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Products of locally dihedral subgroups

Bernhard Amberg [a,*], Andrew Fransman [b], Lev Kazarin [c,1]

[a] Fachbereich Mathematik, Universität Mainz, D-55099 Mainz, Germany
[b] Department of Mathematical Sciences, University of Stellenbosch, Private Bag x1, Matieland 7602, South Africa
[c] Department of Mathematics, Yaroslavl State University, 150000 Yaroslavl, Russia

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ABSTRACT

It is shown that a group $G = AB$ which is a product of two periodic locally dihedral subgroups A and B is soluble.

1. Introduction

If the group $G = AB$ is the product of two its subgroups A and B, i.e. $G = \{ab \mid a \in A, b \in B\}$, the natural question is what can be said about the structure of the factorized group G if the structure of the subgroups A and B is known.

If A and B are abelian, then $G = AB$ is metabelian by a celebrated theorem of N. Itô (see [3], Theorem 2.1.1). But very little is known for groups which are the product of two subgroups having abelian subgroups of finite index. It is therefore natural to study first products of groups possessing abelian subgroups of index at most 2.

* Corresponding author.
E-mail addresses: amberg@mathematik.uni-mainz.de (B. Amberg), af@sun.ac.za (A. Fransman), kazarin@uniyar.ac.ru (L. Kazarin).

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In [10] V.S. Monakhov has proved that a finite group $G = AB$ having cyclic subgroups A_0 and B_0 such that $|A : A_0| \leqslant 2$ and $|B : B_0| \leqslant 2$ is soluble. The solubility of every product of two groups containing cyclic subgroups of index $\leqslant 2$ was shown by B. Amberg and Ya. Sysak in [5].

A group is called *locally dihedral* if it has a local system of dihedral subgroups. An infinite periodic locally dihedral group X is the union of an infinite ascending chain of finite dihedral subgroups and therefore it is locally finite. Such a group has the form $X = X_0\langle i\rangle$, where i is an involution and X_0 is a locally cyclic normal subgroup of X such that $x^i = x^{-1}$ for every $x \in X_0$. Every (locally) cyclic subgroup of X whose order is greater than 4, belongs to X_0 and each non-(locally) cyclic subgroup of X contains its centralizer in X.

It was already shown in [2] that a periodic product of two locally dihedral subgroups is soluble. Here we generalize this as follows.

Theorem 1.1. *Let $G = AB$ be a group, which is a product of two periodic locally dihedral subgroups A and B. Then G is soluble.*

If a soluble group $G = AB$ is the product of two subgroups A and B, which are π-groups for some set of primes π, then G is also a π-group (see [3], Theorem 3.2.6). In particular the group G in Theorem 1.1 is periodic and so locally finite. It may be conjectured that even the product of two (possibly non-periodic) locally dihedral groups and the product of two groups that have (periodic) locally cyclic subgroups of index at most 2 are also soluble.

It was proved in [1] that a group $G = AB$ which is a product of two subgroups A and B having abelian subgroups $A_0 \leqslant A$ and $B_0 \leqslant B$ with minimum condition such that $|A : A_0| \leqslant 2$, $|B : B_0| \leqslant 2$ is also soluble, *provided that A is a group of dihedral type*, i.e. there exists an involution a in A that inverts every element in A_0. It is likely that this result in [1] also holds in the case when the subgroup A is not of dihedral type.

The notation is standard and can for instance be found in [9]. We use extensively the fact that in any group two different involutions generate a dihedral group, the structure of which is well-known (see for instance [9], Chapter 1, Section I).

2. Finite products of dihedral subgroups

This section is devoted to establish bounds on the solubility length of finite products of dihedral groups.

The following slight extension of the theorem of Itô will be used repeatedly. It implies for instance that Theorem 1.1 is true when at least one of the two subgroups A and B is abelian.

Lemma 2.1. *Let N be a normal subgroup of a group G. Suppose that G has two abelian subgroups A and B such that $N \subseteq AB$. Then N is metabelian.*

Proof. Obviously, $NA = A(NA \cap B)$. By the above-mentioned theorem of N. Itô the group NA is metabelian. The result follows. $\square$

The next lemma is a reformulation of a result of O. Kegel (see Lemma 2 in [8]).

Lemma 2.2. *Let the finite group $G = AB$ be the product of subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0B_0 = B_0A_0$, then $A_0^xB_0 = B_0A_0^x$ for all $x \in G$. Assume in addition that A_0 and B_0 are π-groups for a set of primes π. If $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.*

The following lemma gives a bound on the solubility length of a finite 2-group which is the products of two dihedral subgroups.

Lemma 2.3. *Let $G = AB$ be a finite 2-group, which is a product of a subgroups A and B, where A is dihedral and B is either cyclic or a dihedral group. Then $G^{(5)} = 1$.*

Proof. Let $A = A_0\langle c\rangle$, $B = B_0\langle d\rangle$, where A_0 and B_0 are cyclic, c is an involution such that $cac = a^{-1}$ for each $a \in A_0$, $d = 1$ or d is an involution such that $dbd = b^{-1}$ for each $b \in B_0$.

Case 1. A is dihedral, $B = B_0$ is cyclic.

Let K be a largest normal subgroup of G contained in the set A_0B_0. By Lemma 2.1 K is metabelian. We use the standard bar-notation for $\bar{G} = G/K$ and its subgroups. Since $\bar{G}$ has no normal subgroups belonging to $\bar{A}_0\bar{B}_0$, it follows that there exists a normal subgroup $\bar{T}$ in $\bar{G}$ of order 2 not contained in $\bar{A}_0\bar{B}_0$. Then $\bar{T}$ is generated by an element $t = a\bar{c}b$ such that $a \in \bar{A}_0$, $b \in \bar{B}$ and $\bar{c} = Kc$. It follows that $c_1 = a\bar{c} \in \bar{A} \setminus \bar{A}_0$. Since $t = c_1b \in Z(\bar{G})$, the group $F = \langle t, \bar{B}\rangle$ is abelian. Therefore $\bar{G} = \bar{A}_0F$ is a product of two abelian groups and so metabelian by Itô's theorem. Therefore the derived length of G does not exceed 4 in this case.

