A Constant-False-Alarm-Rate Algorithm*

James R. Bunch and Ricardo D. Fierro
Department of Mathematics
University of California, San Diego
La Jolla, California 92093-0112

Submitted by Biswa Datta

ABSTRACT

The constant-false-alarm-rate (CFAR) detection algorithm is used for the detection of an optical target in an image dominated by optical clutter. The algorithm can be used for many aerial images when a clutter-subtraction technique is incorporated. To approximate the assumption of a constant covariance matrix, the digital image scene is partitioned into subimages and the CFAR algorithm is applied to the subimages. For each subimage, local means must be computed; the "best" local mean is the one that minimizes the third moment. The clutter subtraction technique leads to a mathematically tractable algorithm based on hypotheses testing. A test statistic and a threshold level must be computed. The value of the test statistic is subimage-dependent and is compared with the detection threshold which is chosen to specify a performance level for the test. A computationally efficient and stable implementation of the CFAR algorithm is given which may use either the Cholesky decomposition or the *QR* decomposition with a rearrangement of the computations.

1. INTRODUCTION

A multiband constant-false-alarm-rate (CFAR) detection algorithm is used to detect a stationary optical target in an image dominated by optical clutter. The stationary optical target is assumed to have a known signal pattern which

*Support was provided by the Naval Ocean Systems Center, the NSF I/UC Research Center for Ultra-high Speed Integrated Circuits and Systems at UCSD, and NSF grant ECD 89-16669.

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

has nonzero intensities in several signal-plus-noise bands. The detection algorithm relies on the assumption that the optical clutter can be modeled as a Gaussian process with a space-varying mean and a covariance that is slowly varying. Numerical experiments (e.g., [2, 6]) show that this assumption is reasonable for many aerial images after an appropriate clutter subtraction technique is applied. This assumption allows the application of mathematical and statistical techniques, and its applicability to the problem of detection is supported by numerical experiments [2, 7, 8, 10].

In Section 2 the detection problem is presented. In Section 3 a CFAR detection method is presented. In Section 4 a computationally efficient and numerically stable method for determining the test statistic is presented. In Section 5 numerical results are given.

2. THE DETECTION PROBLEM

Given a stationary scene, we wish to determine whether or not a particular optical target is located in the scene. All nontarget images in the scene will be considered to be clutter. Typically, a 512×512 stencil (mesh) will be laid on the scene. The intensity at each of the N pixel elements (mesh points) is measured in m signal-plus-noise bands; let $Y^{(i)}(l, j)$ be the intensity at the (l, j) element for band i. The single-band case is $m = 1$; $m > 1$ is the multiband case; here, m is typically between 2 and 12. See Fig. 1.

Let the $m \times N$ data matrix X represent the digital image scene to be tested. Typically, $m \leqslant 12$, but $N = 512^2 = 262{,}144$. The ith row of X contains the elements of $Y^{(i)}$. Typically, the elements of $Y^{(i)}$ are ordered by the natural ordering, i.e., the pixels are ordered from left to right and from top to bottom.

Let $x(k)$ be the kth column of X, $1 \leqslant k \leqslant N$. Thus, $x(k)$ is an m-vector containing the values of pixel k, $1 \leqslant k \leqslant N$, in the m bands. Then $X = [x(1), \ldots, x(N)]$, $x(k) = [x_1(k), \ldots, x_m(k)]^T$, and $x_i(k) = X_{ik}$. Let $s = [s(1), \ldots, s(N)]^T$ be the N-vector of the known signal pattern and $b = [b(1), \ldots, b(m)]^T$ be the m-vector of unknown signal intensities.

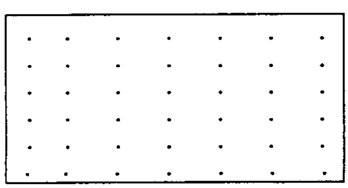

FIG. 1. The discretized image for band i (for a 6×7) stencil.

