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On the Inversion of Integral Transforms Associated with Sturm–Liouville Problems

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Consider the Sturm–Liouville boundary-value problem

(1) $y'' - q(x)\, y = -t^2 y, -\infty < a \leqslant x \leqslant b < \infty$

(2) $y(a) \cos \alpha + y'(a) \sin \alpha = 0$

(3) $y(b) \cos \beta + y'(b) \sin \beta = 0,$

where $q(x)$ is continuous on $[a, b]$. Let $\phi(x, t)$ be a solution of either the initial-value problem (1) and (2) or (1) and (3). In this paper we develop two techniques to invert the integral $F(t) = \int_a^b f(x)\, \phi(x, t)\, dx$, where $f(x) \in L^2(a, b)$; one technique is based on the construction of some biorthogonal sequence of functions and the other is based on Poisson's summation formula. © 1992 Academic Press, Inc.

1. INTRODUCTION

Recently there has been some interest in the study of a certain class of integral transforms which contains the continuous Jacobi, Gegenbauer, Legendre, Laguerre, and Hermite transforms as special cases. The importance of these integral transforms lies not only in their intrinsic properties but also in their connection with sampling theory and signal analysis. They lead to various sampling expansions similar to the one given by the celebrated Whittaker–Shannon–Kotelnikov sampling theorem [2, 15]

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which states that if $\hat{f}(t)$ is band-limited to $[-\pi, \pi]$, then $\hat{f}(t)$ can be expanded in the form

$$\hat{f}(t)=\sum_{n=-\infty}^{\infty} \hat{f}(n)\frac{\sin \pi(t-n)}{\pi(t-n)}. \tag{1.1}$$

Analogously, it was shown in [29] that if $\hat{f}^{(\alpha,\beta)}(t)$ is the continuous Jacobi transform of $f(x)$, i.e., if

$$\hat{f}^{(\alpha,\beta)}(t)=\int_{-1}^{1} f(x)\,\phi_{t-q}^{(\alpha,\beta)}(x)\,dw^{(\alpha,\beta)}(x); \qquad \alpha, \beta>-1, t\in\mathbb{C} \tag{1.2}$$

for some suitable function $f(x)$, where $\phi_t^{(\alpha,\beta)}(x)$ is the Jacobi function, $dw^{(\alpha,\beta)}(x)=2^{-2q}(1-x)^{\alpha}\,(1+x)^{\beta}\,dx$ and $2q=\alpha+\beta+1$ is an integer $\geqslant 0$, then a sampling expansion of $\hat{f}^{(\alpha,\beta)}(t)$ can be given by

$$\hat{f}^{(\alpha,\beta)}(t)=\sum_{n=0}^{\infty} \hat{f}^{(\alpha,\beta)}(n+q)\,S_n^q(t-q), \tag{1.3}$$

where

$$S_n^q(t-q)=\frac{2(n+q)\sin \pi(t-n-q)}{\pi[t^2-(n+q)^2]}. \tag{1.4}$$

When $q=0$, (1.4) is a symmetric version of (1.1). When $\alpha=\beta=\lambda-\frac{1}{2}$, $\hat{f}^{(\alpha,\beta)}(t)$ reduces to the continuous Gegenbauer transform studied in [28] and when $\alpha=0=\beta$, it reduces to the continuous Legendre transform studied in [3, 17, 30]. The continuous Laguerre and Hermite transforms and their sampling expansions have also been investigated in [9–12, 14, 16, 25, 33].

It seems desirable to have a more unified approach to all these integral transforms and their sampling expansions. One such approach which is due to Weiss [31] and Kramer [19] goes as follows: Suppose that $\phi(x, t)$ is a real continuous function on $I\times J$ where I is a bounded interval and J is a domain such that for some sequence $\{t_n\}_{n=0}^{\infty}$ in J, $\{\phi_n(x)=\phi(x, t_n)\}_{n=0}^{\infty}$ is a complete orthogonal family of functions on I with respect to some measure $d\rho$. Then, for $f(x)\in L^2(I, d\rho)$, we have

$$f(x)=\sum_{n=0}^{\infty} \hat{f}(n)\frac{\phi_n(x)}{\|\phi_n(x)\|^2}, \tag{1.5}$$

where

$$\hat{f}(n)=\int_I f(x)\,\phi_n(x)\,d\rho(x). \tag{1.6}$$

If we call the sequence $\{\hat{f}(n)\}_{n=0}^{\infty}$ the discrete ϕ-transform of $f(x)$, then it is natural to call $\hat{f}(t)=\int_I f(x)\,\phi(x,t)\,d\rho$ the continuous ϕ-transform of $f(x)$.

It is our interest to find a sampling expansion and an inversion formula for this transform. As for the sampling expansion, one can easily find from (1.5) that

$$\hat{f}(t)=\sum_{n=0}^{\infty}\hat{f}(n)\,S_n(t), \tag{1.7}$$

where

$$S_n(t)=\frac{1}{\|\phi_n\|^2}\int_I \phi_n(x)\,\phi(x,t)\,d\rho \tag{1.8}$$

are the sampling functions. However, finding an inversion formula for the continuous ϕ-transform is not, in general, as easy.

To see where the function $\phi(x,t)$ arises naturally let us consider the regular Sturm–Liouville boundary-value problem

$$y''-q(x)\,y=-t^2y, \qquad -\infty<a\leqslant x\leqslant b<\infty \tag{1.9}$$

$$y(a)\cos\alpha+y'(a)\sin\alpha=0, \tag{1.10}$$

$$y(b)\cos\beta+y'(b)\sin\beta=0. \tag{1.11}$$

Let $\{t_n^2\}_{n=0}^{\infty}$ denote the eigenvalues of this problem and let $\phi(x,t)$ be a solution of (1.9) together with one of the boundary conditions, say (1.10), then it is well known that $\{\phi_n(x)=\phi(x,t_n)\}_{n=0}^{\infty}$, the eigenfunctions, form a complete orthogonal family on $[a,b]$ with respect to Lebesgue measure. Therefore, the so-called continuous ϕ-transform in this case takes on the form

$$\hat{f}(t)=\int_a^b f(x)\,\phi(x,t)\,dx. \tag{1.12}$$

The function $\phi(x,t)$ with its main property of producing the eigenfunctions when the parameter t is replaced by the t_n, may also arise from some singular Sturm–Liouville boundary-value problems. Since the Legendre, Gegenbauer, Jacobi, Laguerre, and Hermite functions are solutions of Sturm–Liouville boundary-value problems, the importance of (1.12) becomes evident.

