ADVANCES IN APPLIED MATHEMATICS 4, 39–45 (1983)

A Nonlinear Problem Having a Continuous Locus of Singular Points and No Multiple Solutions

MOSTAFA A. ABDELKADER

25 Sh. Champollion, Alexandria, Egypt

The boundary-value problem $\epsilon z'' = (z^2 - t^2)z'$, $\epsilon > 0$, $z(-1) = \alpha$, $z(0) = \beta$, $t \in [-1, 0]$, has been shown to have a solution, and moreover, depending on the choice of α and β, multiple solutions to it exist. We consider the more general equation $f(z, t)z'' = (z^r - t^s)z'$ for a particular non-negative function $f(z, t)$, and integrate the equation exactly. Depending on α and β, we find that either there are no solutions, or that only unique solutions exist. The conclusion is that the presence of a continuous locus of singular points, given by $z^r = t^s$, does not necessarily produce multiple solutions.

1. INTRODUCTION

The class of nonlinear boundary-value problems

$$f(z,t)z'' = (z^2 - t^2)z', \quad t \in [-1,0], \tag{1}$$

$$z(-1) = \alpha, \quad z(0) = \beta, \tag{2}$$

has continuous branches of singular points, depending upon the choice of α and β, for the coefficient of the first derivative in (1) vanishes along the line $\pm t$ in the (t, z) plane. The special problem for which (1) is

$$\epsilon z'' = (z^2 - t^2)z', \tag{3}$$

where ϵ is a positive constant that may tend to zero, has been treated in [1–3] by various approximation methods (since no exact solution to (3) has been found), in which use was made of the equation

$$z' = z'(\tau)\exp\left\{\frac{1}{\epsilon}\int_\tau^t \left[z(s)^2 - s^2\right] ds\right\},$$

where $\tau \in [-1, 0)$, as well as of the equation

$$z' = z'(\tau) + \frac{1}{3\epsilon}\left\{z(t)^3 - z(\tau)^3\right\} - \frac{1}{\epsilon}\int_\tau^t s^2 z'(s)\, ds.$$

0196-8858/83 $7.50

We note in passing that we also have the nonobvious equation

$$z' = \frac{z^3}{3\epsilon} + \exp\left(-\frac{t^3}{3\epsilon}\right)\left\{C - \frac{1}{3\epsilon^2}\int_\tau^t s^2 z(s)^3 \exp\left(\frac{s^3}{3\epsilon}\right) ds\right\},$$

where

$$C = \exp\left(\frac{\tau^3}{3\epsilon}\right)\left\{z'(\tau) - \frac{z(\tau)^3}{3\epsilon}\right\}.$$

The existence of a solution to (3), (2) was proved in [1]; if $0 \leqslant \alpha \leqslant \beta$, or $\beta \leqslant \alpha \leqslant 0$, the solution is unique. It was shown in [2] and proved in [3] that, for other ranges of α and β, the problem (3), (2) has multiple solutions which are characterized by the number of "turning points," i.e., those points at which the solution curves cross the line $\pm t$. The more general case in which the locus of singular points is an arc may be exemplified by

$$f(z,t)z'' = (z^r - t^s)z', \tag{4}$$

where r and s are such that the locus of singular points, given by $z^r = t^s$, has at least one real branch for $t \in [-1, 0]$.

It might be thought that the presence of a continuous locus of singular points must engender multiple solutions for some ranges of values of α and β. So, it is of interest to determine whether multiplicity of the solutions is a general phenomenon associated with the problem (4), (2) when $f(z, t)$ is not identically a constant, and $r \neq s$. We choose a certain functional form for $f(z, t) \geqslant 0$, obtain the exact solution of the problem (4), (2) and, depending on the choice of α and β, find that:

(i) a solution may not exist,

(ii) a unique solution may exist,

(iii) there are no multiple solutions.

These results show that the existence of a continuous locus of singular points does not necessarily produce multiple solutions to the boundary-value problem, so that the existence of such solutions to (3), (2) must be considered as an isolated phenomenon. This calls for an investigation of the conditions guaranteeing the existence of multiple solutions for the problem (4), (2).

2. The Boundary-Value Problem and its Solution

We consider the particular case of (4) given by

$$-tz^r z'' = (1-s)\{z^r - t^s\}z', \quad t \in [-1, 0], \tag{5}$$

$$z(-1) = \alpha, \quad z(0) = \beta, \tag{6}$$

where

$$1 < r = 2p/q, \quad 0 < s = 2P/Q < 1,$$

with p, P positive integers, and q, Q positive odd integers, so that $f(z, t) \geqslant 0$ on $[-1, 0]$, and a real locus of singular points exists.

