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On the Asymptotics of the Dirichlet Eigenfunctions of $\Delta^2 + q$ on the Square

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We obtain an asymptotic formula for the eigenfunctions of

$$\Delta^2 u + q(x)u = \lambda u, \qquad x \in D = (0, \pi) \times (0, \pi),$$

with boundary conditions

$$u|_{\partial D} = \Delta u|_{\partial D} = 0.$$

1. INTRODUCTION

The asymptotics of the eigenfunctions of partial differential operators cannot be satisfactory computed by the so-called WKB method. Here we introduce an alternative approach, inspired by Yu. Karpeshina's article [5], for computing the asymptotics of the eigenfunctions of the problem

$$\begin{gathered} \Delta^2 u + q(x_1, x_2)u = \lambda u, \qquad (x_1, x_2) \in D = (0, \pi) \times (0, \pi), \\ u|_{\partial D} = \Delta u|_{\partial D} = 0, \end{gathered} \tag{1}$$

where q is in $L^\infty(D)$. These asymptotics are needed for attacking inverse problems related to (1), which is the long term goal of this work. We remark that (1) has potential applications in elasticity, since it involves the biharmonic operator Δ^2.

Previous literature related to eigenfunction asymptotics of multidimensional Schrödinger-type operators includes the impressive works of O.

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Hald and J. McLaughlin (see, for example, [3]) on the reconstruction from nodal sets, and of Yu. Karpeshina (see, for example, [5]) on the proof of the Bethe–Sommerfeld conjecture for nonsmooth potentials. In [3] the authors compute the asymptotics for the Dirichlet eigenfunctions of $-\Delta + q$ on a rectangle. Their formula is valid for "most" eigenvalues (i.e., all except a set of zero density), while our formula (see the theorem at the end of the next section) is true for all eigenvalues, but it gives asymptotics of eigenspaces rather than independent eigenfunctions, because of the high multiplicities of the eigenvalues of Δ^2. Karpeshina's work on the other hand is concerned with the case of periodic (or more generally Floquet) boundary conditions.

Let $\{\lambda_n\}_{n=1}^{\infty}$, where it is assumed that $\lambda_n \leq \lambda_{n+1}$ for all n, be the spectrum of (1) and $\phi_n(x_1, x_2)$, $n = 1, 2, 3, \ldots,$ the corresponding normalized eigenfunctions (the problem is, of course, self-adjoint). In the *unperturbed case*, namely when $q(x_1, x_2) \equiv 0$, we denote the spectrum by $\{\mu_n\}_{n=1}^{\infty}$ (again we assume that $\mu_n \leq \mu_{n+1}$ for all n) and the nth eigenfunction by $\psi_n(x_1, x_2)$. In fact, the eigenfunctions ψ_n of the unperturbed case are in one to one correspondence with the set of all ordered pairs (k_1, k_2), where k_1 and k_2 are strictly positive integers, so that the pair (k_1, k_2) corresponds to the eigenfunction

$$\psi_n(x_1, x_2) .= \frac{2}{\pi} \sin(k_1 x_1) \sin(k_2 x_2). \tag{2}$$

The eigenvalue corresponding to ψ_n is, of course,

$$\mu_n = \left(k_1^2 + k_2^2\right)^2 \tag{3}$$

(e.g., $\mu_1 = 4$, $\mu_2 = \mu_3 = 25$, $\mu_4 = 64$, $\mu_5 = \mu_6 = 100$, etc.). The asymptotics of μ_n are (see [1, Sect. VI.4])

$$\mu_n = \left[\frac{4}{\pi} n + O\left(\sqrt{n}\,\right)\right]^2 = \frac{16}{\pi^2} n^2 + O(n^{3/2}) \tag{4}$$

(in fact, one has the better estimate $\sqrt{\mu_n} = (4/\pi)n + O(n^{\alpha})$, where $\alpha > 35/108$—see [2, Sect. 2.7]).

