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Almost sure limit theorems for a stationary normal sequence[☆]

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Abstract

We prove almost sure limit theorems for the maximum of a stationary normal sequence under some conditions.

Keywords: Almost sure limit theorem; Extreme theory; Maximum; Normal comparison inequality; Stationary normal sequence

1. Introduction

The almost sure central limit theorem (ASCLT) was first discovered independently by Schatte [1] and Brosamler [2] for independent identically distributed random variables. Since then, there has been much work in this field. The case of independent non-identically distributed random variables was considered by Berkes and Dehling [3] and Matúla [4]. Recently ASCLT has been extended to the maxima of independent identically distributed random variables by Fahrner and Stadtmüller [5] and Cheng et al. [6].

The general form of the ASCLT is as follows. If $\{X_n, n \geq 1\}$ is a sequence of random variables with partial sums $S_n = \sum_{i=1}^n X_i$ satisfying $a_n(S_n - b_n) \xrightarrow{d} G$ for some numerical sequences $\{a_n\}$, $\{b_n\}$ and distribution function G, then under some conditions we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{\{a_k(S_k - b_k) \leq x\}} = G(x) \quad \text{a.s.}$$

for any continuity point x of G, where I is the indicator function.

Csáki and Gonchigdanzan [7] studied ASCLT for the maximum of a stationary normal sequence:

Theorem A. *Let $Z_1, Z_2, \ldots,$ be a standardized stationary normal sequence with $r_n = \operatorname{cov}(X_1, X_{n+1})$ satisfying $r_n \log n (\log\log n)^{1+\varepsilon} = O(1)$ as $n \to \infty$. Then*

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(1) *If* $n(1-\Phi(u_n)) \to \tau$ *for* $0 \le \tau < \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(M_k \le u_k)} = \exp(-\tau) \quad a.s.$$

where $\{u_n, n \ge 1\}$ *is a sequence of constants.*

(2) *If* $a_n = \sqrt{2\log n}$ *and* $b_n = a_n - \frac{1}{2}\frac{(\log\log n + \log 4\pi)}{a_n}$, *then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(a_k(M_k-b_k)\le x)} = \exp(-\exp(-x)) \quad a.s.$$

We further study ASCLT for the maximum of a stationary normal sequence.

2. Main results

Suppose that $\{X_n, n \ge 1\}$ is a stationary standardized normal sequence with covariance $r_n = \mathrm{cov}(X_1, X_{n+1})$ and $M_n = \max_{1\le i\le n}\{X_i\}$, $M_{k,n} = \max_{k+1\le i\le n}\{X_i\}$. $a \sim b$ stands for $\lim_{x\to\infty} \frac{a(x)}{b(x)} = 1$. $\Phi(x)$ is the standard normal distribution function and $\phi(x)$ is its density function.

Theorem 2.1. *Let* $\{X_n, n \ge 1\}$ *be a stationary standardized normal sequence with* $r_n \to 0$ *as* $n \to \infty$, *and satisfying*

$$\frac{1}{n} \sum_{1\le k\le n} |r_k| \log k \exp\{\gamma |r_k| \log k\} \le \frac{C}{(\log n)^{\varepsilon}} \tag{2.1}$$

for some $\varepsilon > 0$, $\gamma > 2$, $C > 0$.

(1) *If* $n(1-\Phi(u_n)) \to \tau$ *for* $0 \le \tau < \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(M_k \le u_k) = \exp(-\tau)} \quad a.s.$$

(2) *If* $a_n = \sqrt{2\log n}$ *and* $b_n = a_n - \frac{1}{2}\frac{(\log\log n + \log 4\pi)}{a_n}$, *then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(a_k(M_k-b_k)\le x)} = \exp(-\exp(-x)) \quad a.s.$$

Theorem 2.2. *Let* $\{X_n, n \ge 1\}$ *be a stationary standardized normal sequence with covariance sequence* r_n, *and put* $a_n = \sqrt{2\log n}$ *and* $b_n = a_n - \frac{1}{2}\frac{(\log\log n + \log 4\pi)}{a_n}$. *If*

$$\sum_{k=1}^{n-1} |r_k|(n-k)n^{-2/(1+|r_k|)}(\log n)^{1/(1+|r_k|)} \le \frac{C}{(\log n)^{\varepsilon}} \tag{2.2}$$

for some $\varepsilon > 0$, $C > 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(a_k(M_k-b_k)\le x)} = \exp(-\exp(-x)) \quad a.s.$$