The sampling expansion (1.7)–(1.8) of $\hat{f}(t)$ has recently been shown in [33, 36] to be nothing more than the Lagrange interpolation of $\hat{f}(t)$.

As for an inversion formula for (1.12), no general procedure seems to exist; only a few scattered results are known. For the continuous Legendre transform, inversion formulae were found earlier by MacRobert [23, 24] and more recently by Butzer, Stens, and Wehrens [3]. An inversion formula for the continuous Gegenbauer transform was found by Walter in [28]. As for the continuous Jacobi transform, an inversion formula was found by Deeba and Koh for $\alpha+\beta=0$ in [7, 8] and extended by Walter and Zayed [29] to $\alpha+\beta+1$ a nonnegative integer, and by Koornwinder and Walter for general $\alpha, \beta > -1$ [18]; see also [34, 35].

The aim of this paper is to find an inversion formula for (1.12). The rest of this article will be divided into five sections. Section 2 contains some of the preliminary results that will be used in the sequel. In Section 3 we develop a technique to invert (1.12) based on the construction of a biorthogonal sequence. In Section 4 another technique based on the Poisson's summation formula is derived. In Section 5 we give some examples, and in Section 6 discuss the singular case.

2. Preliminaries

Consider the regular Sturm–Liouville boundary-value problem (S–LBVP) given by (1.9)–(1.11), where $q(x)$ is real and continuous on (a, b) and tends to finite limits as $x \to a^+$ and $x \to b^-$. Let $\phi(x, t)$ and $\psi(x, t)$ be solutions of (1.9) such that

$$\phi(a, t) = \sin\alpha, \qquad \phi'(a, t) = -\cos\alpha \tag{2.1}$$

$$\psi(b, t) = \sin\beta, \qquad \psi'(b, t) = -\cos\beta. \tag{2.2}$$

It was shown in [36, Lemma 2.1] that both $\phi(x, t)$ and $\psi(x, t)$ can be chosen as even entire functions in t of exponential type $\leqslant (b-a)$ and as functions in x they are real-valued whenever t^2 is real; they do not vanish identically for any t whether real or complex, and are uniformly bounded for all real t and $a \leqslant x \leqslant b$. Indeed if $\sin\alpha \neq 0$,

$$\phi(x, t) = \cos(t(x-a)) \sin\alpha + 0(|t|^{-1} e^{|\mathrm{Im}\, t|\,(x-a)}) \tag{2.3}$$

and if $\sin\alpha = 0$

$$\phi(x, t) = -\frac{\cos\alpha}{t} \sin(t(x-a)) + 0(|t|^{-2} e^{|\mathrm{Im}\, t|\,(x-a)}) \tag{2.4}$$

[27, p. 10].

At most a finite number of the eigenvalues of problem (1.9)–(1.11) are nonpositive; therefore we can assume, without loss of generality, all are positive; otherwise we choose a fixed number c and consider the boundary-value problem

$$y'' + \{(\lambda - c) - q(x)\} y = 0$$

together with (1.10) and (1.11), where c is larger than the magnitude of all nonpositive eigenvalues. Hence, from now on, we shall denote the eigenvalues by $\{\lambda_n\}_{n=1}^{\infty}$, and set $\lambda_n = t_n^2$, $t_{-n} = -t_n$. We shall also denote the eigenfunction corresponding to the eigenvalue λ_n by $\psi_n(x)$ and since $\phi(x, t)$ is an even function in t, we have

$$\psi_n(x) = \phi(x, t_n) = \phi(x, t_{-n}) = \psi_{-n}(x), \qquad n = 1, 2, \ldots$$

and

$$\begin{aligned}\langle \psi_n, \psi_m \rangle &= \int_a^b \psi_n(x)\, \psi_m(x)\, dx \\ &= \begin{cases} 0 & \text{if } \ n \neq \pm m \\ \|\psi_n\|^2 & n = \pm m \end{cases} \quad m, n = \pm 1, \pm 2, \ldots .\end{aligned}$$

Let $f(x) \in L^2(a, b)$, $\hat{f}(n) = \langle f, \psi_n \rangle$, then

$$f(x) = \sum_{n=1}^{\infty} \hat{f}(n) \frac{\psi_n(x)}{\|\psi_n\|^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty}{}' \hat{f}(n) \frac{\psi_n(x)}{\|\psi_n\|^2}. \tag{2.5}$$

Since $\phi(x, t)$ is bounded in x for each t, it has a similar expansion. In (2.5), the series converges in the sense of $L^2(a, b)$, and $\sum_{n=-\infty}^{\prime\,\infty} = \sum_{n=-\infty, n \neq 0}^{\infty}$. We also have the following estimates [22, pp. 8–13]:

$$t_n = \frac{\pi s_n}{(b-a)} + \frac{c}{(s_n)} + O\left(\frac{1}{n^2}\right) \qquad \text{as} \quad n \to \infty, \tag{2.6}$$

where $s_n = (n-1)$, or $(n - \frac{1}{2})$, or n as (i) $\sin \alpha \sin \beta \neq 0$, (ii) $\sin \alpha = 0$ or $\sin \beta = 0$ but not both, or (iii) $\sin \alpha = \sin \beta = 0$. Similarly in these three cases we have respectively for $\sin \alpha \neq 0$

$$\psi_n(x) = \sin \alpha \cos\left(\frac{(n-1)\,\pi(x-a)}{b-a}\right) + O\left(\frac{1}{n}\right) \qquad \text{as} \quad n \to \infty \tag{2.7}$$

and for $\sin \alpha = 0$,

$$\psi_n(x) = \pm \frac{1}{n} \sin\left(\frac{(n-1)\,\pi(x-a)}{b-a}\right) + O\left(\frac{1}{n^2}\right) \qquad \text{as} \quad n \to \infty.$$

The normalized eigenfunctions are [27, p. 19]

$$\frac{\psi_n(x)}{\|\psi_n\|}=\left(\frac{2}{b-a}\right)^{1/2}\cos\left(\frac{(n-1)\,\pi(x-a)}{b-a}\right)+O\left(\frac{1}{n}\right) \tag{2.8}$$

and

$$\frac{\psi_n(x)}{\|\psi_n\|}=\left(\frac{2}{b-a}\right)^{1/2}\sin\left(\frac{(n-1)\,\pi(x-a)}{b-a}\right)+O\left(\frac{1}{n}\right).$$

3. Inversion Formulae

In this section we develop a technique for inverting any integral transform whose kernel is a solution of the regular Sturm–Liouville differential equation (1.9) and satisfies one of the initial conditions (1.10) or (1.11).