Statistical models of images assume a Gaussian probability density in order to be mathematically tractable. Unfortunately, this assumption is rarely valid for optical images. However, we do not want to abandon Gaussianity. Numerical experiments [2, 6] show that the optical clutter can be modeled as a Gaussian process with a space-varying mean after applying an appropriate clutter subtraction technique.

Let

$$\bar{X} = \frac{1}{w^2}\left[X \circledast W\right],$$

where W is a $w \times w$ all ones matrix and $\circledast$ denotes discrete convolution. $\bar{X}$ is the local nonstationary mean of X. The value of a typical pixel element of $\bar{X}$ is the average of the w^2 pixel elements that are in the square neighborhood (window of size w) of the corresponding pixel element of X. (Elements in the window but outside the image are given value zero.) See Figure 2. Then

$$X_0 \equiv X - \bar{X},$$

the residual image, is the result of the clutter subtraction. Since the third moment of a Gaussian distribution is zero, we will choose a window size w which minimizes the third moment, giving us a distribution as close to Gaussian as possible. This issue is addressed in [9] in a more quantitative approach. In practice [3, 7], the minimum is chosen from $w = 3, 5, 7, 9$. Similarly, for the same window size w chosen above, we form $\bar{s}$ from s, and set $s_0 \equiv s - \bar{s}$.

The hypotheses which the detector algorithm must test are:

$$H_0 : X = X_0 \qquad \text{(clutter only)},$$

$$H_A : X = X_0 + bs_0^T, \qquad \text{(clutter plus signal)}.$$

The GLR (generalized likelihood-ratio) test [1, 11] is now applied. However, in order to make the GLR test mathematically tractable we need the associated covariance matrices to be constant, i.e., we need the

$$M_k \equiv E\left\{\left[x(k) - Ex(k)\right]\left[x(k) - Ex(k)\right]^T\right\}$$

to be equal for $1 \leqslant k \leqslant N$, where $x(k)$ is the kth column of X. Unfortunately,

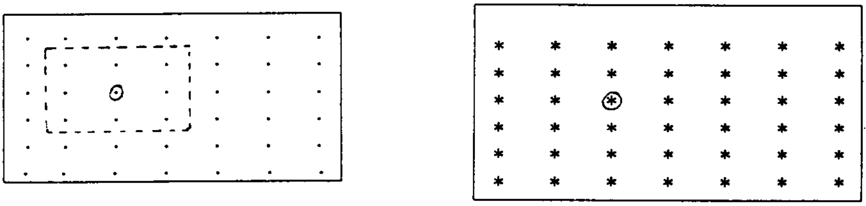

FIG. 2. A window of size $w = 3$ is put on Figure 1 (left) in order to compute the local means (right).

in practice the M_k are not constant [2, 7], and at present the detection problem has not been solved for nonconstant covariance matrices.

This difficulty is circumvented by partitioning the 512×512 image into 256 subimages of size 32×32. Let $X^{(q)}$ be the $m \times n$ data matrix for the qth subimage, $1 \leqslant q \leqslant 256$, where $m \leqslant 12$ and $n = 32^2 = 1024$. Let $X_0^{(q)}$ be the corresponding $m \times n$ residual subimage, and $s^{(q)}$, $s_0^{(q)}$ be the corresponding n-vectors of signals and residual signals. In practice, the covariance matrices for these 32×32 subimages are almost constant [2, 7]. Thus, we may apply the GLR test to each subimage. Also, in practice, the $X_0^{(q)}$ have full rank.

