On *d*-symmetric classical *d*-orthogonal polynomials

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A B S T R A C T

The *d*-symmetric classical *d*-orthogonal polynomials are an extension of the standard symmetric classical polynomials according to the Hahn property. In this work, we give some characteristic properties for these polynomials related to generating functions and recurrence-differential equations. As applications, we characterize the *d*-symmetric classical *d*-orthogonal polynomials of Boas-Buck type, we construct a $(d + 1)$-order linear differential equation with polynomial coefficients satisfied by each polynomial of a *d*-symmetric classical *d*-orthogonal set and we show that the *d*-symmetric classical *d*-orthogonal property is preserved by the derivative operator. Some of the obtained properties appear to be new, even for the case $d = 1$.

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1. Introduction

An orthogonal polynomial set $\{P_n\}_{n\geq 0}$ is called classical according to the Hahn property [1], if both $\{P_n\}_{n\geq 0}$ and its derivative, $\{P'_{n+1}\}_{n\geq 0}$, are orthogonal. These polynomials are exactly Hermite, Laguerre and Jacobi polynomials. They are the more important instances of orthogonal polynomials. One of the reasons is that they satisfy not only a three-term recurrence relation but also other useful properties: They are the eigenfunctions of a second order linear differential equation with polynomial coefficients. They satisfy a Rodrigues-type formula, a Pearson equation, etc., …(For a survey on characterization theorems, see for instance [2].)

Particular interest is devoted to the symmetric classical orthogonal polynomials: Hermite and Gegenbauer. Al-Salam [3] and then von Bachhaus [4], using respectively different techniques, characterized these polynomials as the only orthogonal polynomials defined by the generating function

$$G(2xt - t^2) = \sum_{n=0}^{\infty} c_n P_n(x) t^n, \quad c_n \neq 0, \ n \geq 0.$$

For *d*-orthogonal polynomials, *d* being an arbitrary positive integer, (see Definition 2.1) Douak and Maroni [5] introduced the notion of classical *d*-orthogonal polynomials which means that both the polynomial set $\{P_n\}_{n\geq 0}$ and its derivative $\{P'_{n+1}\}_{n\geq 0}$ are *d*-orthogonal. Some examples of classical *d*-orthogonal polynomials may be found in [6–18,5,19,20]. It is then significant to look for characteristic properties for these polynomials as was done for the case $d = 1$. In this context, for general *d*, Douak and Maroni [18] generalized the Pearson's equation. For the classical *d*-orthogonal polynomials with *d*-symmetry as an additional property (see Definition 2.2), Douak and Maroni [5] proved that the first component sequence is also a classical *d*-orthogonal polynomial. For $d = 2$, Douak and Maroni [5] and recently Boukhemis and Zerouki [21] proved that there are exactly four sets of 2-symmetric classical 2-orthogonal polynomials, via solving a system of recurrence relations. Among

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such polynomials, two families are known: the Humbert polynomials and the Gould–Hopper polynomials and the two other are new sets, but they did not give generating functions. They also stated third-order differential equations satisfied by the four obtained sets [21,19].

In this work, we derive some more results for this class of d-symmetric classical d-orthogonal polynomials for general d. We give some characteristic properties for these polynomials, by means of generating functions and a recurrence-differential equation. Some of these properties appear to be new, even for the case $d = 1$ and some others are an extension of the results of Al-Salam and von Bachhaus to d-orthogonal polynomials.

As an application, we solve the following characterization problem.

(P): *Find all d-symmetric classical d-orthogonal polynomials of the Boas–Buck type.*

We obtain that the Gould–Hopper polynomials and the Humbert polynomials are the only solutions of this problem.

We show afterwards, that all the d-symmetric classical d-orthogonal polynomials satisfy a $(d+1)$-order linear differential equation with polynomial coefficients depending on n.