This technique is based on the construction of a biorthogonal sequence of functions to be used in the derivation of the kernel of the inversion formula. This idea was used earlier by Walter and Zayed [29] and Walter [28].

Since the substitution $\tilde{x}=\pi(x-a)/(b-a)$ transforms the interval $[a, b]$ into the interval $[0, \pi]$ without changing the form of the boundary-value problem, we shall from now on suppose that $a=0$, $b=\pi$.

We denote by $G(t)$ the Wronskian $W(\phi, \psi)$ of ϕ, ψ which is known to be independent of x (cf. [27, p. 7]);

$$G(t)=\phi(x, t)\,\psi'(x, t)-\psi(x, t)\,\phi'(x, t).$$

The continuous Sturm–Liouville transforms will be denoted by

$$F(t)=(\Phi f)(t)=\int_0^\pi f(x)\,\phi(x, t)\,dx \tag{3.1a}$$

and

$$F_\pi(t)=(\Phi_\pi f)(t)=\int_0^\pi f(x)\,\psi(x, t)\,dx. \tag{3.1b}$$

Proposition 3.1. *The map Φ given by* (3.1a) (*respectively Φ_π given by* (3.1b)) *maps $L^2(0, \pi)$ into the Paley–Wiener space B_π of functions in $L^2(\mathbb{R})$ which are even, entire, and of exponential type $\leqslant \pi$. It is continuous and one to one and $F(t)$ (resp. $F_\pi(t)$) admits the sampling representation*

$$F(t)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}{}' \;F(t_n)\,G_n(t), \tag{3.2}$$

where $G_n(t) = 2t_n G(t)/(t^2 - t_n^2)\, G'(t_n)$, *and* $G(t)$ *is the Wronskian given by*

$$G(t) = \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{t_n^2}\right), \qquad \textit{Ref. } [36]. \tag{3.3}$$

Proof. We only prove the results for $F(t)$ since the proof for $F_\pi(t)$ is similar. That $F(t)$ is an even entire function of exponential type η with $0 \leqslant \eta \leqslant \pi$ was shown in [36]. From (2.5), (3.1a), and Parseval's equality, one obtains

$$F(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty}{}' F(t_n) \frac{\langle \phi, \psi_n \rangle}{\|\psi_n\|^2} \tag{3.4}$$

since $F(t_n) = \hat{f}(n)$, $n = \pm 1, \pm 2, \ldots$. From the relation

$$\int_0^\pi F(x)(LG(x))\, dx - \int_0^\pi G(x)(LF(x))\, dx = W_{x=\pi}(F, G) - W_{x=0}(F, G), \tag{3.5}$$

where

$$L = \frac{d^2}{dx^2} - q(x), \tag{3.6}$$

it follows that

$$(t^2 - t_n^2) \int_0^\pi \phi(x, t)\, \psi_n(x)\, dx = W_{x=\pi}(\phi, \psi_n) \tag{3.7}$$

since $W_{x=0}(\phi, \psi_n) = 0$. But since $\phi(x, t)$ and $\psi(x, t)$ are linearly dependent at $t = t_n$, $n = \pm 1, \pm 2, \ldots$ (cf. [27, p. 8]), we have $\phi(x, t_n) = \psi_n(x) = k_n \psi(x, t_n)$; $k_n \neq 0, \infty$. Thus since $\psi(\pi, t_n) = \psi(\pi, t)$, (3.7) becomes

$$(t^2 - t_n^2) \int_0^\pi \phi(x, t)\, \psi_n(x)\, dx = k_n W_{x=\pi}(\phi, \psi) = k_n G(t), \tag{3.8}$$

and

$$\langle \phi, \psi_n \rangle = \int_0^\pi \phi(x, t)\, \psi_n(x)\, dx = \frac{k_n G(t)}{(t^2 - t_n^2)}. \tag{3.9}$$

By differentiating (3.8) with respect to t and then taking the limit as $t \to t_n$ we find that

$$\|\psi_n\|^2 = \int_0^\pi |\psi_n(x)|^2\, dx = \frac{k_n G'(t_n)}{2t_n}, \qquad n = \pm 1, \pm 2, \ldots. \tag{3.10}$$

By substituting (3.9) and (3.10) into (3.4) we obtain (3.2). That $G(t)$ has the form (3.3) was shown in [36].

From the asymptotic formula (2.3), it follows that for $\sin\alpha \neq 0$,

$$F(t) = \sin\alpha \int_0^\pi \cos tx\, f(x)\, dx + O(|t|^{-1}).$$

Hence, $F \in L^2(\mathbb{R})$. Similar calculations follow for $\sin\alpha = 0$. If $F(t) = 0$ almost everywhere, then since F is entire, $F \equiv 0$. In particular, $F(t_n) = 0$, $n = \pm 1, \pm 2, \ldots$, and hence $\hat{f}(n) = 0$, which, since $\psi_n(x)$ is complete in $L^2(0, \pi)$, implies that $f = 0$ a.e. Q.E.D.