In order to obtain the exact solution of (5), we first transform it into a first-order equation. We make the substitutions

$$x = zt^{\sigma}, \qquad \sigma = -s/r, \qquad 2y = t\frac{dx}{dt}, \tag{7}$$

and obtain the nonlinear differential equation

$$\frac{dy}{dx} = g(x) + \frac{G(x)}{y}, \tag{8}$$

where

$$g(x) = \frac{1}{2}(2\sigma + s) + \frac{a}{2}x^{-r}, \quad a = 1 - s,$$
$$G(x) = -\frac{\sigma}{4}(\sigma + s)x - \frac{a\sigma}{4}x^{1-r}.$$

The general solution of (8) is given by

$$2a(1 - r)x^{-r}w^m + s\int w^n\left\{w - \frac{n}{2}\right\}^{-1} dw + sC = 0, \tag{9}$$

where

$$a\left\{w - \frac{n}{2}\right\} = y(x)x^{r-1} - \frac{\sigma}{2}x^r, \tag{10}$$
$$m = \frac{r}{r-1} > 1, \quad n = \frac{1}{r-1} > 0,$$

and C is a constant of integration.

From (9) we have

$$x^r = \frac{bw^m}{I(w) + C}, \tag{11}$$

where

$$I(w) = \int w^n\left\{w - \frac{n}{2}\right\}^{-1} dw, \quad b = \frac{2a}{ns}, \tag{12}$$

and from (7) and (10) we get

$$d(x^r)/d(\log|t|) = 2ar\left\{w - \frac{n}{2}\right\} - sx^r. \tag{13}$$

Integrating (13) parametrically, using (11), we ultimately obtain

$$t = D^{1/s}\{I(w) + C\}^{1/s}, \tag{14}$$

where D is a constant of integration.

From (11) and (14), using (7), we get

$$w = cz^{r-1}, \quad c^m = 1/bD, \tag{15}$$

so that (14) becomes

$$t = \left\{\frac{I(w) + C}{bc^m}\right\}^{1/s}, \tag{16}$$

where c, C are constants of integration.

Differentiating (16), using (15) and (12), we get

$$t^{1-s}z' = 2a\left\{c - \frac{n}{2}z^{1-r}\right\}, \tag{17}$$

which by differentiation yields (5); this verifies the solution. From (15) and (17) we have

$$t^{1-s}z^{r-1}z' = 2a\left\{w - \frac{n}{2}\right\},$$

so that if $z' = 0$, we have $w - (n/2) = 0$, and from (12), $I(w)$ has a pole, and t is infinite. Hence, $|z'| > 0$ on $[-1, 0]$, as was found for (3) in [2] when $\alpha \neq \beta$; but, whereas when $\alpha = \beta$ it was considered in [3] that (3) has the unique solution $z(t) \equiv \alpha$, for (5) no such solution exists, since $z' = 0$ is impossible. Thus, for $\alpha = \beta$, the problem (5), (6) has no solution.

Applying now the boundary conditions (6) to (16), we find for the solution of (5), provided that $\alpha \neq \beta$, the inverse function

$$t = -\left\{\frac{I(w) - I(w_0)}{I(w_1) - I(w_0)}\right\}^{1/s}, \tag{18}$$

where w is given by (15), the function $I(w)$ by (12),

$$w_1 = c\alpha^{r-1}, \quad w_0 = c\beta^{r-1},$$

and the constant c is a non-zero root of the transcendental equation

$$bc^m = I(w_1) - I(w_0), \tag{19}$$
$$b = \frac{2}{s}(1 - s)(r - 1) > 0;$$

we note that $c^m \geqslant 0$ for all c. Any non-zero root of (19) is a function of r, s, α, and β.

3. Necessary Existence Conditions

A solution (18) exists if and only if α and β are such that $I(w_1) > I(w_0)$, and that, as t increases from $t = -1$ to $t = 0$, the function $I(w)$ decreases monotonically from $I(w_1)$ to $I(w_0)$.

From (12) and (19) we have

$$bc^m = \int_{w_0}^{w_1} w^n \left\{ w - \frac{n}{2} \right\}^{-1} dw, \tag{20}$$

provided that the integrand is continuous on $[w_0, w_1]$, i.e., if $w = n/2$ does not belong to this interval. We see from (12) that as w increases, the function $I(w)$ decreases only on the interval $(0, n/2)$, and is otherwise an increasing function. It rises from $-\infty$ at $w = -\infty$, attains a (positive or negative) local maximum at $w = 0$, drops to $-\infty$ at $w = n/2$, and rises to ∞ as $w \to \infty$.