The paper is divided into four sections. Following the introduction, Section 2 is devoted to some needed definitions and results on d-orthogonal polynomials. Then, in Section 3 we give some characterizations of the d-symmetric classical d-orthogonal polynomials by means of generating functions and a recurrence-differential equation. The consequences of this result will be discussed in Section 4. First, we solve the characterization problem (P), then we construct a $(d + 1)$-order differential equation, with polynomial coefficients depending on n, satisfied by each polynomial of a d-symmetric classical d-orthogonal set and finally, we show that the derivative of a d-symmetric classical d-orthogonal polynomial set is also a d-symmetric classical d-orthogonal polynomial set.

2. Preliminary results

In this section, we recall some definitions and results which will be used in the sequel.

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is called a polynomial set (PS, for short) if and only if $\deg P_n = n$ for all non-negative integers n. Throughout this work, we assume that $\{P_n\}_{n\geq 0}$ is a monic polynomial set: $P_n(x) = x^n + \cdots$.

We denote by $\langle u, f \rangle$ the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$.

Definition 2.1 (*[22,23]*). Let $u_0, u_1, \ldots, u_{d-1}$ be d linear functionals and let $\{P_n\}_{n\geq 0}$ be a PS in $\mathcal{P}$. $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial set (d-OPS, for short) with respect to the d-dimensional functional vector

$$\mathcal{U} = {}^t(u_0, u_1, \ldots, u_{d-1}),$$

if, for each integer $k \in \{0, 1, \ldots, d - 1\}$, it satisfies:

$$\begin{cases} \langle u_k, P_m P_n \rangle = 0 & \text{if } m > nd + k, \ n \geq 0, \\ \langle u_k, P_n P_{nd+k} \rangle \neq 0, & n \geq 0. \end{cases} \tag{1}$$

For $d = 1$, we have the well-known notion of orthogonality.

A PS $\{P_n\}_{n\geq 0}$ is called symmetric if it fulfills:

$$P_n(-x) = (-1)^n P_n(x), \quad n \geq 0.$$

Maroni extended this notion to d-symmetric polynomials as follows

Definition 2.2 (*[24]*). A PS $\{P_n\}_{n\geq 0}$ is called d-symmetric if and only if

$$P_n(\omega_d x) = \omega_d^n P_n(x), \quad \text{where } \omega_d = \exp(2i\pi/(d + 1)). \tag{2}$$

When $d = 1$, the 1-symmetric sequence is symmetric.

Douak and Maroni characterized the d-symmetric d-orthogonal polynomials by means of a $(d + 1)$-order recurrence relation:

Proposition 2.3 (*[5]*). *Let $\{P_n\}_{n\geq 0}$ be a d-OPS. Then the following statements are equivalent:*

1. *$\{P_n\}_{n\geq 0}$ is d-symmetric.*
2. *$\{P_n\}_{n\geq 0}$ satisfies a $(d + 1)$-order recurrence relation of the form*

$$\begin{cases} P_n(x) = x^n, & 0 \leq n \leq d, \\ P_{n+d+1}(x) = x P_{n+d}(x) - \gamma_{n+1} P_n(x), & n \geq 0, \ (\gamma_{n+1} \neq 0). \end{cases} \tag{3}$$

Throughout the sequel, $\{Q_n\}_{n\geq 0}$ is defined by $Q_n(x) = \frac{1}{n+1} P'_{n+1}(x)$.

Douak and Maroni defined the classical d-orthogonal polynomials as follows

Definition 2.4 ([5])**.** A d-OPS $\{P_n\}_{n\geq 0}$ is called classical according to the Hahn property if $\{Q_n\}_{n\geq 0}$ is also a d-OPS.