Suppose that for some n we have $\mu_{n-1} < \mu_n = \mu_{n+1} = \cdots = \mu_{n+\nu-1} < \mu_{n+\nu}$, with $\mu_n = N^2$. Then $\nu = \nu(N^2) = \nu(\mu_n)$ (the multiplicity of the eigenvalue N^2) is the number of ways that N can be written as a sum of squares of two strictly positive integers (e.g., $\nu(25) = 2$, $\nu(25^2) = 2$). It follows that (see [4, Sect. 18.7]), for every $\varepsilon > 0$ there is a $c > 0$ such that

$$\nu(\mu_n) < c 2^{(1+\varepsilon)(\ln n / \ln \ln n)}. \tag{5}$$

This implies the slightly weaker estimate

$$\nu(\mu_n) = O(n^\delta), \qquad \text{for any } \delta > 0.$$

Usually authors avoid multiple eigenvalues, but for our analysis they are rather helpful! The main reason is that, on one hand (5) gives a quite satisfactory bound for the multiplicity, while on the other hand, if $\mu_n \neq \mu_{n+1}$, then by (3) and (4)

$$\mu_{n+1} - \mu_n \geq 2\sqrt{\mu_n} + 1 = \frac{8}{\pi}n + O(\sqrt{n}). \tag{6}$$

A standard minimax argument (see [1, Sect. I.4]) implies that the eigenvalues of (1) satisfy

$$|\lambda_n - \mu_n| \leq \|q\|_\infty. \tag{7}$$

2. THE EIGENFUNCTION ASYMPTOTICS

Let $G(x, y; \lambda)$ be the Green's function associated to (1), where we have set $x = (x_1, x_2)$ and $y = (y_1, y_2)$. By definition, $G(x, y; \lambda)$ is the integral kernel of the operator $(L - \lambda)^{-1}$, therefore we have the eigenfunction expansion

$$G(x, y; \lambda) = \sum_{j=1}^{\infty} \frac{\phi_j(x)\phi_j(y)}{\lambda_j - \lambda}.$$

In particular, if Γ is a simple closed curve in the complex plane that encloses $\lambda_n, \lambda_{n+1}, \dots, \lambda_{n+l}$, and no other eigenvalue of (1), then

$$\frac{1}{2\pi i}\oint_\Gamma G(x, y; \lambda)\, d\lambda = \phi_n(x)\phi_n(y) + \phi_{n+1}(x)\phi_{n+1}(y) + \cdots + \phi_{n+l}(x)\phi_{n+l}(y). \tag{8}$$

In the unperturbed case, we denote the Green's function by $K(x, y; \lambda)$. Of course,

$$K(x, y; \lambda) = \sum_{j=1}^{\infty} \frac{\psi_j(x)\psi_j(y)}{\mu_j - \lambda}. \tag{9}$$

This series is dominated by

$$\frac{4}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{|\mu_j - \lambda|}$$

which, by (4), converges as long as $\lambda \neq \mu_j$, for all j. Hence dominated convergence implies the following:

PROPOSITION 1. *Given any $\varepsilon > 0$, there is a constant $M = M(\varepsilon)$ such that, if $|\lambda - \mu_j| \geq M$, for all j, then $|K(x, y; \lambda)| \leq \varepsilon$ (here λ is a complex number).*

It is easy to see that $G(x, y; \lambda)$ is the unique solution (as long as $\lambda \neq \lambda_n$ and $\lambda \neq \mu_n$) of the integral equation

$$G(x, y; \lambda) = K(x, y; \lambda) - \int_D K(x, \xi; \lambda) q(\xi) G(\xi, y; \lambda)\, d\xi. \quad (10)$$

By iterating (10) we obtain a formal (perturbation) series for $G(x, y; \lambda)$, namely

$$G(x, y; \lambda) = \sum_{m=0}^{\infty} (-1)^m G_m(x, y; \lambda), \quad (11)$$

where

$$G_0(x, y; \lambda) = K(x, y; \lambda), \quad (12)$$

$$G_m(x, y; \lambda) = \int_D K(x, \xi; \lambda) q(\xi) G_{m-1}(\xi, y; \lambda)\, d\xi, \qquad m \geq 1. \quad (13)$$

Notice that, if $m \geq 1$, (14) implies that $G_m(x, y; \lambda)$ has the expansion

$$G_m(x, y; \lambda) = \int_D \cdots \int_D K(x, \xi^1; \lambda) q(\xi^1) K(\xi^1, \xi^2; \lambda) \cdots$$
$$q(\xi^m) K(\xi^m, y; \lambda)\, d\xi^1 \cdots d\xi^m, \quad (14)$$

where we have set $\xi^j = (\xi_1^j, \xi_2^j)$.

