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A mean value theorem for cubic fields ☆

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Abstract

Let $r(n)$ denote the number of integral ideals of norm n in a cubic extension K of the rationals, and define $S(x) = \sum_{n \leqslant x} r(n)$ and $\Delta(x) = S(x) - \alpha x$ where α is the residue of the Dedekind zeta function $\zeta(s, K)$ at 1. It is shown that the abscissa of convergence of

$$\int_0^\infty \Delta(e^y)^2 e^{-2y\sigma}\, dy$$

is $\frac{1}{3}$ as expected.

Keywords: Gauss lattice point problem; Dedekind zeta function; Mean value theorem; Integral ideals; Cubic fields

1. Introduction

Let K be an algebraic extension of the rationals of degree k, $r(n)$ denote the number of integral ideals whose norm is the rational integer n, and define

$$S(x) = \sum_{n \leqslant x} r(n).$$

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Then $\zeta(s,K)$, the Dedekind zeta function for the field K satisfies

$$\zeta(s,K)=\sum_{\mathfrak{g}\neq 0}\frac{1}{N(\mathfrak{g})^s}=\sum_{n=1}^{\infty}\frac{r(n)}{n^s},$$

where the first sum is over the integral ideals $\mathfrak{g}$ of K. Moreover,

$$S(x)=\alpha x+\Delta(x),$$

where α is the residue of $\zeta(s,K)$ at its simple pole at $s=1$ and $\Delta(x)$ satisfies $\Delta(x)=o(x)$ as $x\to\infty$. The question of the order of magnitude of $\Delta(x)$ as $x\to\infty$ is a natural generalisation of the problem of Gauss concerning the number of lattice points in a large circle centred at the origin. Landau [9] proved that

$$\Delta(x)\ll x^{\eta}$$

with $\eta=1-2/(k+1)$, improving the classical result $\eta=1-1/k$ (see [17]). More recently, Huxley and Watt [6] (see [5] for the special case of the Gauss lattice point problem) have established that when $k=2$

$$\Delta(x)\ll x^{\frac{23}{73}}(\log x)^{\frac{315}{146}},$$

Müller [12] has shown that when $k=3$

$$\Delta(x)\ll x^{\frac{43}{96}+\varepsilon}$$

and Nowak [13] has obtained

$$\Delta(x)\ll x^{\eta}(\log x)^{\xi}$$

with

$$\eta=1-\frac{2}{k}+\frac{8}{k(5k+2)},\quad \xi=\frac{10}{5k+2}\quad\text{when } 4\leqslant k\leqslant 6,$$

$$\eta=1-\frac{2}{k}+\frac{3}{2k^2},\quad \xi=\frac{2}{k}\quad\text{when } k\geqslant 7.$$

There are a couple of questions that could be considered as generalisations or analogues of the above. The most well known is the Dirichlet divisor problem and its generalisation, the Piltz divisor problem. There one is concerned with the behaviour of

$$\Delta_k(x)=\sum_{n\leqslant x}d_k(n)-xP_k(\log x),$$

where $d_k(n)$ is the coefficient of n^{-s} in the Dirichlet series expansion of $\zeta(s)^k$ and $xP_k(\log x)$ is the residue of $\zeta(s)^k$ at $s=1$. There is a very extensive history of work in connection with $\Delta_k(x)$. Huxley [5] has shown that

$$\Delta_2(x) \ll x^{\frac{23}{73}} (\log x)^{\frac{461}{146}}$$

and Kolesnik [8] has shown that

$$\Delta_3(x) \ll x^{\frac{43}{96}+\varepsilon}.$$

For larger values of k see [7,15].

The other question which could be considered a generalisation of the Gauss lattice point problem is that of the behaviour of $E_k(x) = N_k(x) - \Gamma(1 + 1/k)^k x$ where $N_k(x)$ is the number of non-negative integral lattice points $(n_1, \dots, n_k)$ with $n_1^k + \dots + n_k^k \leqslant x$, but here far less is known. Indeed, the author is not aware of any improvement over the trivial $E_k(x) \ll x^{1-1/k}$ (which can be compared with Weber's bound for Δ) when $k > 2$, although some are doubtless about the possible variants of the van der Corput method, and the only limitation known to how small the error term might be is that obtained by combining the techniques of Montgomery and Vaughan [11] and Vaughan [16] which shows that $E_k(x) = \Omega(x^{\frac{1}{4}})$.