Since the derivative operator takes each d-symmetric PS into a d-symmetric PS, we have

Proposition 2.5 ([5])**.** *Let* $\{P_n\}_{n\geq 0}$ *be a d-symmetric classical d-OPS. Then* $\{Q_n\}_{n\geq 0}$ *is a d-symmetric d-OPS.*

According to Proposition 2.3, the PSs $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ given by Proposition 2.5, satisfy the following $(d+1)$-order recurrence relations:

$$\begin{cases} P_n(x) = x^n, & 0 \leq n \leq d, \\ P_{n+d+1}(x) = xP_{n+d}(x) - \gamma_{n+1}P_n(x), & n \geq 0, \end{cases} \tag{4}$$

and

$$\begin{cases} Q_n(x) = x^n, & 0 \leq n \leq d, \\ Q_{n+d+1}(x) = xQ_{n+d}(x) - \tilde{\gamma}_{n+1}Q_n(x), & n \geq 0, \end{cases} \tag{5}$$

with the regularity conditions $\gamma_n\tilde{\gamma}_n \neq 0$ for $n \geq 1$.

From this, the first author and Douak [16] started to enumerate the d-symmetric d-orthogonal polynomials. They stated the following

Proposition 2.6 ([16])**.** *The coefficients* γ_n *and* $\tilde{\gamma}_n$ *are, respectively, given by*

$$\gamma_{n+2} = \gamma_1 \binom{n+d+1}{d} \prod_{j=0}^{n} \Theta_j, \quad n \geq 0, \tag{6}$$

and

$$\tilde{\gamma}_{n+1} = \gamma_1 \binom{n+d}{d} \prod_{j=0}^{n+1} \Theta_j, \quad n \geq 0, \tag{7}$$

where

$$\Theta_j = \begin{cases} \dfrac{j(\vartheta_j - 1) + 1}{(j+d+2)(\vartheta_{j+1} - 1) + 1}, & j = 1, 2, \ldots, \\ \dfrac{1}{(d+2)(\vartheta_1 - 1) + 1}, & j = 0. \end{cases} \tag{8}$$

with

$$\vartheta_{dn+k} = \begin{cases} 1, & n \geq 0, \\ or \\ \dfrac{n + \lambda_k + 1}{n + \lambda_k}, & n \geq 0, \end{cases} \tag{9}$$

where $n \neq 0$ *if* $k = 0$ *and* λ_k *are d arbitrary parameters satisfying the following conditions:*

$$\begin{cases} \lambda_0 \neq -1, -2, \ldots, \\ \lambda_k \neq 0, -1, -2, \ldots, & k = 1, 2, \ldots, d-1. \end{cases} \tag{10}$$

γ_n *and* $\tilde{\gamma}_n$ *are related as follows:*

$$\tilde{\gamma}_n = \gamma_{n+1} \frac{n}{n+d} \vartheta_n, \quad \vartheta_n \neq 0, \ n \geq 1. \tag{11}$$

It follows that there are exactly 2^d *sets of d-symmetric classical d-orthogonal polynomials.*

This is a generalization of the two following known results:

1. There exist only two families of classical symmetric polynomials [25,4].
2. There exist only four families of 2-symmetric classical 2-orthogonal polynomials [21,5].

3. Characterization theorem

In this section, we give some characteristic properties for the d-symmetric classical d-orthogonal sets related to generating functions and a recurrence-differential equation.

Throughout the sequel of this section, $\{c_n\}_{n\geq 0}$ denotes a sequence of non-zero real numbers and G denotes an analytic function at $z = 0$ with the expansion

$$G(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \neq 0, \ n \geq 0.$$

Theorem 3.1. *Let $\{P_n\}_{n\geq 0}$ be a d-OPS. Then the following statements are equivalent:*

  (i) *$\{P_n\}_{n\geq 0}$ is a d-symmetric classical d-OPS.*
 (ii) *$\{P_n\}_{n\geq 0}$ is generated by*

$$G[(d + 1)xt - t^{d+1}] = \sum_{n=0}^{\infty} c_n P_n(x) t^n. \tag{12}$$

(iii) *$\{P_n\}_{n\geq 0}$ is generated by*

$$\mathfrak{G}_d(x, t) = \sum_{n=0}^{\infty} c_n P_n(x) t^n, \tag{13}$$
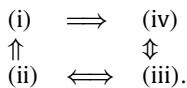
   where $\mathfrak{G}_d$ satisfies

$$t \frac{\partial \mathfrak{G}_d}{\partial t}(x, t) - (x - t^d) \frac{\partial \mathfrak{G}_d}{\partial x}(x, t) = 0. \tag{14}$$

