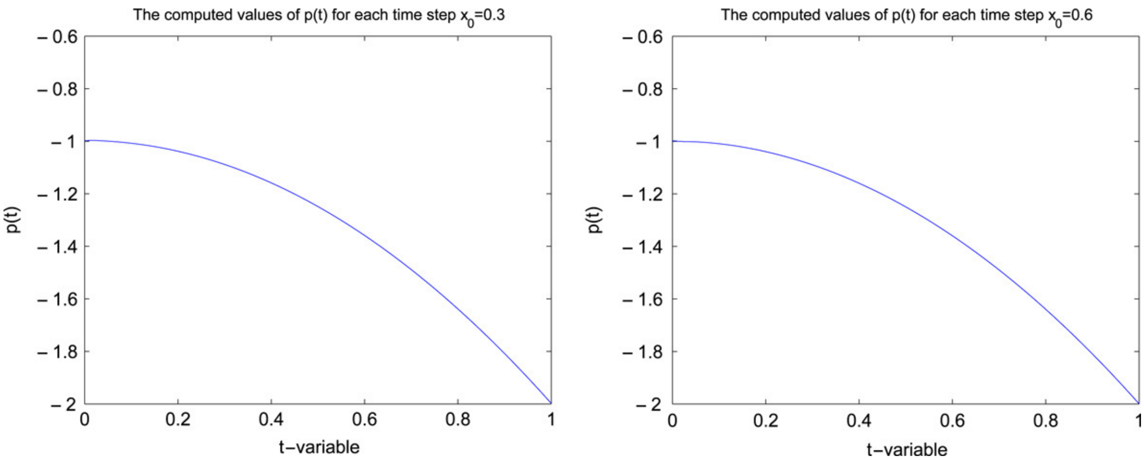

Fig. 8. The estimated value of $p(t)$ of problem 2 at each time step through [0, 1] for $x_0 = 0.3$ (on the left) and $x_0 = 0.6$ (on the right).

References

[1] J.R. Cannon, The One-Dimensional Heat Equation, in: Encyclopedia of Mathematics and Its Applications, vol. 23, Addison-Wesley, Reading, MA, 1984.
[2] J.R. Cannon, S.P. Esteva, van der Hoek Salvador, A Galerkin procedure for the diffusion equation subject to the specification of mass, SIAM J. Numer. Anal. 24 (1987) 499–515.
[3] J.R. Cannon, Yanping Lin, S. Wang, An Implicit finite difference scheme for the diffusion equation subject to mass specification, Int. J. Eng. Sci. 28 (1990) 573–578.
[4] W.A. Day, Extension of a property of the heat equation to linear thermoelasticity and other theories, Quart. Appl. Math. 40 (1982) 319–330.
[5] W.A. Day, A decreasing property of solutions of parabolic equation to linear thermoelasticity and other theories, Quart. Appl. Math. 41 (1983) 319–330.
[6] G. Ekolin, Finite difference methods for a nonlocal boundary value problem for the heat equation, BIT 31 (1991) 245–261.
[7] Y. Liu, C.W. Wu, Solution of quasi-tridiagonal system of linear equations, in: Functional Analysis, Approximation Theory and Numerical Analysis, World Sci. Publishing, River Edge, NJ, 1994, pp. 259–269.
[8] G. Fairweather, J.C. Lopez-Marcos, Galerkin methods for semilinear parabolic problem with non local boundary conditions, Adv. Comput. math. 6 (1996) 243–262.
[9] Y. Liu, Numerical solution of the heat equation with nonlocal boundary conditions, J. Comput. Appl. Math. 110 (1999) 115–127.
[10] J.R. Cannon, Yanping Lin, S. Wang, Determination of source parameter in parabolic equations, Meccanica 7 (1992) 85–94.
[11] J.R. Cannon, Yanping Lin, S. Wang, Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic equations, Inverse Problems 10 (1994) 227–243.

[12] Yanping Lin, An inverse problem for a class of quasilinear parabolic equation, SIAM J. Math. Anal. 22 (1991) 146–156.
[13] A. Friedman, Monotonic decay of parabolic equations with nonlocal boundary conditions, Quart. Appl. Math. 44 (1986) 401–407.
[14] B. Kawohl, Remark on a paper by W.A. day on a maximum principle under nonlocal boundary conditions, Quart. Appl. Math. 44 (1987) 751–752.