THEOREM 1. *The test statistic for subimage q is*

$$r(q) \equiv \frac{c_q^T A_q^{-1} c_q}{\alpha_q},$$

where

$$c_q \equiv X_0^{(q)} s_0^{(q)}$$

is an m-vector,

$$A_q \equiv X_0^{(q)T} X_0^{(q)}$$

is an $m \times m$ symmetric positive definite matrix (if $X_0^{(q)}$ has full rank), and

$$\alpha_q \equiv s_0^{(q)T} s_0^{(q)}.$$

Proof. The theorem follows from a standard statistical argument as in [1] and by using $\det(I - zz^T) = 1 - z^T z$. Cf. [10]. ∎

3. PROBABILITY OF FALSE ALARM

Let r_0 be the detection threshold, depending only on m, n, and PFA (the probability of a false alarm), and not on the covariance matrix. Then the test becomes:

THEOREM 2. *For each subimage q, if $r(q) \geqslant r_0$ then H_A else H_0.*

In order to determine r_0, we need to know the probability density function of the test statistic $r(q)$, given H_0. This makes it possible to determine a rejection region for the test. Given H_0, the probability that the calculated $r(q)$ lies in this region is called the *probability of a false alarm* and denoted PFA; it is the constant false-alarm rate. Then by [11],

$$PFA = \int_{r_0}^{1} f(r \mid H_0)\, dr,$$

where

$$f(r \mid H_0) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)\Gamma\left(\frac{m}{2}\right)}(1-r)^{(n-m-2)/2} r^{(m-2)/2}$$

is the probability distribution function of r given H_0, and Γ is the factorial functi‹n.

In practice, PFA is predetermined to achieve a desired performance level for the test. The closed interval $[r_0, 1]$ defines the rejection region for the test statistic under H_0. If the calculated $r(q)$ is less than r_0, then we accept H_0 (with $1 -$ PFA certainty); otherwise we reject H_0 and accept H_A. To calculate the detection threshold r_0, given PFA, one computes $p \equiv 1 -$ PFA and finds r_0 such that

$$p = \int_0^{r_0} f(r \mid H_0)\, dr$$

using the bisection method on the function

$$g(t) = \int_0^t f(r \mid H_0)\, dr - p,$$

where t is the current estimate of r_0. We will use the continued-fraction representation of the integral in the numerical evaluation of the integral:

$$\int_0^t f(r \mid H_0)\, dr = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)\Gamma\left(\frac{m}{2}\right)} t^{m/2}(1-t)^{(n-m)/2}\left[\frac{1}{1+}\frac{d_1}{1+}\frac{d_2}{1+}\cdots\right],$$

where

$$d_{2j+1} = -\frac{(m+2j)(n+2j)t}{(m+4j)(m+4j+2)},$$

$$d_{2j} = \frac{2j(n-m+2j)t}{(m+4j-2)(m+4j)}.$$

Note that for a fixed m, n, and PFA, r_0 is calculated only once and then used for each subimage q.

4. COMPUTATION OF THE TEST STATISTIC

Given the probabilistic nature of the GLR test, it is not failproof, and conceivably in some applications of the CFAR detection algorithm false conclusions could have dire consequences. Hence the computation $r(q)$ should be performed as accurately as possible so as not to promote false conclusions. The naive computation of $r(q)$, which involves computing the inverse of $A_q = X_0^{(q)}X_0^{(q)T}$, is to be avoided for several reasons. First, the explicit formation of A_q may result in a loss of information, unless higher precision is used [5, p. 142]. Second, if B_q is the computed inverse of A_q, the exact inverse of B_q may not necessarily be close to A_q [13, p. 253]. Third, if z is a solution to the overdetermined system $X_0^{(q)T}z = s_0$, it satisfies the normal equations associated with this system, and hence it follows that

$$r(q) = \left(X_0^{(q)}s_0^{(q)}\right)^T z/\alpha_q.$$

This shows that the explicit inverse need not be computed. If the formation of $A_q \equiv X_0^{(q)}X_0^{(q)T}$ is acceptable (does not result in the severe loss of information or yield a large condition number), the Cholesky decomposition may be employed:

Form $c := X_0^{(q)} s_0^{(q)}$	mn flops
Form $\alpha_q := s_0^{(q)T} s_0^{(q)}$	n flops
Form $A_q := X_0^{(q)} X_0^{(q)T}$	$\frac{1}{2} m^2 n$ flops
Factorize $A_q := GG^T$ (Cholesky)	$\frac{1}{6} m^3$ flops
Solve $Gz = c$ by forward substitution	$\frac{1}{2} m^2$ flops
Then $r(q) := z^T z / \alpha_q$	m flops

A flop is a multiply followed by an add. Only the highest order term is given.