(iv) *$\{P_n\}_{n\geq 0}$ satisfies:*

$$\begin{cases} P_n(x) = x^n, & 0 \leq n \leq d, \\ n c_n P_n(x) + c_{n-d} P'_{n-d}(x) - c_n x P'_n(x) = 0, & n \geq d + 1. \end{cases} \tag{15}$$

d-OPSs generated by (12) were first studied by the first author and Douak [16].

Proof. The outline of the proof is as follows:

    (i) $\Longrightarrow$ (iv)
    $\Uparrow$ $\Updownarrow$
    (ii) $\Longleftrightarrow$ (iii).

The first author and Douak [16] proved the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) and (ii) $\Longrightarrow$ (i). To complete our proof, we need to show the three implications:

    (i) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii).
    (i) $\Longrightarrow$ (iv): By differentiating (4), we get

$$P_{n+d+1}(x) = (n + d + 2)Q_{n+d+1}(x) + (n + 1)\gamma_{n+2} Q_n(x) - (n + d + 1)x Q_{n+d}(x). \tag{16}$$

Using the last equation and (5), we obtain

$$\begin{cases} P_{n+1}(x) = x Q_n(x), & n = 0, 1, \ldots, d - 1, \\ P_{n+d+1}(x) = x Q_{n+d}(x) - [(n + d + 2)\tilde{\gamma}_{n+1} - (n + 1)\gamma_{n+2}] Q_n(x), & n \geq 0. \end{cases} \tag{17}$$

Let α_n be defined by

$$\alpha_n = (n + d + 2)\tilde{\gamma}_{n+1} - (n + 1)\gamma_{n+2}, \quad n \geq 0.$$

These coefficients are necessarily not zero. Indeed, if $\alpha_{n_0} = 0$, for $n_0 \geq 0$, then, on one hand, by virtue of Eq. (11), we have

$$\vartheta_{n_0+1} = \frac{n_0 + d + 1}{n_0 + d + 2},$$

and on the other hand, by means of (9), we get

$$\vartheta_{n_0+1} = \frac{p + \lambda_k + 1}{p + \lambda_k},$$

where $n_0 + 1 = dp + k$ and by identification, we deduce that

$$\lambda_k + d + n_0 + 2 + p = 0,$$

which is contradictory with the conditions (10).

Now, by shifting $n \to n - d - 1$ and replacing Q_n by its expression with P'_n, in (17), we obtain

$$\begin{cases} xP'_n(x) - nP_n(x) = 0, \quad n = 0, 1, \ldots, d, \\ xP'_n(x) - nP_n(x) = \dfrac{n}{n-d}\alpha_{n-d-1}P'_{n-d}(x), \quad n \geq d + 1. \end{cases} \tag{18}$$

Finally, define the sequence $\{c_n\}_{n \geq 0}$ by:

$$\begin{cases} c_0, c_1, \ldots, c_d \in \mathbb{R} \setminus \{0\} \\ c_n = \dfrac{(n-d)}{n\alpha_{n-d-1}}c_{n-d}, \quad n \geq d + 1, \end{cases} \tag{19}$$

to obtain the desired result.

