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The critical exponent for an ordinary fractional differential problem

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ABSTRACT

We consider the Cauchy problem for an ordinary fractional differential inequality with a polynomial nonlinearity with variable coefficient. A nonexistence result is proved and the critical exponent separating existence from nonexistence is found. This is proved in the absence of any regularity assumptions.

1. Introduction

During the last three decades the Cauchy problem

$$\begin{cases}(D_{+}^{\alpha}u)(t)=f[t,u(t)], & t>0, 0<\alpha<1\\ (D_{+}^{\alpha-1}u)(0^{+})=b\end{cases} \tag{1}$$

where D_{+}^{α} is the left-sided Riemann–Liouville fractional derivative has attracted the attention of many researchers and several existence results have appeared. These results have been proved in general in weighted spaces of continuous functions. It has been shown that when the right hand side $f[t,u(t)]$ is regular (with a power singularity at 0) then the solution is also regular (with a power singularity at 0 too) (see [1–7]).

In contrast, one cannot find results on nonexistence for ordinary fractional differential equations and inequalities. In the case of regularity, one can prove nonexistence of global solutions by imitating the existing arguments for partial fractional differential equations and inequalities (see [8–12]). However, in the case of lack of regularity these arguments do not hold. Indeed, the proofs in these papers [6–10] are based on the test function method developed in [13] and the integration by parts formula

$$\int_a^b \varphi(x)(D_{a+}^{\alpha}\psi)(x)\mathrm{d}x=\int_a^b \psi(x)(D_{b-}^{\alpha}\varphi)(x)\mathrm{d}x \tag{2}$$

where D_{b-}^{α} is the right-sided Riemann–Liouville fractional derivative, which is valid for sufficiently good functions: $\varphi\in I_{-}^{\alpha}(L_p)$ and $\psi\in I_{+}^{\alpha}(L_q)$, $p\geq 1$, $q\geq 1$, $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ ($p\neq 1$ and $q\neq 1$ in the case $\frac{1}{p}+\frac{1}{q}=1+\alpha$); see [4,7]. Here,

$$I_{-}^{\alpha}(L_p):=\left\{f: f=I_{-}^{\alpha}g, g\in L_p(a,b)\right\}$$

and

$$I_{+}^{\alpha}(L_q):=\left\{f: f=I_{+}^{\alpha}g, g\in L_q(a,b)\right\},$$

(see the definitions of I_{-}^{α} and I_{+}^{α} below, in the next section). Simple sufficient conditions are: $\varphi(x)$, $\psi(x)\in C([a,b])$ and $(D_{b-}^{\alpha}\varphi)(x)$, $(D_{a+}^{\alpha}\psi)(x)$ exist at every point $x\in[a,b]$ and are continuous (see [7]).

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In this paper, we would like to investigate the nonexistence of global nontrivial solutions for equations and inequalities of the form (1), with $f[t, u(t)] = t^{\beta} |u|^m$, in the space of integrable functions. We find an interval of values for m for which no nontrivial solutions can exist for all time. The upper bound of this interval is shown to be included in this set. It is also shown that this upper bound is sharp in the sense that beyond this bound we have existence of solutions. Therefore this bound is a real "critical exponent".

The paper is organized as follows. In the next section we present the definitions of the fractional integral and the fractional derivative in the Riemann–Liouville sense. An integration by parts result is also presented. Section 3 is devoted to our result on nonexistence of solutions. In Section 4 we establish the sharpness of our bound.

2. Preliminaries

In this section we will prepare some notation and definitions needed in Sections 3 and 4. The Riemann–Liouville left-sided fractional integral and the right-sided fractional integral are given by

$$(I^{\alpha}_{a+}f)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)\mathrm{d}s}{(t-s)^{1-\alpha}}, \quad t > a, \alpha > 0 \tag{3}$$

and

$$(I^{\alpha}_{b-}f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)\mathrm{d}s}{(s-t)^{1-\alpha}}, \quad t < b, \alpha > 0 \tag{4}$$

where $b \in \mathbf{R}^+ \cup \{+\infty\}$, respectively. Here $\Gamma(\alpha) := \int_0^\infty s^{\alpha-1}\mathrm{e}^{-s}\mathrm{d}s$ is the usual Gamma function. For $0 < \alpha < 1$, the Riemann–Liouville left-sided fractional derivative is defined by

$$(D^{\alpha}_{+}f)(t) := \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} f(s)\mathrm{d}s, \quad t > 0. \tag{5}$$

See [4,7,14] for more on fractional integrals and fractional derivatives.