Let

$$\beta = \limsup_{x \to \infty} \frac{\log(\frac{1}{x} \int_0^x \Delta(y)^2 \, dy)}{2 \log x},$$

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it, K)|^2}{|\sigma + it|^2} \, dt$$

and

$$\gamma = \inf\{\sigma : I(\sigma) < \infty\}.$$

Then Ayoub [1] has shown that

$$\beta = \gamma$$

and, more precisely, that when $\sigma > \beta$ one has

$$2\pi \int_0^{\infty} \Delta(y)^2 y^{-2\sigma-1} \, dy = I(\sigma).$$

He has also shown that

$$\beta = \tfrac{1}{4} \quad \text{when } k = 2.$$

The main object of this paper is to show that

$$\beta = \tfrac{1}{3} \quad \text{when } k = 3 \tag{1.1}$$

and that

$$\beta \geqslant \frac{k-1}{2k} \quad \text{for all } k \geqslant 2. \tag{1.2}$$

For the Dirichlet/Piltz divisor problem Hardy [3], Cramér [2] and Heath-Brown [4] have shown that $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{3}{8}$ are indeed the abscissæ of convergence of

$$\int_0^\infty \Delta_k(y)^2 y^{-2\sigma-1}\, dy,$$

when $k = 2$, 3 and 4, respectively. For no larger k has the exact value of the abscissa of convergence been established, although it is known to be at least $\frac{k-1}{2k}$. For further details see [7,15].

The proofs of (1.1) and (1.2) use only the most basic analytic properties of $\zeta(s, K)$, and in fact pertain to the following general situation. Note that an Euler product is not required.

Let f be a non-negative arithmetical function, not identically 0, possessing the listed properties.

Property 1. *For every fixed* $\varepsilon > 0$, *whenever* $x \geqslant 1$, $\sum_{n \leqslant x} f(n)^2 \ll x^{1+\varepsilon}$.

Property 2. *The series* $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$, *which as a consequence of Property* 1 *converges for* $\Re s > 1$, *has an analytic continuation to* $\mathbb{C}$, *is analytic for all* s *except* $s = 1$ *where it has a simple pole with residue* α, *and satisfies a functional equation*

$$F(s) = \rho(s) F(1-s),$$

where

$$\rho(s) = AB^{s-\frac{1}{2}} \left(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \right)^{r_1} \left(\frac{\Gamma(1-s)}{\Gamma(s)} \right)^{r_2}$$

and A *is real*, B *is real and positive, and* r_1 *and* r_2 *are non-negative integers not both* 0. *Note that* $\rho(s)$ *is analytic for all* s *except* $s \in \mathbb{N}$ *when* $r_2 > 0$ *and all* s *except* s *with* $\frac{s+1}{2} \in \mathbb{N}$ *when* $r_2 = 0$, *and in particular whenever* $\Re s < 1$.

Even these properties are susceptible to weakening and generalisation. For example, the methods of this paper are easily adjusted to deal with the case when $F(s)$ has a pole of finite order at $s = 1$ and the only information used about $\rho(s)$ is its analyticity and the vertical order of magnitude given by Lemma 1.

Even the assumption the f be non-negative and A is real could be relaxed to include complex values. In particular, the Dirichlet/Piltz divisor problems could be included also.

Let

$$S(x) = \sum_{n \leqslant x} f(n),$$

$$\Delta(x) = S(x) - \alpha x,$$

$$L(\sigma) = 2\pi \int_{-\infty}^{\infty} \Delta(e^y)^2 e^{-2y\sigma}\, dy,$$

$$\beta = \limsup_{x \to \infty} \frac{\log \frac{1}{x} \int_0^x \Delta(y)^2\, dy}{2 \log x}$$

and

$$k = r_1 + 2r_2.$$

Then we establish the following.

Theorem 1. *If* $k = 2$ *or* 3, *then* $L(\sigma)$ *converges when* $\frac{k-1}{2k} < \sigma < 1$, *and* $\beta \leqslant \frac{k-1}{2k}$.