(iv) $\Longrightarrow$(iii): Multiply (15) by t^n and sum, to get

$$\sum_{n=d}^{\infty} c_n xP'_n(x)t^n - \sum_{n=d}^{\infty} c_{n-d}P'_{n-d}(x)t^n - \sum_{n=d}^{\infty} nc_nP_n(x)t^n = 0. \tag{20}$$

Let

$$\mathfrak{G}_d(x, t) = \sum_{n=0}^{\infty} c_nP_n(x)t^n. \tag{21}$$

By differentiating both members of this equation with respect to x and t, we obtain

$$\begin{cases} \displaystyle\sum_{n=0}^{d-1} c_n xP'_n(x)t^n = \sum_{n=1}^{d-1} nc_nP_n(x)t^n, \\ \dfrac{\partial \mathfrak{G}_d}{\partial x}(x, t) = \displaystyle\sum_{n=0}^{\infty} c_nP'_n(x)t^n, \\ t\dfrac{\partial \mathfrak{G}_d}{\partial t}(x, t) = \displaystyle\sum_{n=0}^{\infty} nc_nP_n(x)t^n. \end{cases} \tag{22}$$

By substituting (22) into (15), we deduce the desired result (14).

(iii) $\Longrightarrow$(ii): In order to solve the partial differential equation (14), we consider the following change of variables.

$$\begin{cases} X = (d + 1)xt - t^{d+1}, \\ T = t^d. \end{cases} \tag{23}$$

Then we have the Jacobian determinant

$$\begin{vmatrix} (d+1)t & 0 \\ (d+1)(x - t^d) & dt^{d-1} \end{vmatrix} = d(d+1)t^d \neq 0. \tag{24}$$

Let

$$F_d(X, T) = \mathfrak{G}_d(x, t).$$

By differentiating this equation with respect to x and t, we obtain

$$\begin{cases} \dfrac{\partial \mathfrak{G}_d}{\partial x}(x, t) = \dfrac{\partial F_d}{\partial X}(X, T)\dfrac{\partial X}{\partial x}(x, t), \\ \dfrac{\partial \mathfrak{G}_d}{\partial t}(x, t) = \dfrac{\partial F_d}{\partial X}(X, T)\dfrac{\partial X}{\partial t}(x, t) + \dfrac{\partial F_d}{\partial T}(X, T)\dfrac{\partial T}{\partial t}(x, t), \end{cases}$$

which gives

$$\begin{cases} \dfrac{\partial \mathfrak{G}_d}{\partial x}(x, t) = (d+1)t\dfrac{\partial F_d}{\partial X}(X, T), \\ \dfrac{\partial \mathfrak{G}_d}{\partial t}(x, t) = (d+1)(x - t^d)\dfrac{\partial F_d}{\partial X}(X, T) + dt^{d-1}\dfrac{\partial F_d}{\partial T}(X, T). \end{cases}$$

Replacing in (14), we obtain the following partial differential equation:

$$\dfrac{\partial F_d}{\partial T}(X, T) = 0.$$

This has as solution

$$F_d(X, T) = G(X),$$

where G is an analytic function. Finally, we have

$$\mathfrak{G}_d(x, t) = G((d + 1)xt - t^{d+1}). \quad \square$$

As examples of polynomial sets having generating functions of type (12), we cite the Humbert PS, the Chebyshev d-OPS of the second kind and the Gould Hopper polynomials. Then, by virtue of Theorem 3.1, they are d-symmetric classical d-orthogonal sets.

For the particular case $d = 1$, we have the following result which appears to be new.

Corollary 3.2. *Let* $\{P_n\}_{n\geq 0}$ *be an OPS. Then the following statements are equivalent:*

 (i) $\{P_n\}_{n\geq 0}$ *is a symmetric classical orthogonal polynomial set.*
 (ii) $\{P_n\}_{n\geq 0}$ *is generated by*

$$g(x, t) = \sum_{n=0}^{\infty} c_n P_n(x) t^n, \tag{25}$$

 where g satisfies

$$t \frac{\partial g}{\partial t}(x, t) - (x - t^2) \frac{\partial g}{\partial x}(x, t) = 0. \tag{26}$$

(iii) $\{P_n\}_{n\geq 0}$ *satisfies:*

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x, \\ nc_n P_n(x) + c_{n-1} P'_{n-1}(x) - c_n x P'_n(x) = 0, \quad n \geq 2. \end{cases} \tag{27}$$