The backward error analysis by Wilkinson (see [5, p. 89]) shows that if the solution to $A_q z = c$ is computed as above, then $(A_q + E_q) z = c$, where $\| E_q \| \leqslant \delta_m \text{ eps } \| A_q \|$, in which δ_m is a small constant that depends on m, eps is the machine precision, and $\| \cdot \|$ is the 2-norm.

Alternatively, the computation of $r(q)$ can be achieved more stably by the QR decomposition of $X_0^{(q)T}$ using a sequence of Householder transformations $\{Q_i : i = 1, \dots, m\}$ (see [5, p. 41]). If $X_0^{(q)T} = QR$, it follows that

$$X_0^{(q)} X_0^{(q)T} = \begin{bmatrix} R^T & 0 \end{bmatrix} Q^T Q \begin{bmatrix} R \\ 0 \end{bmatrix} = R^T R,$$

which implies the sequence of Q_i's need not be accumulated or applied sequentially to any vector. The transformations are performed on $X_0^{(q)}$ implicitly, and R^T is stored in the lower left corner of X_0 to reduce storage requirements. The computations can be ordered as follows, with flop counts:

Form $c := X_0^{(q)} s_0^{(q)}$	mn flops
Form $\alpha_q := s_0^{(q)T} s_0^{(q)}$	n flops
Implicitly transform $X_0^{(q)} \to R^T$	$m^2 n$ flops
Solve $R^T z = c$	$\frac{1}{2} m^2$ flops
$r(q) := z^T z / \alpha_q$	m flops

Triangular systems generally produce solutions that are more accurate than is indicated by the traditional bound for relative accuracy involving the condition number [13, p. 247]. The QR approach above should be used if A_q is ill conditioned. The statistical relevance of these concerns is discussed in [12].

THEOREM 3. *The computation of the test statistic $r(q)$ by the above algorithms using the Cholesky or QR decomposition is stable. If A_q is well conditioned, the Cholesky algorithm may be used.*

5. NUMERICAL RESULTS

We have implemented in the C language the CFAR algorithm described above. Subroutine BISRO determines the detection threshold r_0, given J, n, and PFA. Bisection [4] is used to calculate r_0. For PFA $= 10^{-5}$, Table I gives the number of iterations needed to compute r_0 for various m and n and various values of the stopping criterion TOL (it stops when the change in the function value is $<$ TOL and $|r_{\text{new}} - r_{\text{old}}| <$ TOL $\cdot |r_{\text{new}}|$). Table I shows that as TOL decreases the number of iterations (ITS) increases, but with no significant improvement in the computed value of r_0.

This experiment was repeated for larger values of PFA (10^{-4}, 10^{-3}, 10^{-2}). The results for $m = 5$ are given in Table II. The number of iterations required

TABLE I
DETECTION THRESHOLD r_0 FOR PFA $= 10^{-5}$

		$m = 3$		$m = 4$		$m = 5$	
n	TOL	ITS	r_0	ITS	r_0	ITS	r_0
100	0.010000	10	0.145508	10	0.165039	10	0.184570
	0.001000	13	0.146118	13	0.165649	13	0.183716
	0.000100	17	0.146172	16	0.165665	16	0.183670
	0.000010	20	0.146178	20	0.165671	20	0.183673
256	0.010000	11	0.059082	11	0.066895	11	0.074707
	0.001000	15	0.058990	14	0.067078	14	0.074646
	0.000100	18	0.058964	18	0.067074	18	0.074604
	0.000010	21	0.058967	21	0.067076	21	0.074606
400	0.010000	12	0.037842	12	0.043213	12	0.048096
	0.001000	15	0.037994	15	0.043304	15	0.048187
	0.000100	19	0.038019	18	0.043285	18	0.048176
	0.000010	22	0.038018	22	0.043282	21	0.048176
1024	0.010000	13	0.015015	13	0.016968	13	0.018921
	0.001000	17	0.014977	16	0.017044	16	0.018997
	0.000100	20	0.014970	20	0.017056	20	0.019000
	0.000010	23	0.014969	23	0.017057	23	0.019001