Case 2. A and B are dihedral subgroups of G.

In this case A_0 and B_0 are the cyclic subgroups of A and B with index 2, respectively.

If $H = A^2B^2 = B^2A^2$ is a subgroup of G, then the index of H in G is at most 16, and $H \leqslant \Phi(G)$, the Frattini subgroup of G. If $H = \Phi(G)$, then $G' \leqslant H$ and the derived length of G is at most 3. Clearly, $H \leqslant G'$ and by similar reasons we may assume that $|G : G'| \leqslant 8$. If $|G : G'| = 4$, then by a theorem of O. Taussky (see [6], Chapter III, Satz 11.9, p. 339) G is a 2-group of maximal class, so that G is metabelian. Hence $|G : G'| = 8$. Since G/G' is generated by elements of order 2, the factor-group G/G' is elementary abelian, so that $G' = \Phi(G)$.

Let K be the largest normal subgroup of G belonging to the set A_0B_0. Suppose that $K = 1$.

It follows that there is a normal subgroup T of order 2 in G such that T is generated by the element $t \in G$, which is of the form $t = acb$, adb, or $acdb$ with $a \in A_0$, $b \in B_0$. Since $ac \in A \setminus A_0$, $db \in B \setminus B_0$, we may replace ac by c and db by d, so that $t = cb$, ad, or cd.

If $t = cb$, then the element c centralizes B_0 and $t^2 = 1 = c^2b^2 = b^2$ implies that $b \in Z(B)$ and $c \in C_G(B)$. We have $X = B\langle t\rangle = B \times \langle c\rangle$. Since for any $g \in G$ we have $g = b_1a_1$ for some $a_1 \in A$, $b_1 \in B$, the conjugate of c by g is $c^g = c^{a_1} \in A$. Hence $E = \langle c^G\rangle \subseteq A$. However $E' \leqslant A_0$ is a normal subgroup of G. Therefore $E' \leqslant K = 1$. But $[c, A] = A_0^2$. This implies that $A_0^4 = 1$, i.e. the order of A is at most 8.

If $|A| = 4$, then $G = A_0X$ implies that X is normal in G. Since $K = 1$, the subgroup B_0 has no non-trivial normal subgroups of G. Therefore $B_0 \cap B_0^y = 1$ for some $y \in G$. Hence $|B_0|^2 < |X| = 4|B_0|$ by the structure of X. This implies that $|B_0| \leqslant 2$ and $|G| \leqslant 16$. In this case $G^{(2)} = 1$.

Now we may assume $|A| = 8$ and $G = A_0X$ with $X \cap A_0 = 1$. If $E = A$, then $A' \neq 1$ is normal in G, a contradiction. Thus $|E| = 4$ and the conjugacy class of c consists of c and ca_0, where a_0 is an element of order 2 in A_0. In this case $a_0 \in Z(G)$, which is a contradiction. Hence $G^{(2)} = 1$. Moreover, this implies that $G^{(4)} = 1$ in the general case $K \neq 1$.

By symmetry, the case $t = ad$ also leads to the conclusion $G^{(4)} = 1$.

Assume now that $t = cd$, and let $T = \langle t\rangle \leqslant G'$. Since $t = cd \in Z(G)$, we have that $a^d = a^{-1}$ for each $a \in A_0$ and $b^c = b^{-1}$ for each $b \in B_0$. Then in the factor-group $\bar{G} = G/T$ the subgroups $\bar{A}_0^2 = A_0^2T/T$ and $\bar{B}_0^2 = B_0^2T/T$ permute and generate a normal subgroup $\bar{M}$ in $\bar{G}$. The full preimage M of $\bar{M}$ in G is a product of two abelian subgroups and has derived length at most 2. Recall that $t \in Z(G) \cap G'$, so that $M \leqslant G'$. Then G/M has order 8. This implies by the above considerations that $G' = M$.

Since the derived length of G in the case $K = 1$ is at most 3, the derived length of a 2-group $G = AB$ is at most 5 in the general case when $K \neq 1$. Thus in all cases the lemma is proved. □

Consider now a finite group $G = AB$ that is the product of two dihedral subgroups A and B. It is well-known that for every prime p there exists a Sylow p-subgroup of G which is the product of a Sylow p-subgroup of A and a Sylow p-subgroup of B (see for instance [6], Chapter VI, Satz 4.7, p. 676). In our case the Sylow p-subgroups of G for an odd prime p are products of two cyclic subgroups. This implies that the Sylow p-subgroups of G are metacyclic by a result of B. Huppert (see [6], Chapter III, Satz 11.5, p. 338). Since G is soluble by [10], even the Hall $2'$-subgroups of G are metacyclic (see [6], Chapter VI, Satz 4.8, p. 676). By Lemma 2.3 the Sylow 2-subgroup of G have derived length at most 5. By Theorem 3 in [7] the derived length of G is at most $2 \cdot 2 \cdot 5 + 2 + 5 = 27$. The following theorem gives an even better bound.

Lemma 2.4. *Let $G = AB$ be a finite group, which is a product of a subgroups A and B, where A is dihedral and B is either cyclic or a dihedral group. Then $G^{(7)} = 1$.*

Proof. Assume that the lemma is false and let $G = AB$ be a minimal counterexample to the conclusion of the lemma. Since G/N is likewise a product of two dihedral groups for every normal subgroup N of G, we see that G has a unique minimal normal subgroup of G. In particular, $F(G)$ is a p-group for some prime $p \in \pi(G)$.