Remark. If condition (2.1) is replaced by

$$\frac{1}{n} \sum_{1\le k\le n} |r_k| \log k \le \frac{C}{(\log n)^{\varepsilon}},$$

the conclusions of Theorem 2.1 are still true. If $C/(\log n)^{\varepsilon}$ is replaced by $C/(\log\log n)^{(1+\varepsilon)}$, the results of the theorems still hold.

3. Proof of main result

We need the following lemmas for the proof of the main result.

Lemma 3.1. *Let $\{X_n, n \geq 1\}$ be a stationary standardized normal sequence. Assume that $r_n \to 0$ as $n \to \infty$, and r_n satisfies (2.1). If $n(1 - \Phi(u_n))$ is bounded, then*

$$\sup_{1\leq k\leq n} k \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_k^2 + u_n^2}{2(1+|r_j|)}\right) \leq \frac{A}{(\log n)^{\varepsilon}}$$

for some $\varepsilon > 0$, $A > 0$.

Proof. Since $r_n \to 0$ as $n \to \infty$, we have $\delta \triangleq \sup_{n\geq 1} |r_n| < 1$ (cf. [8], p. 86). By assumption $n(1 - \Phi(u_n)) \leq K$ for some constant $K > 0$. Let $\{v_n, n \geq 1\}$ be a sequence such that $n(1 - \Phi(v_n)) = K$ if $n > K$, $v_n = u_n$ if $n \leq K$. Then clearly $v_n \leq u_n$ and hence

$$k \sum_{j=1}^{n} |r_j| \exp\left(-\frac{u_k^2 + u_n^2}{2(1+|r_j|)}\right) \leq k \sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_k^2 + v_n^2}{2(1+|r_j|)}\right).$$

Thus it is enough to prove the lemma for the sequence $\{v_n, n \geq 1\}$. Since $1 - \Phi(x) \sim \phi(x)/x$, $x \to \infty$, we can see that

$$\exp\left(-\frac{v_n^2}{2}\right) \sim \frac{K\sqrt{2\pi}\, v_n}{n}, \quad v_n \sim \sqrt{2\log n}. \tag{3.1}$$

Take $\beta = 2/\gamma$. Define α to be $0 < \alpha < \min(\beta, \frac{1-\delta}{1+\delta})$. Note that

$$\begin{aligned} k \sum_{j=1}^{n} |r_j| \exp\left(-\frac{v_k^2 + v_n^2}{2(1+|r_j|)}\right) &= k \sum_{j=1}^{[n^\alpha]} |r_j| \exp\left(-\frac{v_k^2 + v_n^2}{2(1+|r_j|)}\right) + k \sum_{j=[n^\alpha]+1}^{[n^\beta]} |r_j| \exp\left(-\frac{v_k^2 + v_n^2}{2(1+|r_j|)}\right) \\ &\quad + k \sum_{j=[n^\beta]+1}^{n} |r_j| \exp\left(-\frac{v_k^2 + v_n^2}{2(1+|r_j|)}\right) \\ &\triangleq T_1 + T_2 + T_3. \end{aligned}$$

Using (3.1)

$$\begin{aligned} T_1 &\leq kn^\alpha \exp\left(-\frac{v_k^2 + v_n^2}{2(1+\delta)}\right) = kn^\alpha \left(\exp\left(-\frac{v_k^2 + v_n^2}{2}\right)\right)^{\frac{1}{1+\delta}} \\ &\leq Ckn^\alpha \left(\frac{v_k v_n}{kn}\right)^{\frac{1}{1+\delta}} \leq Ck^{1-\frac{1}{1+\delta}} n^{\alpha - \frac{1}{1+\delta}} (\log k \log n)^{\frac{1}{2(1+\delta)}} \\ &\leq Cn^{1+\alpha-\frac{2}{1+\delta}} (\log n)^{\frac{1}{1+\delta}}, \end{aligned}$$

where C is a positive constant, whose value is irrelevant.