Accordingly, for a solution (18) to exist, it is necessary that one of the following inequalities hold:

$$\frac{n}{2} < w_0 < w_1,$$
$$0 \leqslant w_1 < w_0 < \frac{n}{2},$$
$$w_0 < w_1 \leqslant 0.$$

In case (19) has a positive root c, and since n is the ratio of positive odd integers, the boundary values α and β must satisfy one of the following inequalities:

$$0 < (n/2c)^n < \beta < \alpha, \tag{21}$$
$$0 \leqslant \alpha < \beta < (n/2c)^n, \tag{22}$$
$$\beta < \alpha \leqslant 0. \tag{23}$$

In case (19) has a negative root c, then α and β must satisfy one of the following inequalities:

$$\alpha < \beta < (n/2c)^n < 0, \tag{24}$$

$$(n/2c)^n < \beta < \alpha \leqslant 0, \tag{25}$$

$$0 \leqslant \alpha < \beta. \tag{26}$$

We see that when α and β have different signs, a solution to the boundary-value problem does not exist.

4. Uniqueness

The transcendental equation (19) for c may be written

$$L(c) = R(c), \tag{27}$$

where

$$L(c) = bc^m, \quad R(c) = I(\alpha^{r-1}c) - I(\beta^{r-1}c),$$

and we have

$$\frac{dL}{dc} = bmc^n, \tag{28}$$

$$\frac{dR}{dc} = c^n\left\{\frac{\alpha}{c - (n/2\alpha^{r-1})} - \frac{\beta}{c - (n/2\beta^{r-1})}\right\}. \tag{29}$$

The function $R(c)$ has two stationary points, one at $c = 0$, and the second at

$$c = \frac{n(\alpha\beta^{1-r} - \beta\alpha^{1-r})}{2(\alpha - \beta)}.$$

Graphs of $R(c)$ may be easily sketched for both signs of c, α, and β, and for the relative orderings of α and β, given in (21)–(26); from these graphs, one deduces the following results:

(a) For inequality (21), a positive root of (27) satisfies

$$(n/2\beta^{r-1}) < c < \infty.$$

As c increases from $(n/2\beta^{r-1})$, $R(c)$ decreases steadily from ∞, attains a relative minimum, and then increases steadily at a rate asymptotically less than that of $L(c)$, as shown by (28) and (29). Hence, $L(c)$ can cross $R(c)$ only at one point.

(b) For inequality (22), a positive root of (27) satisfies

$$0 < c < \left(n/2\beta^{r-1}\right),$$

and since $R(c)$ rises steadily from zero at $c = 0$ to ∞ at $c = n/2\beta^{r-1}$, $L(c)$ can cross it only at one point.

(c) For inequality (23), both $L(c)$ and $R(c)$ tend steadily to ∞ as $c \to \infty$, and they can cross only at one point.

(d) For inequality (24), a negative root of (27) satisfies

$$c < \left(n/2\beta^{r-1}\right) < 0.$$

As c decreases from $(n/2\beta^{r-1})$, $R(c)$ decreases steadily from ∞, attains a relative minimum, and then increases steadily at a rate asymptotically less than that of $L(c)$. Hence, $L(c)$ can cross $R(c)$ only at one point.

(e) For inequality (25), a negative root of (27) satisfies

$$\left(n/2\beta^{r-1}\right) < c < 0,$$

and since, as c increases from $(n/2\beta^{r-1})$, $R(c)$ decreases steadily from ∞ to zero at $c = 0$, $L(c)$ can cross it only at one point.

(f) For inequality (26), both $L(c)$ and $R(c)$ tend steadily to ∞ as $c \to -\infty$, and they can cross only at one point.

To summarize, we have now shown that, in all admissible cases of α and β, there is at most one root to (27), and hence that a solution of the boundary-value problem (5), (6) either does not exist, or that it exists and is unique, i.e., no multiple solutions exist, contrary to what was found for the problem (3), (2).

REFERENCES

1. F. A. HOWES AND S. V. PARTER, A model nonlinear problem having a continuous locus of singular points, *Stud. Appl. Math.* **58** (1978), 249–262.
2. G. KEDEM, S. V. PARTER, AND M. STEUERWALT, The solutions of a model nonlinear singular perturbation problem having a continuous locus of singular points, *Stud. Appl. Math.* **63** (1980), 119–146.
3. N. KOPELL AND S. V. PARTER, A complete analysis of a model nonlinear singular perturbation problem having a continuous locus of singular points, *Adv. Appl. Math.* **2** (1981), 212–238.