Suppose first that $p = 2$ and let $X = O(A)$ and $Y = O(B)$ be the maximal normal subgroups of odd order in A and B. Since the group G is soluble there exists a Hall $2'$-subgroup $H = XY = YX$ of G (see [6], Chapter VI, Satz 4.8, p. 676). By Lemma 2.2 we have $[X^G, Y^G] = 1$, or $O(G) \neq 1$. As $p = 2$, and $O(G) = 1$ it follows that $X = 1$, or $Y = 1$. Therefore the Hall $2'$-subgroup H of G is cyclic. Without loss of generality we may assume that $Y = 1$, so that B is a metacyclic 2-group and $H = O(A)$. Let $\sigma = \pi(G) \setminus \{2\}$ and denote by L the subgroup $O_{\sigma',\sigma}(G)$. Clearly, $L/F(G) = F(G/F(G))$ is the Fitting subgroup of $G/F(G)$. By [6], Chapter III, Satz 4.2 b) it follows that $C_{G/F(G)}(L/F(G)) \leqslant L/F(G)$. Since the Hall σ-subgroup H of G is cyclic, this implies that $H \simeq L/F(G)$ and $G/L \leqslant Aut(H)$ is an abelian group. Therefore the derived length of $G/F(G)$ does not exceed 2. Since $F(G)$ is a subgroup of the Sylow 2-subgroup of G, which is a product of two dihedral subgroups, we obtain by Lemma 2.3 that the derived length of $F(G)$ does not exceed 5. Hence the derived length of G is at most 7, as claimed.

Now let $p > 2$. Then the p-group $F(G)$ is contained in a Sylow p-subgroup of G. Since G is a product of two dihedral groups, a Sylow p-subgroup of G is metacyclic. Thus $F(G)$ is a cyclic or a metacyclic group. In particular, $F(G)/\Phi(F(G))$ is of order p or p^2. The group $G/F(G)$ is a subgroup of the automorphism group of $F(G)$, having no normal p-subgroups. It follows from [6], Chapter III, Sätze 3.17 and 3.18, pp. 274–275, that $G/F(G)$ is isomorphic to a soluble subgroup of the group $GL_2(p)$. By a theorem of Dickson the derived length of $G/F(G)$ does not exceed 4 (see [6], Chapter II, Satz 8.27, pp. 213–214). Therefore the derived length of G in this case is at most 6. The lemma is proved. $\square$

Remark 2.5. It was shown in [4] that there exists a finite 2-group $G = AB$ with derived length 3, which is a product of two dihedral subgroups.

3. The counterexample

From now on we consider a counterexample $G = AB$ for Theorem 1.1. Thus $G = AB$ with periodic locally dihedral subgroups A and B. Then $A = A_0\langle c\rangle$, $B = B_0\langle d\rangle$ for two involutions $c \in A \setminus A_0$ and $d \in B \setminus B_0$, with $cac = a^{-1}$ for each $a \in A_0$ and $dbd = b^{-1}$ for each $b \in B_0$; A_0 and B_0 are locally cyclic normal subgroups of A resp. B. It follows from Lemma 2.1 and Itô's theorem that both subgroups A and B are non-abelian.

Lemma 3.1. *We may assume that G has no nontrivial soluble normal subgroups.*

Proof. Assume that $N \neq 1$ is a soluble normal subgroup of G. Then $R = NA = A(NA \cap B)$ by Dedekind's modular law. Also R is a soluble group and so locally finite. Clearly, if L is a finite normal subgroup of A and S is a finite normal subgroup in $R \cap B$, it follows from [1], Lemma 2.2, that $K = N_R(\langle L, S\rangle) = (K \cap A)(K \cap B)$, moreover $H = \langle L, S\rangle$ is finite. Therefore the finite group $K/C_K(H)$ is the product of two subgroups of dihedral groups. From Lemma 2.4 we conclude that $(K/C_K(H))^{(7)} = 1$. Since $H \cap C_K(H) = Z(H)$ this implies that $H^{(8)} = 1$. Let $R_0 = \langle A_0, R \cap B_0\rangle$. Since $|R : R_0| \leqslant 4$, it follows that $R^{(9)} = 1$. In particular, N is of derived length at most 9. Therefore the product T of all soluble normal subgroups of G is a soluble group of derived length at most 9.

Now $G/T = (AT/T)(BT/T)$ is likewise the product of two periodic locally dihedral subgroups and G/T has no proper soluble normal subgroups. This proves the lemma. $\square$

Lemma 3.2. $A_0 \cap B = 1 = A \cap B_0$.

Proof. Assume that $A \cap B_0 = D \neq 1$. Clearly D is normal in B and since $A_G = \bigcap_{g \in G} A^g = \bigcap_{b \in B} A^b$ contains D, the group A_G is a non-trivial soluble normal subgroup of G. This contradicts Lemma 3.1. Similarly, also $A_0 \cap B = 1$. The lemma is proved. □

4. The special case $A \cap B = \langle c \rangle$

We exclude first the special case when $A \cap B = \langle c \rangle$, where $A = A_0\langle c \rangle$ as defined above. This implies that $AB_0 = A_0B = G$. For the brevity of notation we call this group *a Σ-group*.

As in [1] we denote by $C_H^*(x) = \{y \in H \mid x^y = x^{\pm 1}\}$ the extended centralizer of the element x in the group H. Set also $H^\# = H \setminus \{1\}$.

Lemma 4.1. *Let $G = AB$ be a Σ-group. Then $C_G^*(a) \leqslant A$ for every $a \in A_0^\#$ and $C_G^*(b) \leqslant B$ for every $b \in B_0^\#$.*

Proof. Let $a \in A_0^\#$ and $C_G^*(a) = AB_1 = K$ for $1 < B_1 \leqslant B_0$. Then $M = K_G = \bigcap_{g \in G} K^g = \bigcap_{b \in B} K^b$ contains B_1. By Dedekind's law $M = A_1B_1$ for some $A_1 \leqslant A$. If A_1 is abelian, then M is metabelian by Itô's theorem. This contradicts Lemma 3.1. Hence A_1 is non-abelian. But in this case $A^2 \leqslant M$, so that G/M is a product of an abelian group AM/M of order dividing 4 and an abelian group B_0M/M. Therefore G/M is soluble. Obviously $a \in C_G(M')$, so that $C_G(M') \neq 1$. If $a \in C_G(M') \cap M'$, then $Z(M') = C_G(M') \cap M'$ is a non-trivial soluble normal subgroup of G. If $C_G(M') \cap M' = 1$, then $C_G(M')$ is isomorphic to a subgroup of G/M', which is soluble by the above. In both cases we have a contradiction by Lemma 3.1. The case, when $C_G(b) \neq B$ for some $b \in B_0^\#$ is treated similarly. The lemma is proved. □