Integration by parts together with the asymptotic formulae give us the

COROLLARY. *Let H_0^m be the Sobolev space on $(0, \pi)$ (consisting of the completion of $C_0^m[0, \pi]$ with respect to the norm $\|f\|_m = \sum_{j=0}^m \|f^{(j)}\|$). Then $f \in H_0^m$ implies that $F(t) = O(|t|^{-m})$.*

To derive the inversion formula for (3.1a) or (3.1b) we need

LEMMA 3.1. *Let $G(t)$ be the Wronskian given in (3.3). Then $G(t)$ is an even entire function of exponential type $\leqslant \pi$ with $G(t)/(t^2 - t_n^2) \in L^2(-\infty, \infty)$.*

Proof. That $G(t)$ is an entire function of exponential type follows from the fact that $\phi(x, t)$ and $\psi(x, t)$ and their derivatives are. The estimates (2.3) and (2.4) and similar ones for $\psi(x, t)$ [27, p. 11] may be used to show that the type of $G(t) \leqslant \pi$. They may be used as well to obtain bounds for $G(t)$ on the real axis. Indeed, corresponding to the three conditions in (2.6) we find for real t as $|t| \to \infty$,

$$\begin{aligned} &\text{(i)} \quad G(t) = O(|t|) \\ &\text{(ii)} \quad G(t) = O(1) \qquad\qquad (3.11) \\ &\text{(iii)} \quad G(t) = O(|t|^{-1}). \end{aligned}$$

In each of these cases $G(t)/(t^2 - t_n^2) \in L^2(-\infty, \infty)$. Q.E.D.

The Wronskians for the three cases in (2.6) may be calculated explicitly for the differential equation $y'' + \lambda y = 0$ with (i) $\alpha = \beta = \pi/2$, (ii) $\alpha = 0$, $\beta = \pi/2$, (iii) $\alpha = \beta = 0$. They are (i) $\pi t^2 \prod_{n=1}^\infty (1 - t^2/n^2) = t \sin \pi t$, (ii) $\prod_{n=0}^\infty (1 - t^2/(n + \frac{1}{2})^2) = \cos \pi t$, (iii) $\pi \prod_{n=1}^\infty (1 - t^2/n^2) = (\sin \pi t)/t$.

THEOREM 3.1. *Consider the regular Sturm–Liouville boundary-value problem* (1.9)–(1.11) *on the interval* $[0, \pi]$ *with* $\sin\alpha \sin\beta \neq 0$. *Let*

$$F(t) = \int_0^\pi f(x)\, \phi(x, t)\, dx \tag{3.12}$$

for some $f(x) \in H_0^m$ *the Sobolev space on* $(0, \pi)$, $m > 1$. *Then,*

$$f(x) = \int_{-\infty}^{\infty} F(t)\, K(x, t)\, dt, \tag{3.13}$$

where

$$K(x, t) = \frac{1}{2} \sum_{n=-\infty}^{\infty}{}' \frac{\psi_n(x)}{\|\psi_n\|^2} B_n(t), \tag{3.14}$$

$$B_n(t) = \frac{\sin \pi(t - t_n)}{\pi(t - t_n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iw(t - t_n)}\, dw, \tag{3.15}$$

the sinc function.

Proof. The series defining $K(x, t)$ converges absolutely and uniformly for all $x \in [0, \pi]$ and fixed t on the real line. To see this, first let us observe that $K(x, t)$ can be written in the form

$$K(x, t) = \sum_{n=1}^{\infty} \frac{\psi_n(x)}{\|\psi_n\|^2} \frac{D_n(t)}{\pi(t^2 - t_n^2)},$$

where $D_n(t) = t \sin \pi t \cos \pi t_n - t_n \cos \pi t \sin \pi t_n$. The result now follows from the estimate

$$\psi_n(x) = \sin\alpha \cos t_n x + O\left(\frac{1}{n}\right)$$

and from the fact that $t_n \sin \pi t_n$ is uniformly bounded (cf. (2.6)).

Moreover, $K(x, t)$ is bounded in t for almost all x. This follows from (3.15) since

$$\frac{D_n(t)}{\pi(t^2 - t_n^2)} = \frac{B_n(t) + B_n(-t)}{2} = \frac{1}{\pi} \int_0^\pi \cos wt \cos wt_n\, dw$$
$$= \frac{1}{\pi} \langle \cos wt, \psi_n \rangle + O\left(\frac{1}{n}\right).$$

Hence, $K(x, t)$ is given by the series expansion of $(1/\pi)\cos xt$ plus an L^2 function which is bounded in t; i.e.,

$$\left| K(x, t) - \frac{1}{\pi}\cos xt \right| \leqslant C \left| \sum_{n=1}^{\infty} \frac{\psi_n(x)}{n \, \|\psi_n\|^2} \right|,$$

where C is a constant independent of x and t.

Since $f \in H_0^m(0, \pi)$ for $m > 1$, it follows from (2.5) that $\hat{f}(n) n \in l^2$ and hence $\sum'^{\infty}_{-\infty} |\hat{f}(n)| < \infty$. Since $F(t_n) = \hat{f}(n)$, and $G_n(t) = G(t)\, 2t_n/[(t^2 - t_n^2)\, G'(t_n)]$ is uniformly bounded for all n it follows that the series in (3.2) converges to $F(t)$ absolutely and uniformly on the real line and that $F(t) = O(1/t^2)$ as $|t| \to \infty$. Therefore, the integral in (3.13) is absolutely convergent.

From Lemma 3.1, $G(t)$ is an even entire function of exponential type $\leqslant \pi$ and $G(t)/(t^2 - t_n^2)$ is in $L^2(-\infty, \infty)$, hence by the Paley–Wiener theorem [21, 26],

$$h_n(x) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{A \to \infty} \int_{-A}^{A} G_n(t)\, e^{-itx}\, dt, \qquad n = \pm 1, \pm 2, ... \tag{3.16}$$

exists and

$$G_n(t) = \int_{-\pi}^{\pi} h_n(x)\, e^{itx}\, dx, \qquad n = \pm 1, \pm 2, ... \tag{3.17}$$

Since $G(t_n) = 0$ for $n = \pm 1, \pm 2, ...$, it follows that $G_n(t_m) = 0$ if $n \neq \pm m$, $G_n(t_m) = 1$ if $n = \pm m$ and $G_n(t) = G_{-n}(t)$. Thus,

$$\int_{-\pi}^{\pi} h_n(x)\, e^{it_m x}\, dx = \begin{cases} 0 & \text{if} \quad n \neq \pm m \\ 1 & \text{if} \quad n = \pm m. \end{cases} \tag{3.18}$$