Now, by Proposition 1 we have that, for any $\delta \in (0, 1)$, we can make simultaneously $|K(x, y; \lambda)| \leq 1 - \delta$ and $\pi^2 \|q\|_\infty |K(x, y; \lambda)| \leq 1 - \delta$, by taking $|\lambda - \mu_j| \geq M(\delta)$, for all j. Using this in (14), we obtain

$$|G_m(x, y; \lambda)| \leq (1 - \delta)^{m-1},$$

and hence we have established the following:

PROPOSITION 2. *There is a constant $r > 0$ such that if $|\lambda - \mu_j| \geq r$, for all j, then the series in* (11) *converges absolutely and uniformly in x, y, and λ* (*and therefore it satisfies* (10)).

Next we introduce the following sequence of circles in the complex plane

$$C_n = \{\lambda \in \mathbf{C} : |\lambda - \mu_n| = n\}. \tag{15}$$

If n is sufficiently large, say if $n \geq n_0$, then by (6), C_n encloses exactly $\nu(\mu_n)$ equal eigenvalues of the unperturbed problem ($\nu(\mu_n)$ being, as usual, the multiplicity of μ_n) and, due to (7), exactly $\nu(\mu_n)$ eigenvalues of (1), counting multiplicities. Furthermore, again by (6) we have that

$$\text{if } n \geq n_0, \text{ then } \operatorname{dist}(\mu_j, C_n) > n, \text{ for all } j. \tag{16}$$

Therefore, by Proposition 2, the series in (11) converges absolutely and uniformly in x and y, if $\lambda \in C_n$, $n \geq n_0$ (the convergence is also uniform in λ). Thus, for $n \geq n_0$, (11) gives

$$\frac{1}{2\pi i}\oint_{C_n} G(x, y; \lambda)\, d\lambda = \sum_{m=0}^{\infty} \frac{(-1)^m}{2\pi i}\oint_{C_n} G_m(x, y; \lambda)\, d\lambda.$$

For convenience, let n be such that

$$\mu_{n-1} < \mu_n = \mu_{n+1} = \cdots = \mu_{n+\nu-1} < \mu_{n+\nu}, \tag{17}$$

where $\nu = \nu(\mu_n)$ is the multiplicity of μ_n. Using (8), (9), (11), (12), and the fact C_n encloses exactly $\nu(\mu_n)$ eigenvalues λ_n and exactly $\nu(\mu_n)$ (equal) eigenvalues of the unperturbed problem we get

$$\begin{aligned} &\phi_n(x)\phi_n(y) + \cdots + \phi_{n+\nu-1}(x)\phi_{n+\nu-1}(y) \\ &\quad = \psi_n(x)\psi_n(y) + \cdots + \psi_{n+\nu-1}(x)\psi_{n+\nu-1}(y) \\ &\qquad + \sum_{m=1}^{\infty} \frac{(-1)^m}{2\pi i}\oint_{C_n} G_m(x, y; \lambda)\, d\lambda. \end{aligned} \tag{18}$$

It remains to obtain a bound for the sum in (18).

LEMMA. *For a fixed $\alpha > 0$ we set*

$$f(n) = \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{e^{\alpha \ln k / \ln \ln k}}{|k^2 - n^2|}.$$

Then, as $n \to \infty$,

$$f(n) = O\left(\frac{\ln n}{n} e^{\alpha \ln n / \ln \ln n}\right).$$

Proof. Without loss of generality, $n > 16$. We set

$$f(n) = f_1(n) + f_2(n) + f_3(n),$$

where

$$f_1(n) = \sum_{k=1}^{15} \frac{e^{\alpha \ln k / \ln\ln k}}{n^2 - k^2}, \qquad f_2(n) = \sum_{k=16}^{n-1} \frac{e^{\alpha \ln k / \ln\ln k}}{n^2 - k^2},$$

$$f_3(n) = \sum_{k=n+1}^{\infty} \frac{e^{\alpha \ln k / \ln\ln k}}{k^2 - n^2}$$

(we need f_1 since the quantity $(\ln k)/\ln\ln k$ is increasing in k, as long as $k > e^e$, i.e., $k \geq 16$). It is easy to see that

$$f_1(n) = O\left(\frac{1}{n^2}\right).$$

Now

$$f_3(n) = \frac{e^{\alpha \ln n / \ln\ln n}}{n} \sum_{k=n+1}^{\infty} \frac{e^{\alpha \ln k / \ln\ln k - \alpha \ln n / \ln\ln n}}{(k/n)^2 - 1} \cdot \frac{1}{n}. \tag{19}$$

Next we observe that, if $k > n \geq 16$,

$$0 < \frac{\ln k}{\ln\ln k} - \frac{\ln n}{\ln\ln n} \leq \frac{\ln(k/n)}{\ln\ln k} \leq \frac{\ln(k/n)}{\ln\ln(3k/n)}.$$