We have the following fractional integration by parts formula.

Lemma 1 *(See [4,7]). Let $\alpha > 0, p \geq 1, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\varphi(x) \in L_p(a, b)$ and $\psi(x) \in L_q(a, b)$, then*

$$\int_a^b \varphi(x)(I^{\alpha}_{a+}\psi)(x)\mathrm{d}x = \int_a^b \psi(x)(I^{\alpha}_{b-}\varphi)(x)\mathrm{d}x. \tag{6}$$

Finally, we define the space

$$\mathbf{L}^{\alpha}(a, b) := \left\{ f \in L(a, b) : D^{\alpha}_{a+}f \in L(a, b) \right\}, \quad \alpha > 0. \tag{7}$$

It is established in [4] that a (unique, in the case where the right hand side is Lipschitzian) solution exists for problem (1) in the space $\mathbf{L}^{\alpha}$ when $f[t, u(t)] \in L(a, b)$ (see the discussion below). However, this last assumption on $f[t, u(t)]$ is not necessary.

3. Nonexistence result

In this section, we will state and prove our theorem. We consider the Cauchy problem

$$\begin{cases} (D^{\alpha}_{+}u)(t) \geq t^{\beta} |u(t)|^m, & t > 0, m > 1, 0 < \alpha < 1 \\ (D^{\alpha-1}_{+}u)(0^+) = b \in \mathbf{R} \end{cases} \tag{8}$$

where D^{α}_{+} is defined in (5). Nonexistence of solutions is investigated in the space $\mathbf{L}^{\alpha}$ defined in (7).

Theorem 1. *Assume that $\beta > -\alpha$ and $1 < m \leq \frac{\beta+1}{1-\alpha}$. Then, problem (8) does not admit global nontrivial solutions when $b \geq 0$.*

Proof. Suppose, on the contrary, that a nontrivial solution exists for all time $t > 0$. Let $\varphi(t) \in C^1([0, \infty))$ be a test function satisfying: $\varphi(t) \geq 0$, φ is nonincreasing and such that

$$\varphi(t) := \begin{cases} 1, & t \in [0, T/2] \\ 0, & t \in [T, \infty) \end{cases} \tag{9}$$

for some $T > 0$. From the equation in (8) we have

$$\int_0^\infty (D^{\alpha}_{+}u)(t)\varphi(t)\mathrm{d}t \geq \int_0^\infty t^{\beta} |u(t)|^m \varphi(t)\mathrm{d}t$$

and from the definition (5),

$$\frac{1}{\Gamma(1-\alpha)} \int_0^\infty \varphi(t) \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} u(s)\mathrm{d}s \right) \mathrm{d}t \geq \int_0^\infty t^{\beta} |u(t)|^m \varphi(t)\mathrm{d}t. \tag{10}$$

An integration by parts in (10) yields

$$\frac{1}{\Gamma(1-\alpha)}\left[\varphi(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\right]_0^T - \frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\mathrm{d}t$$
$$\geq \int_0^T t^\beta \left|u(t)\right|^m \varphi(t)\mathrm{d}t$$

or

$$-(I_+^{1-\alpha}u)(0^+) - \frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\mathrm{d}t \geq \int_0^T t^\beta \left|u(t)\right|^m \varphi(t)\mathrm{d}t. \tag{11}$$