Lemma 4.2. *Let $G = AB$ be a Σ-group. Then $O_2(A_0) = 1 = O_2(B_0)$.*

Proof. Suppose that $O_2(B_0) \neq 1$. Then there exists an involution $\nu \in O_2(B) \cap Z(B)$. It follows from $G = A_0B_0 \cup A_0cB_0$ that for $a \in A_0$ the element νac can be expressed in one of the forms: $\nu ac = a_1y$, or $\nu ac = a_1cy$ with $a_1 \in A_0, y \in B_0$. If $\nu ac = a_1y$, then $\nu = a_1yac$ is an involution. This implies that y is inverted by $aca_1 = ca^{-1}a_1 \in A$. By Lemma 4.1 it follows that $a_1 = a$. Hence $a^{-1}\nu a \in B$. If $\nu ac = a_1cy$, then $\nu acy^{-1} = a_1c$ is an involution and the element $y^{-1}\nu$ is inverted by the involution ac. By Lemma 4.1 this is possible only when $a = 1$ or $y = \nu$. In this case $\nu ac = a_1c\nu$, which implies $\nu a = a_1\nu$. Therefore $a_1, a_1^\nu \in A_0$, thus $(a_1a_1^\nu)^\nu = a_1^\nu a_1 = a_1a_1^\nu$, so that $\nu \in C_B^*(a_1a_1^\nu)$. Then, by Lemma 4.1, $a_1a_1^\nu = 1$ and $a_1^\nu = a_1^{-1}$. Again by Lemma 4.1 this implies $a_1 = 1$, which is not true. Hence the only possibility is that $\nu^a \in B_0$ for each $a \in A_0$, a contradiction. Therefore $O_2(B_0) = 1$. By symmetry, $O_2(A_0) = 1$. The lemma is proved. □

Lemma 4.3. *Every Σ-group is soluble.*

Proof. Assume there exists a nonsoluble Σ-group G. Then by Lemma 4.2 we have $O_2(A_0) = O_2(B_0) = 1$. It follows that $C_G(c) = C_A(c)C_B(c) = \langle c \rangle$. We prove that A_0B_0 is a subgroup.

Assume that for some $a \in A_0, b \in B_0$ we have $ba = a_1b_1c$ with $a_1 \in A_0$, $b_1 \in B_0$. Then the element $a_1^{-1}ba = b_1c$ is an involution. Hence $baa_1^{-1} = ba_2$ is an involution. Thus $ba_2 = bcca_2$ is a product of two involutions. Since the involutions bc and ca_2 are conjugate to c, this is a contradiction. Therefore $B_0A_0 = A_0B_0$ is a subgroup of G of index 2. By Itô's theorem G is soluble. This contradiction proves the lemma. □

5. The general case

Let $G = AB$ be a counterexample for Theorem 1.1, $A = A_0\langle c \rangle$ with $cac = a^{-1}$ for each $a \in A_0$, $B = B_0\langle d \rangle$ with $dbd = b^{-1}$ for each $b \in B_0$, c, d are involutions and A_0 and B_0 are locally cyclic.

Lemma 5.1. *We have $A \cap B = 1$, so that no element in $A^{\#}$ is conjugate to an element in $B^{\#}$.*

Proof. By Lemma 4.3 we may assume that $A \cap B = 1$. If $x \in A$ is conjugate with $y \in B$ by an element $g \in G$, then $g = ab$, where $a \in A$, $b \in B$. Therefore $x^g = y$ implies that $x^a = y^{b^{-1}} \in A \cap B = 1$. $\square$

Lemma 5.2. *For every $a \in A_0^{\#}$ we have that $C_G^*(a) \cap B_0 = 1$ and $C_G^*(b) \cap A_0 = 1$ for every $b \in B_0^{\#}$.*

Proof. Assume that $C_G^*(a) \cap B_0 \neq 1$. Since $A \leqslant C_G^*(a)$ it follows that $K = C_G^*(a) = AB_1$ for some $B_1 \leqslant B$ and $L = B_0 \cap B_1 \neq 1$. This implies that the normal subgroup $M = K_G = \bigcap_{g \in G} K^g = \bigcap_{b \in B} K^b$ contains L.

If $M \cap B_1$ contains an element in $B \setminus B_0$, then $B^2 \leqslant M$ and $G/M = (AM/M)(BM/M)$ is a product of a locally dihedral or abelian group AM/M and a group BM/M of order at most 4. Therefore G/M and also G/M' is soluble. Hence $C_G(M')/Z(M') = C_G(M')/M' \cap C_G(M') \simeq C_G(M')M'/M'$ is also soluble. Thus $C_G(M')$ is soluble. Since $a \in C_G(M')$, the group $C_G(M')$ is a non-trivial soluble normal subgroup of G. This contradicts Lemma 3.1.

Therefore, $M \cap B_1 \leqslant B_0$. By Lemma 3.1 this is a contradiction. If $M = (A \cap M)L$ and $A \cap M \leqslant A_0$, then M is metabelian by Itô's theorem, again a contradiction. If $A \cap M$ contains an element in $A \setminus A_0$, then $A^2 \leqslant M$ and G/M is soluble. This implies that either $C_G(M')$ is soluble, or $M' \cap C_G(M') = Z(M')$ is non-trivial. Both cases cannot occur.

Then there is no element of M in $A \setminus A_0$ and in $B \setminus B_0$. Since M is not factorized, there exists an element $g \in M$ of one of the forms $g = a_1cb$, $g = a_1db$ or $g = a_1cdb$ with $a_1 \in A_0$, $b \in B_0$. Since A and B are locally dihedral, we can replace c by a_1c and d by db. Thus we may suppose that $g = cb$, $g = a_1d$, or $g = cd$.