Since $1 + \alpha - \frac{2}{1+\delta} < 0$, we get $T_1 \leq n^{-\sigma}$ for some $\sigma > 0$, uniformly for $1 \leq k \leq n$. Writing

$$\delta_n = \sup_{m\geq n} |r_m|, \quad p = [n^\alpha], \quad q = [n^\beta],$$

using (3.1) again, we have

$$\begin{aligned} T_2 &\leq k \exp\left(-\frac{v_k^2 + v_n^2}{2}\right) \sum_{j=p+1}^{q} \exp\left(\frac{|r_j|(v_k^2 + v_n^2)}{2(1+|r_j|)}\right) \\ &\leq kn^\beta \exp\left(-\frac{v_k^2 + v_n^2}{2}\right) \exp\left(\frac{\delta_p(v_k^2 + v_n^2)}{2}\right) \end{aligned}$$

$$= kn^{\beta}\left(\exp\left(-\frac{v_k^2+v_n^2}{2}\right)\right)^{1-\delta_p} \le Ckn^{\beta}\left(\frac{\sqrt{\log k \log n}}{kn}\right)^{1-\delta_p}$$
$$\le Cn^{-1+\beta+2\delta_p}\log n.$$

Noting that $\delta_n = \sup_{m\ge[n^{\alpha}]}|r_m| \to 0$ as $n\to\infty$, we get $T_2 \le n^{-\eta}$ for some $\eta > 0$. Finally, again using (3.1), we have

$$T_3 = k\sum_{j=q+1}^{n}|r_j|\left(\exp\left(-\frac{v_k^2+v_n^2}{2}\right)\right)^{\frac{1}{1+|r_j|}} \le Ck\sum_{j=q+1}^{n}|r_j|\left(\frac{\sqrt{\log k\log n}}{kn}\right)^{\frac{1}{1+|r_j|}}$$
$$\le Cn^{-1}\log n\sum_{j=q+1}^{n}|r_j|\exp(2|r_j|\log n).$$

Since $j > q$, we have $\log j \ge \beta\log n$, and hence

$$T_3 \le Cn^{-1}\sum_{j=q+1}^{n}|r_j|\log j\exp(2\beta^{-1}|r_j|\log j)$$
$$\le Cn^{-1}\sum_{j=1}^{n}|r_j|\log j\exp(2\beta^{-1}|r_j|\log j)$$
$$\le Cn^{-1}\sum_{j=1}^{n}|r_j|\log j\exp(\gamma|r_j|\log j) \le \frac{C}{(\log n)^{\varepsilon}}.$$

The proof is completed. □

The following lemma is Theorem 2.1 and Corollary 2.1 in Li and Shao [9]:

Lemma 3.2. (1) *Let $\{X_n, n\ge 1\}$ and $\{Y_n, n\ge 1\}$ be sequences of standard normal variables with covariance matrices $R^1 = (r_{ij}^1)$ and $R^0 = (r_{ij}^0)$ respectively. Put $\rho_{ij} = \max(|r_{ij}^1|, |r_{ij}^0|)$. Then we have*

$$P\left(\bigcap_{j=1}^{n}\{X_j\le u_j\}\right) - P\left(\bigcap_{j=1}^{n}\{Y_j\le u_j\}\right)$$
$$\le \frac{1}{2\pi}\sum_{1\le i<j\le n}(\arcsin(r_{ij}^1)-\arcsin(r_{ij}^0))^{+}\exp\left(-\frac{u_i^2+u_j^2}{2(1+\rho_{ij})}\right)$$

for any real numbers u_i, $i = 1,2,\ldots,n$.

(2) *Let $\{X_n, n\ge 1\}$ be standard normal variables with $r_{ij} = \mathrm{cov}(X_i, X_j)$. Then*

$$\left|P\left(\bigcap_{j=1}^{n}(X_j\le u_j)\right) - \prod_{j=1}^{n}P(X_j\le u_j)\right| \le \frac{1}{4}\sum_{1\le i<j\le n}|r_{ij}|\exp\left(-\frac{u_i^2+u_j^2}{2(1+|r_{ij}|)}\right)$$

for any real numbers u_i, $i = 1,2,\ldots,n$.