From (3.15) we also have

$$e^{it_n u} \chi_{[-\pi, \pi]}(u) = \int_{-\infty}^{\infty} B_n(t)\, e^{itu}\, dt. \tag{3.19}$$

Since $B_n(-t) = B_{-n}(t)$, it follows from Parseval's equality that

$$\int_{-\infty}^{\infty} G_m(t)\, B_n(t)\, dt = \int_{-\pi}^{\pi} h_m(x)\, e^{it_n x}\, dx = \begin{cases} 0 & \text{if} \quad m \neq \pm n \\ 1 & \text{if} \quad m = \pm n. \end{cases} \tag{3.20}$$

Therefore, from (3.2), (3.14), and (3.13) we obtain

$$\int_{-\infty}^{\infty} F(t)\, K_N(x, t)\, dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2} \sum_{n=-\infty}^{\infty}{}' F(t_n)\, G_n(t) \right) \left(\frac{1}{2} \sum_{n=-N}^{N}{}' \frac{\psi_n(x)}{\|\psi_n\|^2} B_n(t) \right) dt$$

which upon using (3.20) and (2.5), and taking the limit as $N \to \infty$, yields

$$\int_{-\infty}^{\infty} F(t)\, K(x, t)\, dt = \frac{1}{2} \sum_{n=-\infty}^{\infty}{}' F(t_n) \frac{\psi_n(x)}{\|\psi_n\|^2} = f(x).$$

Interchanging the integration and the summation signs is permissable because of the dominated uniform convergence of the series involved.

Q.E.D.

Remarks. (1) For the cases where $\sin\alpha \sin\beta = 0$, the theorem still holds except that the inversion formula (3.13) now is

$$f(x) = \lim_{N \to \infty} \int_{-\infty}^{\infty} F(t)\, K_N(x, t)\, dt,$$

where

$$K_N(x, t) = \frac{1}{2} \sum_{n=-N}^{N}{}' \frac{\psi_n(x)}{\|\psi_n\|^2} B_n(t)$$

and is valid for Sobolev spaces H_0^m with $m > 2$.

(2) One can easily verify that if zero is an eigenvalue, then the kernel $K(x, t)$ can be written in the form

$$K(x, t) = \frac{\psi_0(x)}{\|\psi_0\|^2} B_0(t) + \frac{1}{2} \sum_{n=-\infty}^{\infty}{}' \frac{\psi_n(x)\, B_n(t)}{\|\psi_n(x)\|^2}, \qquad (3.21)$$

where $B_0(t) = \sin \pi t / \pi t$, and

$$F(t) = F(0)\, G_0(t) + \tfrac{1}{2} \sum_{n=-\infty}^{\infty}{}' F(t_n)\, G_n(t).$$

(3) An inversion formula for the integral (3.1b) can also be derived in a similar way.

The integral operator with kernel $K(x, t)$ is also a right inverse of our operator Φ defined by (3.1a). Indeed we have

COROLLARY 3.1. *Let $F(t)$ be an even entire function in the Paley–Wiener space satisfying $F(t)=O(|t|^{-s})$ for some $s>3$, let* $\sin\alpha\sin\beta\neq 0$, *and let*

$$f(x)=\int_{-\infty}^{\infty} K(x,t)\,F(t)\,dt.$$

Then $f\in L^2(0,\pi)$ and

$$F(t)=\int_0^{\pi}\phi(x,t)\,f(x)\,dx.$$

Proof. We denote by $H(t)$ the function given by

$$H(t)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}{}' \;F(t_n)\,G_n(t) \tag{3.22}$$

since $G_n(t)=\int_0^{\pi}\phi(x,t)\,\psi_n(x)\,dx$, it follows from the asymptotic formulae that G_n is uniformly bounded in both n and t, and hence this series converges uniformly on $\mathbb{R}$. It also converges uniformly on bounded sets in $\mathbb{C}$ and hence $H(t)$ is an entire function. Furthermore, the series for $t^2H(t)$ is given by

$$t^2H(t)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}{}' \;F(t_n)(t_n^2G_n(t)+k_nG(t))$$

by (3.8), which again converges uniformly on bounded sets in $\mathbb{C}$. The sum of the terms involving $k_nG(t)$ converges since k_n satisfies

$$k_n^2\int_0^{\pi}\psi^2(x,t_n)\,dx=\|\psi_n\|^2$$

and $\|\psi(\cdot,t_n)\|$ is bounded below by a positive constant, and hence is bounded. Also since $G_n(t)$ may be given by (3.17) where $\|h_n\|$ is bounded, it follows that $H(t)$ is the Fourier transform of an L^2 function with support on $[0,\pi]$, i.e., a Paley–Wiener function.

By the uniqueness theorem for such functions [1, p. 163], it follows from the fact that $H(t_n)=F(t_n)$ for all n and

$$|t|^{\beta}\,|H(t)-F(t)|\leqslant Ce^{\pi\,|\mathrm{Im}\,t|}$$

for $\beta=2>\frac{1}{2}$, that

$$H(t)=F(t).$$

We now may substitute the series (3.22) into the integral (3.13) and interchange the integral and summation to obtain

$$f(x)=\frac{1}{2}\sum_{n=-\infty}^{\infty}{}' F(t_n)\frac{\psi_n(x)}{\|\psi_n\|^2}.$$

This is then substituted into (3.12) and the process repeated to obtain the conclusion. Q.E.D.

4. Inversion by a Poisson Summation Formula

The inversion formula (3.13) involves a kernel given by a series related to, but itself not, a solution of the differential Eq. (1.9). It would be more desirable to obtain a closed form expression which is a solution. In order to do so, we need a version of a Poisson summation formula. The classical version of this formula is given by

$$\sum_{n=-\infty}^{\infty} \phi(w+2\pi n)=\sum_{n=-\infty}^{\infty} \hat{\phi}(n)\, e^{iwn}, \qquad w\in\mathbb{R}, \tag{4.1}$$

where ϕ is a function in $L^1(\mathbb{R})$ satisfying

$$\phi(w)=O(1+|w|)^{-1-\varepsilon}, \qquad \varepsilon>0 \tag{4.2}$$

and $\hat{\phi}$ is its Fourier transform which also satisfies (4.2). The proof involves finding the Fourier series of the periodic function on the left of (4.1).