Thus, (19) implies

$$f_3(n) < \frac{e^{\alpha \ln n / \ln\ln n}}{n} \sum_{k=n+1}^{\infty} \frac{e^{\ln(k/n)/\ln\ln(3k/n)}}{(k/n)^2 - 1} \cdot \frac{1}{n}. \tag{20}$$

The sum in (20) is a Riemann sum dominated by the integral

$$\int_{1+(1/n)}^{\infty} \frac{e^{\ln x / \ln\ln(3x)}}{x^2 - 1}\, dx = \ln n + O(1).$$

Hence

$$f_3(n) = O\left(\frac{\ln n}{n} e^{\alpha \ln n / \ln\ln n}\right).$$

Finally

$$f_2(n) = \frac{e^{\alpha \ln n / \ln \ln n}}{n} \sum_{k=16}^{n-1} \frac{e^{\alpha \ln k / \ln \ln k - \alpha \ln n / \ln \ln n}}{1 - (k/n)^2} \cdot \frac{1}{n}$$
$$< \frac{e^{\alpha \ln n / \ln \ln n}}{n} \sum_{k=16}^{n-1} \frac{1}{1 - (k/n)^2} \cdot \frac{1}{n},$$

thus

$$f_2(n) < \frac{e^{\alpha \ln n / \ln \ln n}}{n} \int_0^{1-(1/n)} \frac{dx}{1 - x^2} = \frac{e^{\alpha \ln n / \ln \ln n}}{n} \ln n$$

and this finishes the proof. ∎

The lemma, together with (5), implies that if $\lambda \in C_n$, where C_n is given by (15), then, for any $\varepsilon > 0$, there is a $c > 0$ (depending only on ε) such that

$$\sum_{j=1}^{\infty} \frac{\nu(\mu_j)}{|\mu_j - \lambda|} < c\left(\frac{\ln n}{n}\right) 2^{(1+\varepsilon)(\ln n / \ln \ln n)}.$$

Thus, if $\lambda \in C_n$, (9) implies

$$|K(x, y, \lambda)| < c\left(\frac{\ln n}{n}\right) 2^{(1+\varepsilon)(\ln n / \ln \ln n)}, \tag{21}$$

where c is independent of x, y, and λ. Since, by (15), the length of C_n is $2\pi n$, (21) applied to (14) gives

$$\left| \frac{1}{2\pi i} \oint_{C_n} G_m(x, y; \lambda)\, d\lambda \right| < \pi^{2m} \|q\|_\infty^m c^{m+1} \left[\left(\frac{\ln n}{n}\right) 2^{(1+\varepsilon)(\ln n / \ln \ln n)} \right]^{m+1} n. \tag{22}$$

Therefore, if n is sufficiently large,

$$\left| \sum_{m=1}^{\infty} \frac{(-1)^m}{2\pi i} \oint_{C_n} G_m(x, y; \lambda)\, d\lambda \right| < c_0 \|q\|_\infty \left(\frac{\ln^2 n}{n}\right) 2^{(2+\varepsilon)(\ln n / \ln \ln n)}.$$

Using this in (18), we obtain the following:

THEOREM. *Let $\phi_n(x)$ be the nth eigenfunction of* (1) *and $\psi_n(x)$ be, as in* (2), *the nth eigenfunction of the corresponding unperturbed problem, where n* *satisfies* (17), *with* μ_n *as in* (3). *Then, there is an* n_0 (*depending on* $\|q\|_\infty$) *such that, if* $n \geq n_0$, *the quantity*

$$\left|\phi_n(x)\phi_n(y) + \cdots + \phi_{n+\nu-1}(x)\phi_{n+\nu-1}(y) - \psi_n(x)\psi_n(y) - \cdots - \psi_{n+\nu-1}(x)\psi_{n+\nu-1}(y)\right|$$

is bounded by

$$c_0\|q\|_\infty\left(\frac{\ln^2 n}{n}\right)2^{(2+\varepsilon)(\ln n/\ln\ln n)},$$

where $\varepsilon > 0$ *is arbitrary, and* c_0 *depends only on* ε.

Remark. For any $\delta > 0$ we have

$$\left(\frac{\ln^2 n}{n}\right)2^{(2+\varepsilon)(\ln n/\ln\ln n)} = O\left(\frac{1}{n^{1-\delta}}\right).$$

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