The core of the proof of Theorem 1 lies in establishing the inequality

$$\int_T^{2T} |F(\sigma + it)|^2\, dt \ll T^{1+\varepsilon} \quad \left(1 - \frac{1}{k} \leqslant \sigma < 1\right)$$

and in fact we prove also the more precise estimate

$$\int_T^{2T} |F(\sigma + it)|^2\, dt = C(\sigma)T + O(T^{1-v}),$$

where $v = v(\varepsilon) > 0$ and $C(\sigma) = \sum_{n=1}^{\infty} f(n)^2 n^{-2\sigma}$, uniformly for $1 - \frac{1}{k} + \varepsilon \leqslant \sigma \leqslant 1 - \varepsilon$ and any fixed $\varepsilon > 0$. As an easy consequence of this latter approximation we have

Theorem 2. *If* $k \geqslant 2$, *then* $L(\sigma)$ *diverges when* $0 < \sigma < \frac{k-1}{2k}$, *and* $\beta \geqslant \frac{k-1}{2k}$.

Our third theorem follows immediately on combining the previous two.

Theorem 3. *When* $k = 3$ *the abscissa of convergence of*

$$\int_0^\infty \Delta(e^y)^2 e^{-2y\sigma}\, dy$$

is $\frac{1}{3}$.

With more precise information concerning the mean square of $f(n)$ in place of Property 1, it would be possible to give a more exact description of the nature of the singularity of $L(\sigma)$ at $\frac{1}{3}$.

2. Preparatory lemmata

The first lemma is a standard consequence of Stirling's formula for the gamma function.

Lemma 1. *Suppose that* $\sigma_0, \sigma_1 \in \mathbb{R}$, $\sigma_0 < \sigma_1$, *and* $s = \sigma + it$. *Then*

$$\rho(s) \ll |t|^{k(\frac{1}{2}-\sigma)}$$

uniformly in the region $\{s : |t| \geqslant 1, \sigma_0 \leqslant \sigma \leqslant \sigma_1\}$.

Note that the implicit constant here may depend on A, B, k.

The second lemma is trivial when $\sigma \geqslant 1 + \delta$ and δ is any sufficiently small (in terms of k and ε) positive real number, and is immediate from the functional equation when $\sigma \leqslant -\delta$. Then in the range $-\delta < \sigma < 1 - \delta$ it follows from a general convexity principle for Dirichlet series, see [14, Sections 5.65, 9.41].

Lemma 2. *Suppose that* $\sigma_0, \sigma_1 \in \mathbb{R}$, $\sigma_0 < \sigma_1$, *and* $s = \sigma + it$. *Then*

$$F(s) \ll |t|^{\mu(\sigma)+\varepsilon}$$

uniformly in the region $\{s : |t| \geqslant 1, \sigma_0 \leqslant \sigma \leqslant \sigma_1\}$, *where*

$$\mu(\sigma) = \begin{cases} k(\frac{1}{2} - \sigma) & (\sigma \leqslant 0), \\ k\,\frac{1-\sigma}{2} & (0 < \sigma \leqslant 1), \\ 0 & (\sigma > 1) \end{cases}$$

and the implicit constant may depend on A, B, k, ε.

We now come to the main tool of the proof.

Lemma 3. *Suppose that* $0<\sigma<1$, $-1<\phi<-\sigma$, $0<\psi<1-\sigma$, $j\geqslant\frac{3}{2}k+1$, $x\geqslant 1$, $y\geqslant 1$. *Then*

$$F(s) = \sum_{n\leqslant x} f(n)n^{-s}(1-n/x)^j + \rho(s)\sum_{n\leqslant y} f(n)n^{s-1} - R(s) - U(s) - V(s),$$

where

$$R(s) = \alpha\lambda_j(x, 1-s),$$

$$U(s) = \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \rho(s+w)\sum_{n>y} f(n)n^{s+w-1}\lambda_j(x,w)\,dw,$$

$$V(s) = \frac{1}{2\pi i}\int_{\psi-i\infty}^{\psi+i\infty} \rho(s+w)\sum_{n\leqslant y} f(n)n^{s+w-1}\lambda_j(x,w)\,dw$$

and

$$\lambda_j(x,z) = \frac{x^z j!}{z(z+1)\cdots(z+j)}.$$

Proof. By moving the vertical path to $\Re w = -\infty$ when $u\geqslant 1$ and to $\Re w = +\infty$ when $u<1$ it is readily seen that for $j\geqslant 1$, $u>0$, $\theta>1$, one has

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \lambda_j(u,w)\,dw = \begin{cases} (1-1/u)^j & (u\geqslant 1), \\ 0 & (u<0). \end{cases}$$