Lemma 3.3. *Let $\{X_n, n\ge 1\}$ be a stationary standardized normal sequence. Assume that $r_n\to 0$ as $n\to\infty$, and r_n satisfies (2.1). If $n(1-\Phi(u_n))$ is bounded, then for $1\le k<n$*

$$|P(M_k\le u_k, M_{k,n}\le u_n) - P(M_k\le u_k)P(M_{k,n}\le u_n)| \le \frac{C}{(\log n)^{\varepsilon}},$$

and

$$P(M_{k,n}\le u_n) - P(M_n\le u_n) \le \frac{k}{n} + \frac{C}{(\log n)^{\varepsilon}}.$$

Proof. In Lemma 3.2(1) we take $Y_j = X'_j, 1 \le j \le k$, $Y_j = X_j, k+1 \le j \le n$, where $X_j, 1 \le j \le k$ and $X'_j, 1 \le j \le k$ have identical distributions, but $X'_j, 1 \le j \le k$ are independent of $X_j, k+1 \le j \le n$. Hence

$$|P(M_k \le u_k, M_{k,n} \le u_n) - P(M_k \le u_k)P(M_{k,n} \le u_n)| \le \frac{1}{4}\sum_{i=1}^{k}\sum_{j=k+1}^{n}|r_j|\exp\left(-\frac{u_k^2+u_n^2}{2(1+|r_j|)}\right)$$
$$\le \frac{1}{4}k\sum_{j=1}^{n}|r_j|\exp\left(-\frac{u_k^2+u_n^2}{2(1+|r_j|)}\right).$$

From Lemma 3.1 we get the first result.

Note that

$$P(M_{k,n} \le u_n) - P(M_n \le u_n) \le |P(M_{k,n} \le u_n) - (\Phi(u_n))^{n-k}| + |P(M_n \le u_n) - (\Phi(u_n))^n| + |(\Phi(u_n))^{n-k} - (\Phi(u_n))^n| \triangleq L_1 + L_2 + L_3.$$

From the elementary fact that $x^{n-k} - x^n \le \frac{k}{n}, 0 \le k \le n$, we have $L_3 \le \frac{k}{n}$. By Lemma 3.2(2), we have

$$L_i \le Kn\sum_{j=1}^{n}|r_j|\exp\left(-\frac{u_n^2}{1+|r_j|}\right), \quad i = 1, 2.$$

Thus by Lemma 3.1 we have $L_i \le \frac{C}{(\log n)^\varepsilon}$. The proof is completed. □

The following lemma is from [5]:

Lemma 3.4. *Let $\{\xi_n, n \ge 1\}$ be a sequence of bounded random variables, i.e. there exists some $M \in (0,\infty)$ such that $|\xi_k| \le M$ a.s. for all $k \in N$, satisfying $E\xi_k \to \mu$ as $k \to \infty$. Suppose furthermore that*

$$\mathrm{Var}\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\xi_k\right) \le \frac{C}{(\log n)^\varepsilon}$$

for some $\varepsilon > 0$. Then we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\xi_k \to \mu \quad a.s.\ as\ n \to \infty.$$

Proof of Theorem 2.1. First, we claim that under the assumptions of Lemma 3.1, we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}(I_{(M_k \le u_k)} - P(M_k \le u_k)) = 0 \quad \text{a.s.} \tag{3.2}$$

By Lemma 3.4 it is sufficient to prove

$$\mathrm{Var}\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I_{(M_k \le u_k)}\right) \le \frac{C}{(\log n)^\varepsilon} \tag{3.3}$$

for some $\varepsilon > 0$. Let $\xi_k = I_{(M_k \le u_k)} - P(M_k \le u_k)$, then

$$\mathrm{Var}\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I_{(M_k \le u_k)}\right) = E\left(\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\xi_k\right)^2$$
$$= \frac{1}{(\log n)^2}\left(\sum_{k=1}^{n}\frac{1}{k^2}E|\xi_k|^2 + 2\sum_{1\le k<l\le n}\frac{E|\xi_k\xi_l|}{kl}\right)$$
$$\triangleq T_1 + T_2.$$