If $g = a_1d$, then $a_1M = dM$ and in the factor-group $\bar{G} = G/M$, the subgroups $\bar{A} = AM/M$ and $\bar{B} = BM/M$ have a non-trivial intersection $\langle dM \rangle \leqslant \bar{A}_0 \cap \bar{B}$. It follows that the normal closure of dM in $\bar{G}$ coincides with the normal closure of dM in $\bar{B}$, hence it contains $\bar{B}^2$. In this case G/M is soluble. By the symmetry between A and B, the same conclusion follows, when $g = cb$.

Suppose that $g = cd$. Then $cM = dM$ and $\bar{G} = G/M$ is the product of two locally dihedral subgroups $\bar{A} = AM/M$ and $\bar{B} = BM/M$ such that $\bar{A} \cap \bar{B} = \langle cM \rangle = \langle dM \rangle$. Hence G/M is a Σ-group, which is soluble by Lemma 4.3. This contradiction proves the lemma. $\square$

For any group X denote by $O(X)$ the largest normal subgroup of X having no involutions.

Lemma 5.3. *Either $A_0 = O(A)$, or $B_0 = O(B)$.*

Proof. Assume that the lemma is false. Then there exist involutions $\tau \in Z(A)$ and $\mu \in Z(B)$. Clearly, the subgroup $D = \langle \tau, \mu \rangle$ is dihedral. By [1], Lemma 2.2 (iii) we have $H = N_G(D) = A_1B_1$, where $A_1 \leqslant A$, $B_1 \leqslant B$. By Lemma 3.1 it follows that $H \neq G$. Since A_1 normalizes D, this implies that $U = A_1D = A_1B_2$ for some $B_2 \leqslant B_1$ is a soluble group, which is a product of two periodic soluble subgroups. By [3], Theorem 3.2.6, the group U is periodic. In particular, D is a finite dihedral group. Now we may use Lemma 2.12 in [1] and Lemma 5.2 to deduce that D is even a dihedral group of order 8.

Since μ and τ are not conjugate, there exists an involution $\nu \in Z(D)$ different from τ and μ. Clearly, $\nu \in C_G(\tau) \cap C_G(\mu)$. Since $C_G(\tau) \geqslant A$ and $C_G(\mu) \geqslant B$ and $A \cap B = 1$, this implies that $C_G(\tau) > A$ or $C_G(\mu) > B$. Without loss of generality suppose that $F = C_G(\mu) > B$. By Lemma 5.2 we have $F \cap A_0 = 1$ and $|F : B| = 2$. Since A is locally dihedral and $A = A_0(F \cap A)$ with $|F \cap A| = 2$, we may assume without loss of generality that $F = B\langle c \rangle$ and $c = \nu$.

Since B is normal in F, also the subgroup $Z(B)$ is normal in F. We claim that for every $a \in O(A)^{\#}$ the element $\mu^a \in F$. Clearly, $\mu ac \in A_0F = A_0B \cup A_0cB$. Hence either $\mu ac = a_1cb$ or $\mu ac = a_1b$ with $a_1 \in A_0$, $b \in B$.

If $\mu ac = a_1cb$, then $\mu acb^{-1} = a_1c$ is an involution. Hence ac inverts $b^{-1}\mu$. If $b^{-1}\mu \in B_0$, then $C_G^*(b^{-1}\mu)$ contains ac. Then $C_G^*(b^{-1}\mu) = F_1 \geqslant B$ and by Lemma 5.2 it follows that $|F_1 : B| = 2$. Therefore $B' = B_0^2$ is normal in $F_2 = \langle F, F_1 \rangle = A_2B$, where $A_2 \leqslant A$. Clearly, F_2 is soluble. If $c \neq ac$, then A_2

contains $a \in A_0^\#$ and $AF_2 = G$ with $a, c \in A \cap F_2$. The normal closure in G of the subgroup $A_0 \cap F_2$ is a subgroup of F_2. By Lemma 3.1 this is a contradiction. Hence $ac = c$, or $b = \mu$. Since $a \neq 1$ by the choice of a, this means that $\mu = b$.

Thus $\mu ac = a_1 c\mu = a_1 \mu c$, which forces $\mu a = a_1 \mu$. Hence $a \in A_0 \cap A_0^\mu$ and $C_G^*(a_0)$ for some $a_0 \in A_0 \cap A_0^\mu$ contains $\mu \in B_0$. By Lemma 5.2 this is a contradiction.

Now $b \in B \setminus B_0$ is an involution. Then $b = a_1 c\mu ac$ implies that $\mu aca_1 c\mu = \mu aa_1^{-1}\mu = a^{-1}a_1$. By Lemma 5.2 this means that $a_1 = a$. Hence $\mu^a = cbc \in F$, as claimed.

Assume that $\mu ac = a_1 b$. Then $\mu = a_1 bac$ is an involution. Therefore $aca_1 = a^{-1}a_1 c$ inverts b. If $b \in B_0$, then $a = a_1$ and again $\mu^a = bc \in F$. Suppose that $b \in B \setminus B_0$. Then b is an involution and $a_1^{-1}\mu ac$ is an involution. This implies that $aca_1^{-1} = aa_1 c$ inverts μ. By Lemma 5.2 this means that $a_1 = a^{-1}$ and $\mu ac = a^{-1}b$. If $d = b\mu$ then $\mu ac\mu = a^{-1}d$ is an involution. Since $d = b\mu$ is an involution, d inverts a.

In this case $L = C_G^*(a) = A(L \cap B)$, where $L \cap B = \langle d\rangle$ with $d \in B \setminus B_0$. Observe that the involution $\mu(ac)\mu$ is conjugate to ac. Since a is of odd order, this means that ac is conjugate with c. On the other hand, $a^{-1}d$ is an involution, which is conjugate to d. Therefore c and d are conjugate, which is impossible.