Thus, by the absolute convergence of the above integral and the uniform convergence of $F(s+w)$ when $\Re w = \theta$ one has

$$\frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} F(s+w)\lambda_j(u,w)\,dw = \sum_{n\leqslant x} f(n)n^{-s}(1-n/x)^j.$$

By Lemma 2, when $\Re(s+w) > -1$,

$$F(s+w) - \frac{\alpha}{s+w-1} \ll (1+|\Im(s+w)|)^{\frac{3}{2}k+\varepsilon}.$$

Hence, as $j\geqslant\frac{3}{2}k+1$, we may move the vertical path to the line $\Re w = \phi$, picking up the residues of the integrand at $w = 1-s$ and 0, so that

$$\sum_{n\leqslant x} f(n)n^{-s}(1-n/x)^j = R(s) + F(s) + \frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} F(s+w)\lambda_j(u,w)\,dw.$$

By the functional equation in Property 2,

$$F(s+w) = \rho(s+w)F(1-s-w).$$

Moreover, as $-\Re(s+w) = -\sigma - \phi > 0$ we may write

$$F(1-s-w) = \sum_{n \leqslant y} f(n)n^{s+w-1} + \sum_{n>y} f(n)n^{s+w-1}.$$

The second series here gives rise to $U(s)$, and we treat the part arising from the first, namely

$$\frac{1}{2\pi i}\int_{\phi-i\infty}^{\phi+i\infty} \rho(s+w) \sum_{n \leqslant y} f(n)n^{s+w-1}\lambda_j(u,w)\,dw,$$

by moving the path of integration to the line $\Re w = \psi$, picking up a further residue at $w = 0$, obtaining

$$-\rho(s)\sum_{n \leqslant y} f(n)n^{s-1} + V(s).$$

This completes the proof of the lemma. □

Our penultimate lemma is Corollary 3 of Montgomery and Vaughan [10].

Lemma 4. *Suppose that* $a_n \in \mathbb{C}$, X, $Y \in \mathbb{R}$, $Y > X$ *and* $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$. *Then*

$$\int_X^Y \left|\sum_{n=1}^{\infty} a_n n^{it}\right|^2 dt = (Y-X)\sum_{n=1}^{\infty} |a_n|^2 + O\left(\sum_{n=1}^{\infty} n|a_n|^2\right).$$

Below we record the intimate connection afforded by Plancherel's identity between the mean squares of $F(s)$ and $\Delta(x)$.

Lemma 5. *Suppose that* $0 < \sigma < 1$,

$$L(\sigma) = 2\pi \int_{-\infty}^{\infty} \Delta(e^y)^2 e^{-2y\sigma}\,dy$$

and

$$I(\sigma) = \int_{-\infty}^{\infty} \left|\frac{F(\sigma+it)}{\sigma+it}\right|^2 dt.$$

Then $L(\sigma)$ *converges if and only if* $I(\sigma)$ *converges, and*

$$L(\sigma) = I(\sigma)$$

when convergence occurs. Moreover, if

$$\gamma = \inf\{\sigma : I(\sigma) \text{ converges}\},$$

then $\beta = \gamma$.

Proof. This follows by a standard argument, as exemplified by Titchmarsh [15, Theorem 12.5], or Ayoub [1, pp. 25–26]. □

Finally, we establish a convexity principle for mean values. This has some similarity to the main result of Section 7.8 of Titchmarsh [15]. However, the form given in Lemma 6 is more convenient for our purposes, applies to a wider class of functions and has a much shorter proof. Titchmarsh's conclusion is also an easy deduction from it.

Lemma 6. *Suppose that* $\sigma_0 < \sigma_1$, $T \geqslant 1$ *and that* $G(s)$ *is analytic for all* $s \in \mathscr{S} = \{s : \sigma_0 \leqslant \Re s \leqslant \sigma_1\}$ *and satisfies* $G(s) \ll \exp\left(\frac{(\Im s)^2}{2T^2}\right)$ *uniformly in* $\mathscr{S}$, *i.e. the implicit constant depends at most on* σ_0, σ_1 *and* T. *Let*

$$J(\sigma, T) = \int_{-\infty}^{\infty} |G(\sigma + it)|^2 \exp\left(\frac{2(\sigma^2 - t^2)}{T^2}\right) dt.$$