TABLE II
DETECTION THRESHOLD r_0 FOR $m = 5$

		PFA = 10^{-2}		PFA = 10^{-3}		PFA = 10^{-4}		PFA = 10^{-5}	
n	TOL	ITS	r_0	ITS	r_0	ITS	r_0	ITS	r_0
100	0.010000	10	0.145508	10	0.192383	9	0.232422	9	0.275391
	0.001000	13	0.144653	13	0.191528	13	0.234253	12	0.273682
	0.000100	17	0.144768	16	0.191574	16	0.234238	16	0.273727
	0.000010	20	0.144767	29	0.191576	19	0.234243	19	0.273718
256	0.010000	11	0.058105	11	0.077637	11	0.097168	10	0.114258
	0.001000	15	0.058014	14	0.078064	14	0.096985	14	0.115051
	0.000100	18	0.058002	17	0.078041	17	0.096931	17	0.115013
	0.000010	21	0.058000	21	0.078039	20	0.096936	20	0.115006
400	0.010000	12	0.037354	11	0.050293	11	0.062988	11	0.074707
	0.001000	15	0.037323	15	0.050446	14	0.062927	14	0.074890
	0.000100	29	0.037336	18	0.050426	18	0.062870	18	0.074848
	0.000010	22	0.037335	21	0.050425	21	0.062867	21	0.074851
1024	0.010000	13	0.014771	13	0.019897	12	0.025146	13	0.029541
	0.001000	17	0.014671	16	0.019913	16	0.024918	16	0.029770
	0.000100	20	0.014674	19	0.019903	19	0.024912	19	0.029776
	0.000010	23	0.014675	23	0.019902	22	0.024911	22	0.029775

to compute the threshold r_0 for each value of TOL is similar for the various values of PFA. If the event of a false alarm has serious consequences, then a small PFA value (10^{-5}) could be chosen without increasing the amount of computation significantly.

Subroutine WSKUV computes the third moment for window size w. Subroutine MINELT determines the optimal window size among $w = 3, 5, 7, 9$. Subroutine LCMN forms the $m \times n$ residual matrix $X_0^{(q)}$ of subimage q. Subroutine SIGBAR creates the corresponding residual signal pattern. Next the test statistic $r(q)$ is computed by RVAL (using the QR decomposition) or by AGLT (using the Cholesky decomposition as long as $X_0^{(q)}$ is very well conditioned). Now we can determine whether there is an optical target in subimage q.

In the following simulation, we set $m = 3$, $N = 64^2 = 4096$, and $n = 8^2 = 64$. There are 64 subimages of size 8×8; the target was placed in subimage $q = 28$. The computed detection threshold is $r_0 = 0.3437$. The results of the simulation can be seen in Table III. Note that the results for QR and Cholesky are identical here (indicating that the A_q were well conditioned in this case).