We shall need a version of (4.1) involving nonharmonic Fourier series. Accordingly let $\{t_n\}_{n=-\infty}^{\infty}$ be a sequence such that

$$\{e^{it_n w}\}_{n=-\infty}^{\infty}, \qquad |w|\leqslant\pi,$$

is a Riesz basis of $L^2[-\pi,\pi]$ [32, p. 196].

We shall suppose that the t_n satisfy the conditions of the last sections, i.e.,

$$t_n=0(|n|),$$

$$t_{-n}=-t_n, \qquad n=1,2,...,$$

but with $t_0=0$. Then $\{h_n(w)\}$ given by (3.16) (modified to include t_0), is the sequence biorthogonal to $\{e^{it_n w}\}$ [32, p. 149]. Hence, $f\in L^2(-\pi,\pi)$ may be expressed as

$$f(w)=\sum_{n=-\infty}^{\infty} \langle f, h_n\rangle\, e^{it_n w}, \qquad |w|\leqslant\pi, \tag{4.3}$$

with convergence in the sense of $L^2(-\pi,\pi)$.

The series in (4.3) may be extended to all of $\mathbb{R}$ in two ways. One way is by extending it periodically to obtain

$$f^*(w) = \sum_{n=-\infty}^{\infty} \langle f, h_n \rangle \left(\sum_{k=-\infty}^{\infty} e^{it_n(w-2k\pi)} \chi_\pi(w-2k\pi) \right), \tag{4.4}$$

where χ_π is the characteristic function of $[-\pi, \pi]$. The other involves extending it by the analytic continuation of $e^{it_n w}$, i.e.

$$f^+(w) = \sum_{n=-\infty}^{\infty} \langle f, h_n \rangle \, e^{it_n w}, \qquad w \in \mathbb{R}. \tag{4.5}$$

The two series converge in L^2_{loc} and hence in the sense of distributions on $\mathbb{R}$. Thus by taking $f(w) = H(w)$ for $-\pi < w < \pi$, the Heaviside function, and differentiating, we obtain a convergent expansion of $\delta^+(w)$ from (4.5).

We may also extend δ periodically to obtain δ^*. This is not the same as the derivative of H^*, but rather we have

$$\delta^*(w) - \delta^*(w-\pi) = DH^*(w),$$

where D is the derivative operator. Hence the difference between this and δ^+ is

$$\delta^+(u) - \delta^*(u) + \delta^*(u-\pi) = D[H^+(u) - H^*(u)]. \tag{4.6}$$

Now let ϕ be as in (4.2). Then

$$\langle \delta^+, \phi \rangle = \sum_{n=-\infty}^{\infty} C_n \hat{\phi}(t_n), \tag{4.7}$$

where $C_n = it_n \int_0^\pi h_n$, while

$$\begin{aligned} \langle \delta^* - \delta^*_\pi, \phi \rangle &= \sum_{k=-\infty}^{\infty} \phi(2\pi k) - \phi((2k+1)\pi) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \, \phi(\pi k). \end{aligned} \tag{4.8}$$

We combine these last three formulae into

$$\begin{aligned} &\sum_k (-1)^k \, \phi(\pi k) - \sum_n C_n \hat{\phi}(t_n) \\ &\quad = \langle H^* - H^+, \phi' \rangle \\ &\quad = -\sum_n \int_0^\pi h_n \sum_{|k| \geq 1} \int_{(2k-1)\pi}^{(2k+1)\pi} [e^{it_n w} - e^{it_n(w-2\pi k)}] \, \phi'(w) \, dw, \end{aligned} \tag{4.9}$$

which is our substitute for the Poisson summation formula.

Remark. If $\{t_n\}$ are integers or if ϕ has support on $[-\pi, \pi]$ then the expression on the right hand side vanishes. If as in the case of Sturm–Liouville problems

(i) $t_n = n + O(1/n)$, then $(1 - e^{-it_n 2\pi k}) = O(k/n)$.

We now return to our Sturm–Liouville problem. In the three cases we have considered in (2.6), it is possible to show [27, p. 11] that

(i) $G(t) \sim \sin\alpha \sin\beta\, (t \sin \pi t)$ for $\sin\alpha\sin\beta \neq 0$, (4.10a)

(ii) $G(t) \sim \cos\alpha \sin\beta \cos \pi t$ for $\sin\alpha = 0$, $\sin\beta \neq 0$ and similarly for $\sin\alpha \neq 0$, $\sin\beta = 0$, and (4.10b)

(iii) $G(t) \sim \cos\alpha \cos\beta\, (\sin \pi t/t)$ for $\sin\alpha = 0$ and $\sin\beta = 0$. (4.10c)

Thus in the second case we have a function of sine type [32, p. 171] while in the other two cases $G(t)/(t + t_1)$ and $tG(t)$ are functions of sine type. The zeros of a function of sine type are such that $\{e^{it_n w}\}$ is a Riesz basis of $L^2(-\pi, \pi)$. Thus by removing the function $e^{-it_1 w}$ in case (i) and inserting the function $\{1\}$ in case (iii) we obtain a Riesz basis in all three cases. The Fourier transform of $e^{it_n w}\chi_\pi(w)$ is exactly $B_n(t)$ of (3.15) which is then a Riesz basis of the Paley–Wiener space B_π.

We shall for the moment consider case (ii) only since we can use the notation of the last section without modification. The dual basis of $B_n(t)$ then is just given by

$$S_n(t) = \frac{G(t)}{G'(t_n)(t - t_n)}, \qquad n = \pm 1, \pm 2, ...,$$

and

$$G_n(t) = S_n(t) + S_{-n}(t), \qquad n = 1, 2, ...$$

Hence, for any sequence $\{a_n\} \in l^2$,

$$\sum_{n=1}^{\infty} a_n G_n(t)$$

converges to some even function in B_π. This enables us to construct the weight function

$$r(t) = t^4 \sum_{n=1}^{\infty} \frac{G_n(t)}{t_n^4 \, \|\psi_n\|^2 \, C_n}, \tag{4.11}$$

where $C_n = it_n \int_0^\pi h_n(w)\, dw$.