Hence for every $a \in O(A_0)$ it follows that $\mu^a \in F$. This implies that the subgroup $R = \langle \mu^{O(A)}\rangle$ is an $O(A)$-invariant subgroup of F. From the structure of $F = B\langle c\rangle = (B_0\langle d\rangle)\langle c\rangle$ we conclude that R is elementary abelian of order at most 8. Since no element in $O(A)^\#$ centralizes μ, it follows that $O(A)$ is finite. Since A_0 is locally cyclic with finite subgroup $O(A)$, this means that $O_2(A)$ and A are Chernikov groups.

Since A is infinite, this implies that $O_2(A)$ is quasicyclic. Therefore there exists $a_0 \in O_2(A)$ such that $a_0^2 = \tau$ and $a_0 \in N_G(\langle \tau, c\rangle)$. Recall that $c \in Z(\langle \tau, \mu\rangle)$ and $D = \langle \tau, \mu\rangle$ is dihedral of order 8. The subgroup $S = \langle a_0, \mu\rangle$ normalizes D and is a proper subgroup of G. Since D is normal in $S = \langle a_0, \mu\rangle$, also $\langle c\rangle = Z(D)$ is normal in S. But $a_0^c = a_0^{-1}$, a contradiction. The lemma is proved. □

6. Proof of Theorem 1.1

Let $G = AB$ be a counterexample for Theorem 1.1, $A = A_0\langle c\rangle$ with $cac = a^{-1}$ for each $a \in A_0$, $B = B_0\langle d\rangle$ with $dbd = b^{-1}$ for each $b \in B_0$, where c, d are involutions, A_0 and B_0 are locally cyclic.

Lemma 6.1. *If $C_A^*(b) = 1$ for every $b \in B_0^\#$, then there exist involutions $c_1 \in A \setminus A_0$ and $d_1 \in B \setminus B_0$ such that $c_1 d_1 = d_1 c_1$. Moreover, if d and c do not commute, then either $dc = acb$ or $dc = adb$ for $a \in A_0$, $b \in B_0$.*

Proof. Assume that $dc \neq cd$. If $dc = ab$ with $a \in A_0$, $b \in B_0$, then $cdc = cab$ is an involution. Hence b is inverted by ca, which is a contradiction. If $dc = acdb$ with $a \in A_0$, $b \in B_0$, then $dcd = acdbd = acb^{-1}$ and b is inverted by ac, which is not the case. Hence either $dc = acb$ or $dc = adb$ with $a \in A_0$, $b \in B_0$. In the first case $dcd = acbd$ is an involution and bd is an involution. The product of two involution is an involution only, when they commute. Hence we may replace ac by c_1 and bd by d_1, obtaining the required conclusion $c_1 d_1 = d_1 c_1$. The case $dc = adb$ is treated similarly. The lemma is proved. □

Lemma 6.2. *Either $C_A^*(b) \neq 1$ for some $b \in B_0^\#$ or $C_B^*(a) \neq 1$ for some $a \in A_0^\#$.*

Proof. Suppose that $C_A^*(b) = C_B^*(a) = 1$ for every $a \in A_0^\#$ and $b \in B_0^\#$. By Lemma 6.1 we may assume that there exist involutions $c \in A \setminus A_0$, $d \in B \setminus B_0$ such that $cd = dc$.

We prove first that for every $a \in A_0^\#$ we have $dac = a^{-1}db$ for some $b \in B_0$. Also, for every $b \in B_0^\#$ we have $dbc = acb^{-1}$.

If d and ac commute, then $dac = acd$ implies that $da = ad$, which is not the case. By Lemma 6.1 it follows that $dac = a_1 cb$ or $dac = a_1 db$ for some $a_1 \in A_0$, $b \in B_0$. Assume that $dca^{-1} = dac = a_1 cb$. This implies that $dc = a_1 cba$ is an involution. Hence $aa_1 c$ inverts b. Since $b \neq 1$, this is a contradiction. Therefore $dac = a_1 db$. Since db is an involution, $a_1^{-1}dac$ is an involution and d commutes with $aa_1 c$. This is only possible when $aa_1 = 1$, so that $dac = a^{-1}db$, as required. Moreover, $dacd = a^{-1}b^{-1}$ and the element $a^{-1}b^{-1}$ is an involution in $A_0 B_0$, as claimed.

The equality $dbc = acb^{-1}$ holds by the symmetry between A and B.

Let $a^{-1}b^{-1}$ be an involution, where $a \in A_0, b \in B_0$. By what was proved above such an involution exists. Consider the element $dca^{-1}b^{-1} = d(ac)b^{-1} = (dac)b^{-1}$. It follows from the above considerations that $dac = a^{-1}db_1$ for some $b_1 \in B_0$. Since $dacd = a^{-1}b_1^{-1}$ is also an involution, we obtain that $ab = b^{-1}a^{-1}$ and ab_1 are involutions. Since $b^{-1}a^{-1}ab_1 = b^{-1}b_1$ is a product of the involutions ab and $ab_1 = b_1^{-1}a^{-1}$, we conclude that $b^{-1}b_1$ is inverted by $b^{-1}a^{-1}$ and thus by a^{-1}. Hence $C^*_{A_0}(b^{-1}b_1) \neq 1$. By Lemma 5.2 this means that $b_1 = b$. Therefore $dac = a^{-1}db$ and

$$dca^{-1}b^{-1} = \big(dca^{-1}\big)b^{-1} = (dac)b^{-1} = a^{-1}dbb^{-1} = a^{-1}d.$$

On the other hand, $dbc = acb^{-1}$, which implies

$$dca^{-1}b^{-1} = dacb^{-1} = d\big(acb^{-1}\big) = d(dbc) = bc.$$

Hence $a^{-1}d = bc$ and $a^{-1}c = bd \in A \cap B$. By Lemma 4.3 this is a contradiction. The lemma is proved. □

Lemma 6.3. *If $C^*_B(a) \neq 1$ for some $a \in A_0^\#$, then B_0 contains no involutions.*

Proof. Suppose that $C^*_B(a) \neq 1$ for some $a \in A_0^\#$. By Lemma 5.2 we have $C_G(a)^* \cap B_0 = 1$. Since $F = C^*_G(a)$ contains A, it follows that $F = A(F \cap B)$ and $F \cap B$ is of order 2. But every element $y \in B \setminus B_0$ is an involution. By Lemma 5.3 either A_0 or B_0 contain no involutions. If $A_0 = O(A_0)$, then we may choose an element $c \in A \setminus A_0$ commuting with $y \in F \cap B$, so that we may take $y \in F \cap B$ as d. If $B_0 = O(B_0)$, then all involutions in B are conjugate and we may take y as d.