Then, whenever $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, *we have*

$$J(\sigma, T) \leqslant J(\sigma_0, T)^{\frac{\sigma_1 - \sigma}{\sigma_1 - \sigma_0}} J(\sigma_1, T)^{\frac{\sigma - \sigma_0}{\sigma_1 - \sigma_0}}.$$

Proof. Let

$$L(\sigma, y) = \int_{\sigma - i\infty}^{\sigma + i\infty} G(s) \exp(s^2 T^{-2} - 2\pi y s)\, ds.$$

Then, by Cauchy's theorem,

$$L(\sigma_0, y) = L(\sigma, y) = L(\sigma_1, y) \tag{2.1}$$

for every $\sigma \in [\sigma_0, \sigma_1]$. Moreover,

$$\begin{aligned} &L(\sigma, y) \exp(2\pi y \sigma) \\ &\quad = \int_{-\infty}^{\infty} iG(\sigma + it) \exp((\sigma^2 - t^2)T^{-2} + 2i\sigma t T^{-2} - 2\pi i y t)\, dt \end{aligned}$$

and this is simply the Fourier transform of

$$iG(\sigma + it) \exp((\sigma^2 - t^2)T^{-2} + 2i\sigma t T^{-2}).$$

Hence, by Plancherels' identity,

$$\int_{-\infty}^{\infty} |L(\sigma, y)|^2 \exp(4\pi y\sigma)\, dy = J(\sigma, T).$$

By (2.1), the integral on the left is

$$\int_{-\infty}^{\infty} (|L(\sigma_0, y)|^2 \exp(4\pi y\sigma_0))^{\frac{\sigma_1-\sigma}{\sigma_1-\sigma_0}}(|L(\sigma_1, y)|^2 \exp(4\pi y\sigma_1))^{\frac{\sigma-\sigma_0}{\sigma_1-\sigma_0}}\, dy.$$

By Hölder's inequality, this is bounded by

$$J(\sigma_0, T)^{\frac{\sigma_1-\sigma}{\sigma_1-\sigma_0}} J(\sigma_1, T)^{\frac{\sigma-\sigma_0}{\sigma_1-\sigma_0}},$$

which completes the proof of the lemma. □

3. The main theorem

Theorem 0. (i) *Suppose that* $1 - 1/k < \sigma < 1$ *and* $\delta = \frac{k}{4}(\sigma - 1 + \frac{1}{k})$. *Then, whenever* $T \geqslant 1$, *we have*

$$\int_T^{2T} |F(\sigma + it)|^2 dt = T \sum_{n=1}^{\infty} f(n)^2 n^{-2\sigma} + O(T^{1-\delta}).$$

(ii) *For each fixed* $\varepsilon > 0$, *whenever* $T \geqslant 1$ *we have*

$$\int_T^{2T} |F(1 - 1/k + it)|^2\, dt \ll T^{1+\varepsilon}.$$

Proof. We first suppose that $\frac{1}{2} \leqslant \sigma < 1$. By Lemma 4 and Property 1, when $\frac{1}{2} < \sigma$,

$$\int_T^{2T} \left| \sum_{n \leqslant x} f(n) n^{-\sigma-it} \right|^2 dt = T \sum_{n=1}^{\infty} f(n)^2 n^{-2\sigma} + O((Tx^{-1} + 1)x^{2-2\sigma+\varepsilon})$$

and, when $\frac{1}{2} = \sigma$,

$$\int_T^{2T} \left| \sum_{n \leqslant x} f(n) n^{-\sigma-it} \right|^2 dt \ll (Tx^{-1} + 1)x^{1+\varepsilon}.$$

By Lemmas 1 and 4 and Property 1,

$$\int_T^{2T} \left| \rho(\sigma+it) \sum_{n\leqslant y} f(n) n^{\sigma+it-1} \right|^2 dt \ll T^{2k(\frac{1}{2}-\sigma)} \sum_{n\leqslant y} f(n)^2 n^{2\sigma-2}(T+n)$$
$$\ll T^{2k(\frac{1}{2}-\sigma)}(Ty^{-1}+1)y^{2\sigma+\varepsilon}.$$