Since in this case we have

$$\phi(x, t) = -\cos\alpha \frac{\sin tx}{t} + O(|t|^{-2}),$$

it follows that $\|\psi_n\|^2 = O(n^{-2})$, and since

$$\frac{C_n}{it_n} = \frac{1}{2} G_n(0) = \frac{G(0)\, 2t_n}{-2t_n^2 G'(t_n)}$$

it follows from the asymptotic formula for $G'(t)$ [27, p. 11] and that of t_n, that $C_n = O(1)$. Hence, $r(t)/t^4$ is a Paley–Wiener function in B_π.

LEMMA 4.1. *Let $f \in L^2(0, \pi)$ with expansion in terms of $\{\psi_n\}$ given by*

$$f = \sum_{n=1}^{\infty} F(t_n)\, \psi_n / \|\psi_n\|^2;$$

then

$$f(x) = \sum_{n=-\infty}^{\infty}{}' \; C_n [t_n^4 F(t_n)][r(t_n)/t_n^4][\phi(x, t_n)], \tag{4.12}$$

where $t^4F(t)$, $r(t)/t^4$, and $\phi(x, t)$ are entire functions of exponential type $\leqslant \pi$. Furthermore, if $f \in H_0^4(0, \pi)$, each of these functions is in $L^2(\mathbb{R})$ as well.

Proof. We observe that

$$F(t_n)\, \psi_n(x)/\|\psi_n\|^2 = C_n t_n^4 F(t_n)[1/(C_n t_n^4 \|\psi_n\|^2)]\, \phi(x, t_n)$$

which by (4.11) and the sampling property of $G_n(t)$ gives us the first conclusion. The second follows from the fact that if $f \in H_0^4(0, \pi)$, integration by parts gives us the fact $t^4F(t)$ is in $L^2(\mathbb{R})$. Q.E.D.

We now define a function ϕ by means of its Fourier transform

$$\hat{\phi}(t) = F(t)\, r(t)\, \phi(x, t) \tag{4.13}$$

and use our modified Poisson summation formula (4.9) applied to it. Clearly $\hat{\phi}$ is of exponential type $\leqslant 3\pi$ and belongs to $L(\mathbb{R})$ since it is the product of two $L^2(\mathbb{R})$ functions with a bounded function. Thus (4.9) becomes

$$\sum_{k=-2}^{2} \phi(k\pi)(-1)^k - \sum_{n=-\infty}^{\infty}{}' \; C_n F(t_n)\, r(t_n)\, \phi(x, t_n) = \langle H^* - H^+, \phi' \rangle. \tag{4.14}$$

By the Fourier inversion formula $\phi(w) = (1/2\pi) \int_{-\infty}^{\infty} e^{-iwt} \hat{\phi}(t)\, dt$ ($\hat{\phi} \in L(\mathbb{R})$), and we have for the left side of (4.14)

$$\phi(0) - \phi(\pi) - \phi(-\pi) + \phi(2\pi) + \phi(-2\pi) - f(x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - e^{-i\pi t} - e^{i\pi t} + e^{-i2\pi t} + e^{i2\pi t})\, \hat{\phi}(t)\, dt - f(x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - 2\cos \pi t + 2\cos 2\pi t)\, F(t)\, r(t)\, \phi(x, t)\, dt - f(x). \qquad (4.15)$$

The right side of (4.14) may be expressed, by using the evenness of ϕ, as

$$\begin{aligned} \langle H^* - H^+, \phi' \rangle &= \int_{\pi}^{3\pi} [H^*(w) - H^+(w)]\, \phi'(w)\, dw \\ &\quad + \int_{-3\pi}^{-\pi} [H^*(w) - H^+(w)]\, \phi'(w)\, dw \\ &= \int_{-\pi}^{\pi} \{[H(w) - H^+(w - 2\pi)]\, \phi'(w - 2\pi) \\ &\quad + [H(w) - H^+(w + 2\pi)]\, \phi'(w + 2\pi)\}\, dw \\ &= \phi(\pi - 2\pi) - \phi(0 - 2\pi) + \phi(\pi + 2\pi) - \phi(0 + 2\pi) \\ &\quad - \int_{-\pi}^{\pi} [H^+(w - 2\pi) - H^+(-w + 2\pi)]\, \phi'(w - 2\pi)\, dw \\ &= \phi(\pi) - 2\phi(2\pi) - \int_{-\pi}^{\pi} u^+(w - 2\pi)\, \phi'(w - 2\pi)\, dw, \end{aligned} \qquad (4.16)$$

where $u^+(w - 2\pi) = H^+(w - 2\pi) - H^+(-w + 2\pi)$ is the extension of the sgn function. We may replace ϕ and ϕ' in this expression by the inverse Fourier integral since $t\hat{\phi}(t) \in L(\mathbb{R})$ as well. This gives us

$$\begin{aligned} \langle H^* - H^+, \phi' \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{-i\pi t} - 2e^{-2\pi i t})\, \hat{\phi}(t)\, dt \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\pi}^{\pi} u^+(w - 2\pi)\, e^{-i(w - 2\pi)t}(it)\, dw \right\} \hat{\phi}(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [\cos \pi t - 2\cos 2\pi t - g(t)]\, \hat{\phi}(t)\, dt, \end{aligned} \qquad (4.17)$$

where

$$g(t) = \int_{-\pi}^{\pi} u^+(w - 2\pi)\, e^{-i(w - 2\pi)t} it)\, dw.$$

We define $u(t)$ as

$$u(t)=\frac{1}{2\pi}(1-3\cos\pi t+4\cos 2\pi t+g(t))\,r(t). \tag{4.18}$$

This gives us our inversion formula.

THEOREM 4.2. *Let $f\in H_0^m(0,\pi)$, $m>4$; let*

$$F(t)=\int_0^\pi f(x)\,\phi(x,t)\,dx;$$

then $f(x)$ is given by the convergent integral

$$f(x)=\int_{-\infty}^{\infty}\phi(x,t)\,F(t)\,u(t)\,dt,$$

where $u(t)$ is given by (4.18).

The remainder of the proof merely requires that we rewrite (4.14) as

$$f(x)=\sum_{k=-2}^{2}\phi(k\pi)(-1)^k-\langle H^*-H^+,\phi'\rangle \tag{4.19}$$

and substitute (4.15), (4.17), and (4.18) into it.