Assume that B_0 contains an involution ν. Recall that B is locally dihedral. Since $F \cap B = \langle d\rangle$ and $G = AB = FB$, it follows from Lemma 2.2(i) in [1] that $E = C_G(d) = C_F(d)C_B(d) = C_A(d)\langle d, \nu\rangle$. Assume that $C_{A_0}(d) > 1$. Then $E \cap F = (E \cap A)\langle x\rangle$ is normal in E, but $E \cap A_0$ has index 2 in $E \cap F$, so that $1 \neq C_{A_0}(d) = E \cap A_0 = O(E \cap A_0)$ is normal in E. Therefore $C^*(a_1) \cap B_0$ contains ν for some $a_1 \in (E \cap A_0) \setminus 1$, contradicting Lemma 5.2. Therefore $C_{A_0}(d) = 1$.

This means that d inverts every element in A_0, so that cd centralizes A_0. Now $G = AB = AF = \langle A_0, cd\rangle\langle d\rangle B$ with locally dihedral subgroups F and B having an intersection $\langle d\rangle$, i.e. G is a Σ-group. By Lemma 4.3 this is a contradiction. Hence B_0 contains no involutions. The lemma is proved. □

From now on we assume that $F = C^*_G(a') \neq A$ for some $a' \in A_0^\#$. By what was proved above $F = A\langle d\rangle$ and $B_0 = O(B)$.

Lemma 6.4. $O(A) = A_0$.

Proof. Assume that μ is an involution in A_0. Obviously, $\mu \in Z(A)$. Since $|F : A| = 2$ and A is normal in F, we may assume that $\mu \in Z(F)$. Since $G = FB_0$, we have that $G = AB_0 \cup AdB_0$. Therefore for every $b \in B_0^\#$ either $db\mu = adb_1$ or $db\mu = ab_1$ with $a \in A$, $b_1 \in B_0$. We prove that $b^{-1}\mu b \in F$ for every $b \in B$.

Indeed, if $db\mu = ab_1$, then $\mu = dbab_1$. By Lemma 3.2 we have $a \neq 1$. It follows that $b_1db = b_1b^{-1}d$ inverts a. If $a \in A_0$, then $b_1 = b$ and $db\mu = ab$. This implies that $b\mu b^{-1} = da \in F$.

Assume that $a = b^{-1}d\mu b_1^{-1}$ is an involution. Then $b_1^{-1}db\mu$ is also an involution. It follows that $b_1^{-1}b^{-1}d\mu = \mu b^{-1}b^{-1}d$ and so $b_1 = b^{-1}$. Hence $db\mu = ab^{-1}$ and so $ab^{-1}\mu = db$ is an involution. Therefore $ab^{-1}\mu ab^{-1}\mu = 1$. Thus μa inverts b, a contradiction. Hence a is not an involution and $b\mu b^{-1} \in F$.

Assume now that $bd\mu = adb_1$. Then $a^{-1}bd\mu = db_1$ is an involution and bd inverts $a^{-1}\mu$. If $a^{-1}\mu \in A_0^\#$, then $b = 1$, a contradiction. Suppose that $\mu = a$. Then $b\mu d = bd\mu = \mu db_1 = \mu b_1^{-1}d$ implies $b\mu = \mu b_1$. Hence $b \in B_0 \cap B_0^\mu$. By Lemma 5.2 this is a contradiction. Therefore $a^{-1}\mu \in A \setminus A_0$ is an involution. Note that in the case a is also an involution. On the other hand, $bd\mu = b\mu d = ab_1^{-1}d$ and $b\mu = ab_1^{-1}$. Hence $b\mu b_1$ is an involution and μ inverts $b_1^{-1}b$. This implies that $b = b_1$ and $b\mu b^{-1} \in A \leqslant F$.

Hence $\langle \mu^{B_0} \rangle$ is a B_0-invariant subgroup of F. This contradiction proves the lemma.

Lemma 6.5. *$C_G(b) \leqslant B$ for every $b \in B_0^{\#}$.*

Proof. Since $O(A) = A_0$ and $O(B) = B_0$, we have that $A = A_0\langle c \rangle$ and $B = B_0\langle d \rangle$. Since $C_G^*(a) = F = A\langle d \rangle$ by Lemma 6.3 and $O(A) = A_0$, it follows that all Sylow 2-subgroups in F of order 4 are conjugate and we may choose $c \in A \setminus A_0$ and $d \in B \setminus B_0$ so that $cd = dc$. If $L = C_A(b)^* \neq 1$, then $A = A_0L$ with L of order 2. Since $C_G^*(b) \cap F$ contains d and $F = A_0\langle c, d \rangle$, and every element in $A \setminus A_0$ inverts the elements in A_0, we may assume that $S = \langle c, d \rangle = F \cap C_G^*(b)$. It is obvious that $C_{A_0}(c) = C_{B_0}(d) = 1$. If $C_{A_0}(d) = 1$, then cd centralizes A_0. Since $G = AB = A_0\langle cd \rangle(B_0S)$ and $cd \in (A_0\langle cd \rangle) \cap (B_0S)$ it follows that $(cd)^G \leqslant \bigcap_{g \in G}(B_0S)^g = \bigcap_{x \in A_0\langle cd \rangle}(B_0S)^x$. Hence G has a non-trivial normal soluble subgroup, a contradiction. A similar assertion holds for $C_{B_0}(c)$.