Since $|\xi_k| \leq 1$, it follows that

$$T_1 \leq \frac{1}{(\log n)^2} \sum_{k=1}^{n} \frac{1}{k^2} \leq \frac{C}{(\log n)^2}. \tag{3.4}$$

Note that for $l > k$

$$\begin{aligned} |E\xi_k\xi_l| &= |\mathrm{Cov}(I_{(M_k \leq u_k)}, I_{(M_l \leq u_l)})| \\ &\leq |\mathrm{Cov}(I_{(M_k \leq u_k)}, I_{(M_l \leq u_l)} - I_{(M_{k,l} \leq u_l)})| + |\mathrm{Cov}(I_{(M_k \leq u_k)}, I_{(M_{k,l} \leq u_l)})| \\ &\leq 2E|I_{(M_l \leq u_l)} - I_{(M_{k,l} \leq u_l)}| + |\mathrm{Cov}(I_{(M_k \leq u_k)}, I_{(M_{k,l} \leq u_l)})|. \end{aligned}$$

By Lemma 3.3, we have

$$|E\xi_k\xi_l| \leq \frac{3k}{l} + \frac{C}{(\log l)^\varepsilon}.$$

Hence

$$\begin{aligned} T_2 &\leq \frac{6}{(\log n)^2} \sum_{1 \leq k < l \leq n} \frac{1}{kl} \frac{k}{l} + \frac{2C}{(\log n)^2} \sum_{1 \leq k < l \leq n} \frac{1}{kl(\log l)^\varepsilon} \\ &\leq \frac{C}{(\log n)^2} \log n + \frac{C}{(\log n)^2} \sum_{l=2}^{n} \frac{1}{l(\log l)^\varepsilon} \sum_{k=1}^{l-1} \frac{1}{k} \\ &\leq \frac{C}{\log n} + \frac{C}{(\log n)^2} \sum_{l=2}^{n} \frac{\log l}{l(\log l)^\varepsilon} \\ &\leq \frac{C}{\log n} + \frac{C}{(\log n)^2} (\log n)^{2-\varepsilon} \\ &\leq \frac{C}{\log n} + \frac{C}{(\log n)^\varepsilon}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we establish (3.3).

The proof of (1). By the conditions of the theorem and Theorem 4.5.2(ii) in Leadbetter et al. [8], we have

$$P(M_n \leq u_n) \to \exp(-\tau).$$

Clearly this implies $\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} P(M_k \leq u_k) = \exp(-\tau)$, which is, by (3.2), equivalent to

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(M_k \leq u_k)} = \exp(-\tau) \quad \text{a.s.}$$

The proof of (2). By Theorem 4.5.2(iii) in Leadbetter et al. [8], we have $n(1-\Phi(u_n)) \to \exp(-x)$ for $u_n = \frac{x}{a_n} + b_n$. Thus from (1) we obtain

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I_{(a_k(M_k - b_k) \leq x)} = \exp(-\exp(-x)) \quad \text{a.s.} \quad \square$$

Proof of Theorem 2.2. Let $u_n = x/a_n + b_n$. By the definition of a_n and b_n, we have

$$u_n^2 = 2\log n - \log\log n + O(1) \quad \text{as } n \to \infty.$$

Using Lemma 3.2(1), for $1 \leq k < n$

$$|P(M_k \leq u_k, M_{k,n} \leq u_n) - P(M_k \leq u_k)P(M_{k,n} \leq u_n)| \leq \frac{1}{4} \sum_{j=1}^{n-1} (n-j)|r_j| \exp\left(-\frac{u_n^2}{1+|r_j|}\right).$$

Since (2.2) holds, we get

$$|P(M_k \leq u_k, M_{k,n} \leq u_n) - P(M_k \leq u_k)P(M_{k,n} \leq u_n)| \leq \frac{C}{(\log n)^\varepsilon}.$$

Similar to Lemma 3.3, we have

$$P(M_{k,n} \leq u_n) - P(M_n \leq u_n) \leq \frac{k}{n} + \frac{C}{(\log n)^{\varepsilon}}.$$

According to the condition of Theorems 2.2 and 9.2.1 in Berman [10], we have

$$P(M_n \leq u_n) \to \exp(-\exp(-x)).$$

The rest of proof is similar to that of Theorem 2.1. □

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