Similar results are possible in the other two cases (i) and (iii) with appropriate modification of the definitions.

5. EXAMPLES

(1) Consider the regular Sturm–Liouville problem:

$$y''=-t^2y \qquad 0\leqslant x\leqslant\pi$$
$$y(0)=0=y(\pi).$$

In this case $\alpha=0=\beta$ and $\phi(x,t)=\sin tx/t$. The eigenvalues are n^2, $n=\pm1,\pm2,\ldots$ and the eigenfunctions $\psi_n(x)=\sin nx/n$, for $n=\pm1,\pm2,\ldots$. Therefore,

$$K(x,t)=\frac{2t\sin\pi t}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,n\sin nx}{(t^2-n^2)}$$
$$=\frac{2t\sin\pi t}{\pi^2}\left(\frac{\pi\sin tx}{2\sin\pi t}\right)=\frac{t\sin tx}{\pi}, \qquad 0\leqslant x\leqslant\pi,\ t\in\mathbb{R},$$

and consequently Theorem 3.1 gives the known inversion formula that if

$$F(t)=\int_0^\pi f(x)\frac{\sin tx}{t}\,dx, \qquad f\in H_0^2(0,\pi);$$

then

$$f(x)=\int_{-\infty}^{\infty} F(t)\frac{t\sin tx}{\pi}\,dt, \qquad 0\leqslant x\leqslant \pi.$$

With a slight change of variable these become the Fourier sine transform pairs.

(2) Consider

$$y''=-t^2y, \qquad 0\leqslant x\leqslant\pi$$
$$y'(0)=0=y'(\pi).$$

The same consideration lead to the Fourier cosine transform. Thus, by Theorem 3.1 we have that if

$$F(t)=\int_0^\pi f(x)\cos xt\,dx, \qquad f\in H_0^2(0,\pi),\ t\in\mathbb{R};$$

then

$$f(x)=\frac{2}{\pi}\int_0^\infty F(t)\cos tx\,dt, \qquad 0\leqslant x\leqslant\pi.$$

(3) Consider again the problem

$$y''=-t^2y, \qquad 0\leqslant x\leqslant\pi$$
$$y(0)=0, \qquad y'(\pi)=0,$$

which corresponds to $\alpha=0$, $\beta=\pi/2$. Then $\phi(x,t)=(\sin tx)/t$ again, but the eigenvalues now are $(n+\frac{1}{2})^2$, $n=0,\pm1,\pm2,\ldots$, while the eigenfunctions are

$$\psi_n(x)=\frac{\sin(n+\frac{1}{2})x}{n+\frac{1}{2}}.$$

Therefore,

$$K_N(x,t)=\sum_{n=-N}^{N}\frac{\sin(n+\frac{1}{2})x}{\pi}\,\frac{(n+\frac{1}{2})\sin\pi(t-n-\frac{1}{2})}{\pi(t-n-\frac{1}{2})},$$

which may be shown to converge weakly to $(t/\pi)\sin\pi x$. This gives formally the same results as in example 1.

6. The Singular Case

The argument given in Theorem 3.1 can be extended to singular Sturm–Liouville poblems but under further restrictions since, for example, the spectrum is not always discrete and even if it is, the eigenvalues λ_n are not necessarily of $O(n^2)$ as $n \to \infty$, hence the canonical product given in (2.10) may not converge.

In some cases of interest the eigenvalues are $O(n^2)$ but do not satisfy the other asymptotics of the regular case. This happens, for example, in the case of Jacobi polynomials as mentioned in the introduction ($t_n = n(n+\alpha+\beta+1)$). The inverse transform has already been found in this case [18, 29] by the same methods.

In other singular cases, the series method still works even though the eigenvalues are merely $O(n)$. This happens with the Laguerre functions as may be seen from straightforward calculations.

In order to illustrate the method of series in the singular case, we apply it in one case and show that the result is the same as that obtained by direct calculations [3].

Example 4. Consider the singular Sturm–Liouville problem

$$\begin{gathered} y'' + \tfrac{1}{4}(\sec^2 x)\, y = -t^2 y, \qquad -\pi/2 \leqslant x \leqslant \pi/2 \\ |y(\pm\pi/2)| < \infty. \end{gathered} \tag{6.1}$$

The eigenvalues are $(n+\frac{1}{2})^2$ and the corresponding eigenfunctions are $P_n(\sin x)\sqrt{\cos x}$, where $P_n(z)$ is the Legendre polynomial of degree n. The function $\phi(x, t)$ in this case is chosen as

$$\phi(x, t) = P_{t-1/2}(\sin x)\sqrt{\cos x},$$

where

$$P_t(z) = {}_2F_1\left(-t, t+1; 1, \left(\frac{1-z}{2}\right)\right)$$

is the Legendre function. It is known that $\|P_n\|^2 = (n+\frac{1}{2})^{-1}$ and hence the inverse kernel (3.14) will have partial sums given by

$$\begin{aligned} K_N(x, t) &= \sum_{n=0}^{N} \left(n+\frac{1}{2}\right) \frac{(n+\frac{1}{2})(-1)^n \cos \pi t}{\pi(t^2 - (n+\frac{1}{2})^2)} P_n(\sin x)\sqrt{\cos x} \\ &= \sum_{n=0}^{N} \left(n+\frac{1}{2}\right)^2 \frac{(-1)^n \cos \pi t}{\pi(t^2 - (n+\frac{1}{2})^2)} P_n(\sin x)\sqrt{\cos x}. \end{aligned}$$

The corresponding series does not converge pointwise but does converge weakly to

$$t \sin \pi t \, P_{t-1/2}(\sin x) \sqrt{\cos x}$$

with respect to Paley–Wiener functions $F(t) = O(t^{-2})$. This still holds if we let $w = \sin x$. Then we find that

$$\int_{-\infty}^{\infty} F(t) \, t \sin \pi t \, P_{t-1/2}(-w) \, dt$$

$$= \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^2 P_n(w)(-1)^n \int_{-\infty}^{\infty} F(t) \frac{\cos \pi t}{\pi(t^2 - (n + \frac{1}{2})^2)} \, dt.$$

This is exactly the inverse given in [3] for Legendre transforms.

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