The implication (i) $\Longrightarrow$ (iii) is well known. In fact, the symmetric classical orthogonal polynomial sets of Hermite polynomials, $\{H_n\}_{n\geq 0}$, and Gegenbauer polynomials, $\{C_n^\lambda\}_{n\geq 0}$, satisfy, respectively [26, p.188, p.279]:

$$nH_n(x) + nH'_{n-1}(x) - xH'_n(x) = 0,$$

and

$$nC_n^\lambda(x) + (C_{n-1}^\lambda)'(x) - x(C_n^\lambda)'(x) = 0.$$

4. Applications

4.1. Boas–Buck polynomials

A polynomial set $\{P_n\}_{n\geq 0}$ is said to be of Boas–Buck type (or generalized Appell polynomials) if it is generated by [27]:

$$A(t)B(xtC(t)) = \sum_{n=0}^{\infty} P_n(x) t^n, \tag{28}$$

where A, B and C are three formal power series satisfying:

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad B(t) = \sum_{n=0}^{\infty} b_n t^n, \qquad C(t) = \sum_{n=0}^{\infty} c_n t^n \tag{29}$$

and $a_0 b_n c_0 \neq 0$ for all $n \in \mathbb{N}$.

If $C(t) = 1$, (respectively, $B(t) = \exp(t)$), the corresponding PS is called a Brenke type (respectively, of Sheffer type zero).

Our purpose here is to solve the following problem:

(P): *Find all d-symmetric classical d-orthogonal polynomials of Boas–Buck type.*

That is to say: Find all d-OPSs, $\{P_n\}_{n\geq 0}$, generated by

$$A(t^{d+1})B(xtC(t^{d+1})) = \sum_{n=0}^{\infty} P_n(x) t^n, \tag{30}$$

where A, B and C are three formal power series satisfying (29).

The general solution of problem (P) is given in the following result.

Theorem 4.1. *The Gould–Hopper polynomials and the Humbert polynomials are the only d-symmetric classical d-OPSs of Boas–Buck type.*

To prove this theorem, we need the following result

Lemma 4.2 (*[27, Proposition p33]*)**.** *The only functions of the form*

$$K(x, t) = H\{a(t) + xg(t)\},$$

with $H(0) = 1$, that generate Boas–Buck polynomials are those with

$$H(t) = \exp(t) \quad or \quad H(t) = (1 - t)^\lambda, \quad \lambda \neq 0, 1, 2, \ldots.$$

Proof of Theorem 4.1. Let $\{P_n\}_{n \geq 0}$ be a d-symmetric classical d-OPS. By virtue of Theorem 3.1, $\{P_n\}_{n \geq 0}$ is generated by

$$G[(d + 1)xt - t^{d+1}] = \sum_{n=0}^{\infty} c_n P_n(x) t^n. \tag{31}$$

Then using the Lemma 4.2, we deduce that $\{P_n\}_{n \geq 0}$ is of Boas–Buck type if and only if it is of one the following two sets:

1. The Gould–Hopper polynomials, $\{\widehat{H}_n(., d)\}_{n \geq 0}$, generated by [28]

$$\exp\left(xt - \frac{t^{d+1}}{d!(d+1)^2}\right) = \sum_{n=0}^{\infty} \widehat{H}_n(x, d) \frac{t^n}{n!}. \tag{32}$$

2. The Humbert polynomials, $\{\Pi_{n,d}^\lambda\}_{n \geq 0}$, generated by [29]

$$(1 - (d + 1)xt + t^{(d+1)})^{-\lambda} = \sum_{n=0}^{\infty} \Pi_{n,d}^\lambda(x) t^n, \quad \lambda > -1/2. \quad \square \tag{33}$$

Corollary 4.3. (1) *The Gould–Hopper polynomials are the only d-symmetric classical d-OPS of Brenke type.*
(2) *The Gould–Hopper polynomials are the only d-symmetric classical d-OPS of Sheffer type zero.*

4.2. A $(d + 1)$-order differential equation

Each classical orthogonal polynomial set satisfies a second order differential equation. This result is not yet generalized to classical d-orthogonal polynomials for an arbitrary positive integer d. Our purpose in this subsection is to give a $(d+1)$-order differential equation satisfied by each classical d-orthogonal sequence of polynomials with d-symmetry as an additional condition. We apply the obtained result to some known d-symmetric classical d-orthogonal polynomials.