TABLE III
TARGET DETECTED IN SUBIMAGE $q = 28$

q	Third moment $w = 3$	5	7	9	Optimal w	$r(q)$ QR	Cholesky	Target detected?
1	0.634	1.118	1.238	1.333	3	0.1507	0.1507	No
2	1.362	1.389	1.720	1.920	3	0.2482	0.2482	No
3	0.596	0.600	0.510	0.492	9	0.0390	0.0390	No
4	0.970	0.853	0.807	0.613	9	0.0610	0.0610	No
5	0.930	0.674	0.732	0.982	5	0.0164	0.0164	No
6	1.055	1.487	1.528	1.516	3	0.0329	0.0329	No
7	1.209	1.170	1.275	1.320	5	0.0519	0.0519	No
8	1.227	0.774	0.722	0.761	7	0.0429	0.0429	No
9	0.677	0.786	0.839	0.883	3	0.0638	0.0638	No
10	0.742	0.728	0.675	0.550	9	0.0168	0.0168	No
11	1.289	1.480	1.603	1.684	3	0.0277	0.0277	No
12	0.861	0.674	0.919	0.906	5	0.0803	0.0803	No
⋮					⋮			⋮
25	0.866	1.023	1.055	1.018	3	0.0105	0.0105	No
26	1.407	1.315	1.260	1.169	9	0.0398	0.0398	No
27	0.531	0.532	0.569	0.579	3	0.0329	0.0329	No
28	11.13	11.05	11.75	11.75	5	0.8638	0.8638	Yes
29	0.828	0.752	0.969	1.030	5	0.0613	0.0613	No
30	0.494	0.179	0.136	0.113	9	0.0207	0.0207	No
31	0.595	0.505	0.514	0.545	5	0.1151	0.1151	No
32	0.741	0.989	1.013	0.994	3	0.0361	0.0361	No
33	0.798	0.504	0.286	0.348	7	0.0455	0.0455	No
34	0.646	0.648	0.767	0.741	3	0.1156	0.1156	No
35	0.968	1.314	1.259	1.311	3	0.0194	0.0194	No
36	0.772	0.834	0.836	0.895	3	0.1671	0.1671	No
⋮					⋮			⋮
60	1.181	1.066	1.100	0.929	9	0.0041	0.0041	No
61	0.818	1.104	0.925	1.045	3	0.0152	0.0152	No
62	1.482	1.586	1.494	1.635	3	0.0485	0.0485	No
63	1.034	0.620	0.643	0.564	9	0.0427	0.0427	No
64	0.379	0.505	0.748	0.904	3	0.0483	0.0483	No

REFERENCES

1 T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Wiley, 1985.

2 J. Y. Chen, Adaptive Detection of Optical and Radar Targets Using Reference Processes, Ph.D. Dissertation, Univ. of Southern California, Apr. 1986.

3 J. Y. Chen and I. S. Reed, A detection algorithm for optical targets in clutter, *IEEE Trans. Aerospace Electron. Systems*, Jan. 1987, pp. 46–59.

4 J. E. Dennis and R. B. Schnabel, *Numerical Method for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, 1983.

5 G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd ed., Johns Hopkins U.P., 1989.

6 B. R. Hunt and T. M. Cannon, Non-stationary assumptions of Gaussian models of images, *IEEE Trans. Systems Man. and Cybernet.*, Dec. 1976, pp. 876–881.

7 A. Margalit, Adaptive Detection of Stationary Optical and IR Targets Using Correlated Scenes, Ph.D. Dissertation, Univ. of Southern California, Nov. 1984.

8 A. Margalit, I. S. Reed, and R. M. Gagliardi, Adaptive optical targets using correlated images, *IEEE Trans. Aerospace Electron. Systems* 21:394–405 (May 1985).

9 I. S. Reed, R. M. Gagliardi, H. M. Shoa, and A. Margalit, Optical Target Detection Using Dual Scene Observations, USCEE Report, May 1983.

10 I. S. Reed and X. Yu, Adaptive multiple-band CFAR detection of an optical pattern with unknown spectral distribution, *IEEE Trans. Acoust. Speech Signal Process.* 38:1760–1770 (Oct. 1990).

11 J. A. Rice, *Mathematical Statistics and Data Analysis*, Wadsworth & Brooks, Pacific Grove, Calif., 1988.

12 G. W. Stewart, Collinearity and least squares regression, *Statist. Sci.* 2:69–100 (1987).

13 J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon, Oxford, 1965.

Received 15 August 1991; final manuscript accepted 31 January 1992