Trivially,

$$\int_T^{2T} |R(\sigma+it)|^2\, dt \ll x^{2-2\sigma}T^{-j}.$$

Suppose $T\leqslant|t|\leqslant 2T$. Then, by Lemma 1 and Schwarz's inequality,

$$U(s)^2 \ll \int_{\phi-i\infty}^{\phi+i\infty} (T+|\Im w_1|)^{k(\frac{1}{2}-\sigma-\phi)} |\lambda_j(x,w_1)|\, |dw_1|$$
$$\times \int_{\phi-i\infty}^{\phi+i\infty} (T+|\Im w_2|)^{k(\frac{1}{2}-\sigma-\phi)} |\lambda_j(x,w_2)| \left| \sum_{n>y} f(n) n^{s+w_2-1} \right|^2 |dw_2|.$$

Therefore, by Lemma 4 and Property 1,

$$\int_T^{2T} |U(\sigma+it)|^2\, dt \ll T^{k(\frac{1}{2}-\sigma-\phi)} x^{\phi} \int_{\phi-i\infty}^{\phi+i\infty} (T+|\Im w|)^{k(\frac{1}{2}-\sigma-\phi)} |\lambda_j(x,w)|$$
$$\times \int_T^{2T} \left| \sum_{n>y} f(n) n^{s+w-1} \right|^2 dt|dw|$$
$$\ll T^{2k(\frac{1}{2}-\sigma-\phi)} x^{2\phi} \sum_{n>y} f(n)^2 n^{2\sigma+2\phi-2}(T+n)$$
$$\ll T^{2k(\frac{1}{2}-\sigma)} \left(\frac{T^k}{xy}\right)^{-2\phi} (T/y+1) y^{2\sigma+\varepsilon}.$$

A similar argument applied to $V(s)$ gives

$$\int_T^{2T} |V(\sigma+it)|^2\, dt \ll T^{k(\frac{1}{2}-\sigma-\psi)}\, x^{\psi} \int_{\psi-i\infty}^{\psi+i\infty} (T+|\Im w|)^{k(\frac{1}{2}-\sigma-\psi)} |\lambda_j(x,w)|$$
$$\times \int_T^{2T} \left| \sum_{n\leqslant y} f(n) n^{s+w-1} \right|^2 dt|dw|$$

$$\ll T^{2k(\frac{1}{2}-\sigma-\psi)} x^{2\psi} \sum_{n \leqslant y} f(n)^2 n^{2\sigma+2\psi-2}(T+n)$$

$$\ll T^{2k(\frac{1}{2}-\sigma)} \left(\frac{xy}{T^k}\right)^{2\psi} (T/y+1) y^{2\sigma+\varepsilon}.$$

It follows that, when $\frac{1}{2} < \sigma < 1$,

$$\int_T^{2T} |F(\sigma+it)|^2 \, dt = T \sum_{n=1}^{\infty} f(n)^2 n^{-2\sigma} + O(E + (TE)^{\frac{1}{2}}),$$

where

$$E = (T/x+1)x^{2-2\sigma+\varepsilon} + T^{2k(\frac{1}{2}-\sigma)}(T/y+1)y^{2\sigma+\varepsilon} \times \left(\left(\frac{T^k}{xy}\right)^{-2\phi} + \left(\frac{xy}{T^k}\right)^{2\psi} \right). \tag{3.1}$$

Now suppose that $1 - \frac{1}{k} < \sigma < 1$. Let $\eta = \sigma - 1 + 1/k$ and $\delta = k\eta/4$. Choose $x = T^{\frac{k}{2}+\delta}$ and $y = T^{\frac{k}{2}-\delta}$. Then

$$E \ll T^{1+\frac{2}{k}\delta - k\eta - 2\eta\delta + \varepsilon(k/2+\delta)} \ll T^{1+\frac{2}{k}\delta - k\eta}$$

on choosing ε sufficiently small. Thus,

$$E \ll T^{1-\frac{1}{2}k\eta}$$

gives the first part of the theorem.