Notice first that

$$\begin{aligned}\frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\mathrm{d}t &\leq \frac{1}{\Gamma(1-\alpha)}\int_0^T \left|\varphi'(t)\right|\int_0^t (t-s)^{-\alpha}\frac{\varphi(s)^{1/m}}{\varphi(s)^{1/m}}\left|u\right|(s)\mathrm{d}s\mathrm{d}t\\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_0^T \frac{\left|\varphi'(t)\right|}{\varphi(t)^{1/m}}\int_0^t (t-s)^{-\alpha}\varphi(s)^{1/m}\left|u\right|(s)\mathrm{d}s\mathrm{d}t\\
&\leq \frac{1}{\Gamma(1-\alpha)}\int_{T/2}^T \frac{\left|\varphi'(t)\right|}{\varphi(t)^{1/m}}\int_0^t (t-s)^{-\alpha}\varphi(s)^{1/m}\left|u\right|(s)\mathrm{d}s\mathrm{d}t\end{aligned} \tag{12}$$

and second that, by the definitions (3), (4) and Lemma 1 together with (12), we may write

$$\frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\mathrm{d}t \leq \int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)(t)\varphi(t)^{1/m}\left|u\right|(t)\mathrm{d}t.$$

Next, we multiply by $t^{\beta/m}.t^{-\beta/m}$ inside the integral in the right hand side and take into account the fact that for $-\alpha < \beta < 0$ we have $t^{-\beta/m} < T^{-\beta/m}$ and for $\beta > 0$ we have $t^{-\beta/m} < 2^{\beta/m}T^{-\beta/m}$ (because $T/2 < t < T$); that is,

$$t^{-\beta/m} < \max\{1, 2^{\beta/m}\}T^{-\beta/m}.$$

It appears that

$$\frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\mathrm{d}t \leq \max\{1, 2^{\beta/m}\}T^{-\beta/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)(t)t^{\beta/m}\varphi(t)^{1/m}\left|u\right|(t)\mathrm{d}t. \tag{13}$$

A simple application of the Young inequality with m and m' such that $\frac{1}{m}+\frac{1}{m'}=1$ gives

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)}\int_0^T \varphi'(t)\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s\mathrm{d}t\\
&\quad\leq \frac{1}{m}\int_{T/2}^T t^\beta\varphi(t)\left|u\right|^m(t)\mathrm{d}t + \frac{\left(\max\{1, 2^{\beta/m}\}\right)^{m'}}{m'}T^{-\beta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\mathrm{d}t\\
&\quad\leq \frac{1}{m}\int_0^T t^\beta\varphi(t)\left|u\right|^m(t)\mathrm{d}t + \frac{\left(\max\{1, 2^{\beta/m}\}\right)^{m'}}{m'}T^{-\beta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\mathrm{d}t.\end{aligned} \tag{14}$$

The first term in the right hand side of (14) can be subtracted from the right hand side of (11) and we obtain

$$-(I_+^{1-\alpha}u)(0^+) + \frac{\left(\max\{1, 2^{\beta/m}\}\right)^{m'}}{m'}T^{-\beta m'/m}\int_{T/2}^T \left(I_{T-}^{1-\alpha}\frac{\left|\varphi'\right|}{\varphi^{1/m}}\right)^{m'}(t)\mathrm{d}t \geq \left(1-\frac{1}{m}\right)\int_0^T t^\beta\left|u(t)\right|^m\varphi(t)\mathrm{d}t.$$

If $b \geq 0$ and we make the change of variables $\sigma = t/T$, we find

$$\begin{aligned}&\frac{1}{m'}\int_0^T t^\beta\left|u(t)\right|^m\varphi(t)\mathrm{d}t\\
&\quad\leq \frac{\left(\max\{1, 2^{\beta/m}\}\right)^{m'}}{m'}T^{-\beta m'/m}\int_{1/2}^1 \left(\frac{1}{\Gamma(1-\alpha)}\int_{\sigma T}^T (s-\sigma T)^{-\alpha}\frac{\left|\varphi'(s)\right|}{\varphi(s)^{1/m}}\mathrm{d}s\right)^{m'} T\mathrm{d}\sigma.\end{aligned}$$

Another change of variables $s = rT$ gives

$$\int_0^T t^\beta |u(t)|^m \varphi(t)\mathrm{d}t \le \frac{\left(\max\{1, 2^{\beta/m}\}\right)^{m'}}{\Gamma^{m'}(1-\alpha)} T^{1-\alpha m'-\beta m'/m} \int_{1/2}^1 \left(\int_\sigma^1 (r-\sigma)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} \mathrm{d}r\right)^{m'} \mathrm{d}\sigma. \tag{15}$$