Therefore, we may assume that $C_{A_0}(d) \neq 1 \neq C_{B_0}(c)$. Also $C_{A_0}(cd) \neq 1 \neq C_{B_0}(cd)$. It follows from [1], Lemma 2.2(ii) that $C_G(cd) = C_A(cd)C_B(cd)$. Since $C_G(cd)/\langle cd \rangle = XY$ with locally dihedral subgroups X and Y such that $X \cap Y = S/\langle cd \rangle$, it follows that $C_G(cd)/\langle cd \rangle$ is a Σ-group. By Lemma 4.3 the group $R = C_G(cd) = (A \cap R)(B \cap R)$ is a soluble group.

Recall that the groups $A_0 \cap R$ and $B_0 \cap R$ are periodic and locally cyclic. Since $A_0 \cap R$ and $B_0 \cap R$ are permutable, it follows that $R_0 = (A_0 \cap R)(B_0 \cap R)$ is a group with minimum condition for p-subgroups for every prime p (see [11], Theorems 3.5 and 3.2). There exists a non-trivial normal subgroup L of R_0 contained in $A_0 \cap R$, say (see [11] or [3], Theorem 7.1.2). Clearly, we may assume that L is finite, so that also $|R_0 : C_{R_0}(L)|$ is finite. In particular, there exists a non-trivial element $x \in L$ such that $C_{A_0}(x) \neq 1 \neq C_{B_0}(x)$. This contradicts Lemma 5.2. The lemma is proved. □

Lemma 6.6. *For every $b \in B_0^{\#}$ there exists $a \in A_0^{\#}$ such that $cbc = dab^{-1}$.*

Proof. We have

$$G = A_0B_0 \cup A_0cB_0 \cup A_0dB_0 \cup A_0cdB_0.$$

Hence we consider the following four cases.

(1) If $dbc = ab_1$, then $db = ab_1c$. Hence ca inverts b, which is impossible.
(2) If $dbc = adb_1$, then $b^{-1}cd = ab_1^{-1}d$. It follows that $b^{-1}cd = ab_1^{-1}d$ and $cd = bab_1^{-1}d$. Hence $b_1^{-1}db = db_1b$ inverts a. Since $a \neq 1$, it follows that $b_1 = b^{-1}$ and thus $dbc = adb^{-1} = abd$. This implies that $b^{-1}(dc)b = ac$, so that dc is conjugate with c. On the other hand, $G = (A_0\langle c, d \rangle)B_0 = (A_0\langle dc \rangle)B$ and $A_0\langle dc \rangle \cap B = 1$. Hence dc and d are not conjugate.
(3) If $dbc = acdb_1$, then $cdbc = cacdb_1 = a^{-1}db_1$ is an involution. Hence db_1 inverts a, which is possible only when $b_1 = 1$ or $a = 1$. In the first case $cdb = a^{-1}d$ and $cb^{-1} = a^{-1}$. This means that $b \in F$, a contradiction. If $dbc = cdb_1$, then $cb = b_1c$ and $b \in B_0 \cap B_0^c$, which is possible only when $b = b_1 = 1$.
(4) If $dbc = acb_1$, then $dbcb_1^{-1} = ac$ is an involution. Hence $cdb_1bc = cb_1^{-1}dbc = b_1^{-1}db = db_1b$. Since $cd = dc$, we have that $c(b_1b)c = b_1b$. This implies that $b_1 = b^{-1}$ and $dbc = acb^{-1}$. It follows that $b^{-1}(dc)b = dbcb = ac$. Hence c is conjugate with dc.

It follows that $dcbc = cdbc = cacb^{-1}$ and $cbc = da^{-1}b^{-1}$. Since $dcbc = cb^{-1}cd$, we have also $cb^{-1}c = a^{-1}b^{-1}d$. Replacing a by a^{-1}, we obtain $cbc = dab^{-1}$, as required. The lemma is proved. □

Now we can *finish the proof of Theorem* 1.1.

Let $b \in B_0$ be an element of order at least 5. By Lemma 6.6 for every $b \in B_0$ we have $cbc = dab^{-1}$ with some $a \in A_0$. Hence $cb^{-1}c = da_1b$ with $a_1 \in A_0$. Since ab^{-1} is conjugate with db, this is an involution. Similarly a_1b is an involution. Hence ba_1 is an involution and $aa_1 = ab^{-1}ba_1$ is a product of two involutions. Therefore aa_1 is inverted by ab^{-1} and so by b^{-1}. Thus either $aa_1 = 1$ or we have that $b^{-1} \in C_G^*(aa_1) \cap B_0 = 1$ by Lemma 5.2 and $b = 1$, a contradiction. Hence $aa_1 = 1$ and $cb^{-1}c = da^{-1}b$.

Obviously, $(cbc)^2 = da_2b^{-2}$ for some $a_2 \in A_0$. On the other hand,

$$(cbc)^2 = \left(cdb^{-1}c\right)(cdbc) = a^{-1}bab^{-1} = (cbc)(cbc) = dab^{-1}dab^{-1}.$$

It follows that $a^{-1}b = dab^{-1}d = dadb$. This implies $dad = a^{-1}$. Hence $(cbc)^2 = a^{-1}bab^{-1} = da_2b^{-2}$. Therefore $da^{-1}ba = a_2b^{-1}$ and $a^{-1}b^{-1}ad = a_2b^{-1}$. Thus $a^{-1}(a_2^{-1}b^{-1})a = b^{-1}d$ is an involution. In this case $a_2^{-1}b^{-1}$ is also an involution. Recall that ab^{-1} is an involution. As above, this implies that $a_2 = a^{-1}$.

However, $a_2b^{-2} = a^{-1}b^{-2}$ is an involution. Since $b^3 \neq 1$, the element $b^{-1}aa^{-1}b^{-2} = b^3$ is inverted by ba^{-1} and thus by a^{-1}. Then $a^{-1} \in C_G^*(b^3) \cap A_0 = 1$ by Lemma 5.2, a contradiction. Hence $B_0 \cap B_0^{a^{-1}}$ contains b^3. By Lemma 6.5 this is a contradiction. The theorem is proved. □

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