Finally, suppose that $\sigma = 1 - 1/k$. Then

$$\int_T^{2T} |F(1-1/k+it)|^2 \, dt \ll E,$$

where E satisfies (3.1). The choice $x = y = T^{\frac{k}{2}}$ gives

$$E \ll T^{1+k\varepsilon}$$

and completes the proof of the theorem. □

4. The proof of Theorem 1

When $k = 2$ or 3 we have $1 - 1/k \leqslant \frac{k+1}{2k}$. Hence, by Theorem 0,

$$\int_T^{2T} \left| F\left(\frac{k+1}{2k} + it\right)\right|^2 dt \ll T^{1+\varepsilon}.$$

Therefore, by the functional equation in Property 2 and Lemma 1,

$$\int_T^{2T} \left| F\left(\frac{k-1}{2k} + it\right)\right|^2 dt \ll T^{2+\varepsilon}.$$

Thus, by Lemma 6 with $\sigma_0 = (k-1)/(2k)$ and $\sigma_1 = (k+1)/(2k)$, whenever $\sigma_0 < \sigma < \sigma_1$,

$$\int_T^{2T} |F(\sigma + it)|^2 \, dt \ll T^{2+\varepsilon - k(\sigma - \sigma_0)}.$$

Thus

$$\int_{-\infty}^{\infty} \left| \frac{F(\sigma + it)}{\sigma + it} \right|^2 dt < \infty.$$

Theorem 1 now follows from Lemma 5.

5. The proof of Theorem 2

Suppose first that $k \geqslant 3$, so that $\frac{k+1}{2k} \leqslant 1 - \frac{1}{k}$. Let η be any real number with $0 < \eta < \frac{1}{3k}$. We apply Lemma 6 with $G(s) = F(s)$, $\sigma_0 = \frac{k+1}{2k} + \eta$, $\sigma = 1 - \frac{1}{k} + 2\eta$, $\sigma_1 = 1 - \frac{1}{k} + 3\eta$. Then

$$J\left(1 - \frac{1}{k} + 2\eta, T\right) \leqslant J\left(\frac{k+1}{2k} + \eta, T\right)^{\frac{\eta}{\frac{1}{2} - \frac{3}{2k} + 2\eta}} J\left(1 - \frac{1}{k} + 3\eta, T\right)^{\frac{\frac{1}{2} - \frac{3}{2k} + \eta}{\frac{1}{2} - \frac{3}{2k} + 2\eta}}.$$

By part (i) of Theorem 0,

$$T \ll J\left(1 - \frac{1}{k} + 2\eta, T\right)$$

and

$$J\left(1 - \frac{1}{k} + 3\eta, T\right) \ll T.$$

Therefore,

$$T \ll J\left(\frac{k+1}{2k} + \eta, T\right).$$

By Lemma 2, there is a positive number $c = c(k)$ such that

$$\int_{cT(\log T)^{1/2}}^{\infty} \left| F\left(\frac{k+1}{2k} + \eta + it\right) \right|^2 \exp\left(2(\sigma^2 - t^2)T^{-2}\right) dt \ll 1.$$

Hence, for every sufficiently large U,

$$U(\log U)^{-1/2} \ll \int_0^U \left| F\left(\frac{k+1}{2k} + \eta + it\right) \right|^2 dt. \tag{5.1}$$

By Lemma 1, there is a positive number $d = d(k, \eta)$ such that, whenever $|t| \geqslant d$,

$$\left| \rho\left(\frac{k+1}{2k} + \eta + it\right) \right| \ll \left| \frac{k-1}{2k} - \eta + it \right|^{-\frac{1}{2} - k\eta}.$$

Hence, for U sufficiently large,

$$\begin{aligned} U(\log U)^{-1/2} &\ll \int_d^U \left| F\left(\frac{k+1}{2k} + \eta + it\right) \right|^2 dt \\ &\ll \int_d^U \left| F\left(\frac{k-1}{2k} - \eta + it\right) \right|^2 \left| \frac{k-1}{2k} - \eta + it \right|^{-1-2k\eta} dt. \end{aligned}$$

Therefore,

$$U^{2k\eta}(\log U)^{-1/2} \ll \int_d^U \left| \frac{F(\frac{k-1}{2k} - \eta + it)}{\frac{k-1}{2k} - \eta + it} \right|^2 dt,$$

and so the integral

$$\int_0^{\infty} \left| \frac{F(\frac{k-1}{2k} - \eta + it)}{\frac{k-1}{2k} - \eta + it} \right|^2 dt$$

diverges. This is true for every η with $0 < \eta < \frac{1}{3k}$. Hence $\beta \geqslant \frac{k-1}{2k}$ as required.

When $k = 2$ the above argument fails because now $\frac{k+1}{2k} > 1 - \frac{1}{k}$. However, the lower bound

$$U \ll \int_{-U}^{U} \left| F\left(\frac{3}{4} + \eta + it\right) \right|^2 dt$$

is immediate from Theorem 0 and so can be used in place of (5.1).

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