At this point it is clear that we may assume that the integral term in the right hand side of (15) is bounded, that is

$$\int_{1/2}^1 \left(\int_\sigma^1 (r-\sigma)^{-\alpha} \frac{|\varphi'(r)|}{\varphi(r)^{1/m}} \mathrm{d}r\right)^{m'} \mathrm{d}\sigma \le C_1$$

for some positive constant C_1, for otherwise we consider $\varphi^\lambda(t)$ with some sufficiently large λ. Therefore

$$\int_0^T t^\beta |u(t)|^m \varphi(t)\mathrm{d}t \le C_2 T^{1-\alpha m'-\beta m'/m}$$

with $C_2 := \frac{\left(\max\{1,2^{\beta/m}\}\right)^{m'} C_1}{\Gamma^{m'}(1-\alpha)}$. Suppose that $m < \frac{\beta+1}{1-\alpha}$; then $1 - \alpha m' - \beta m'/m < 0$ and thus

$$\lim_{T\to\infty} \int_0^T t^\beta |u(t)|^m \varphi(t)\mathrm{d}t = 0.$$

We reached a contradiction since the solution is not supposed to be trivial.

In the case $m = \frac{\beta+1}{1-\alpha}$, the relation (15) ensures that

$$\lim_{T\to\infty} \int_0^T t^\beta |u(t)|^m \varphi(t)\mathrm{d}t \le C_2. \tag{16}$$

We use the Hölder inequality in the right hand side of (13) to find

$$\left(\frac{T}{2}\right)^{-\beta/m} \int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}}\right)(t) t^{\beta/m} \varphi(t)^{1/m} |u|(t)\mathrm{d}t$$

$$\le \left(\frac{T}{2}\right)^{-\beta/m} \left[\int_{T/2}^T \left(I_{T-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}}\right)^{m'} (t)\mathrm{d}t\right]^{\frac{1}{m'}} . \left[\int_{T/2}^T t^\beta \varphi(t) |u|^m (t)\mathrm{d}t\right]^{\frac{1}{m}}.$$

Therefore

$$\int_0^T t^\beta |u(t)|^m \varphi(t)\mathrm{d}t \le C_3 \left[\int_{T/2}^T t^\beta \varphi(t) |u|^m (t)\mathrm{d}t\right]^{\frac{1}{m}}$$

for some positive constant C_3, with

$$\lim_{T\to\infty} \int_{T/2}^T t^\beta \varphi(t) |u|^m (t)\mathrm{d}t = 0$$

due to the convergence of the integral in (16). This is again a contradiction. The proof of the theorem is complete. □

4. Sharpness of the bound

In [4] the following equation has been treated in detail:

$$(D_{a+}^\alpha u)(t) = \lambda (t-a)^\beta [u(t)]^m, \quad t > a, m > 0, m \ne 1$$

for $\alpha > 0$, $\lambda, \beta \in \mathbf{R}, \lambda \ne 0$. This equation has the solution

$$u(t) = \left[\frac{\Gamma\left(\frac{\alpha+\beta}{m-1}+1\right)}{\lambda\Gamma\left(\frac{\alpha m+\beta}{m-1}+1\right)}\right]^{\frac{1}{m-1}} (t-a)^{\frac{\alpha+\beta}{1-m}}.$$

In this case the right hand sides becomes

$$f[t, u(t)] = \lambda \left[\frac{\Gamma\left(\frac{\alpha+\beta}{m-1}+1\right)}{\lambda\Gamma\left(\frac{\alpha m+\beta}{m-1}+1\right)}\right]^{\frac{m}{m-1}} (t-a)^{\frac{\alpha m+\beta}{1-m}}.$$

It is then clear that $f[t, u(t)]$ is integrable if $\frac{\alpha m+\beta}{1-m} > -1$. Since $m > 1$, we have $\frac{\alpha+\beta}{1-m} > \frac{\alpha m+\beta}{1-m} > -1$. Therefore $u(t)$ also is integrable. This is exactly the condition $m > \frac{\beta+1}{1-\alpha}$. Notice also that the initial condition is zero in this example. That is, $b = 0$. Hence the bound $m_c = \frac{\beta+1}{1-\alpha}$ is a critical exponent.

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