Theorem 4.4. *Let $\{P_n\}_{n \geq 0}$ be a d-symmetric classical d-OPS. Then each polynomial P_n, for $n > d$, satisfies a $(d + 1)$-order differential equation of the form:*

$$\left[D^{d+1} - B_n(xD - n) \prod_{j=0}^{d-1}(A_{n-1-j}(xD - n + d + 1) + n - j)\right] y = 0, \tag{34}$$

where

$$\begin{cases} D = \mathrm{d}/\mathrm{d}x, \qquad A_n = 1 - \gamma_{n-d+1} B_n, \\ \dfrac{1}{B_n} = \dfrac{n}{n - d}\{(n + 1)\tilde{\gamma}_{n-d} - (n - d)\gamma_{n-d+1}\}, \quad n > d. \end{cases} \tag{35}$$

(γ_n and $\tilde{\gamma}_n$ are given, respectively, by the $(d + 1)$-order recurrence relations corresponding to $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$.)

This theorem is deduced from Theorem 3.1 and the following result of the first author and Douak [16].

Lemma 4.5 (*[16, Theorem 4.1]*)**.** *Let $\{P_n\}_{n \geq 0}$ be a d-OPS generated by (12). Then each polynomial P_n, for $n > d$, satisfies a $(d + 1)$-order differential equation of the form (34).*

Theorem 4.4 generalizes the case $d = 2$ treated by Douak and Maroni [20, Proposition 3.2].
   Next, we apply Theorem 4.4 to some known polynomial sets.

(1) The Gould–Hopper polynomials [28]: Douak [17] showed that these polynomials are d-symmetric classical d-orthogonal, with

$$\gamma_{n+1}^0 = \tilde{\gamma}_{n+1}^0 = \frac{1}{d+1}\binom{n+d}{d}.$$

Consequently, the differential equation (34) is reduced to

$$\widehat{H}_n^{(d+1)}(x, d) - (d+1)! x \widehat{H}_n'(x, d) + (d+1)! n \widehat{H}_n(x, d) = 0. \tag{36}$$

This equation was first given by Douak [17]. In particular, for $d = 1$ we get again the second-order differential equation satisfied by the Hermite polynomials (see [30,31], for instance).

(2) The Chebyshev d-OPS of the first kind [15,32]: These polynomials are generated by

$$\frac{1 - \mathrm{d}bt^{d+1}}{1 + bt^{d+1} - xt} = \sum_{n=0}^{\infty} T_n(x, d)t^n, \quad b \neq 0, \tag{37}$$

and they are d-symmetric classical d-orthogonal with

$$\gamma_{n+1}^0 = \tilde{\gamma}_{n+1}^0 = b.$$

The Eq. (34) is reduced to

$$\left[D^{d+1} + \frac{1}{(d+1)^{d+1}b}(n - xD)\prod_{j=0}^{d-1}(\mathrm{d}xD + n + (d+1)j)\right] y = 0, \quad n > d+1. \tag{38}$$

In particular, we have
- For $d = 1$ and $b = 1$: The differential equation satisfied by the Chebyshev polynomials of the first kind [30,31].
- For $d = 2$: The third-order differential equation satisfied by the Chebyshev 2-OPS of the first kind [20,21].

(3) The Chebyshev d-OPS of the second kind [19]: These polynomials are generated by

$$\frac{1}{1 + bt^{d+1} - xt} = \sum_{n=0}^{\infty} U_n(x, d)t^n, \quad b \neq 0. \tag{39}$$

They were characterized in [19] as the self-associated d-symmetric classical d-orthogonal polynomials. Only for the case $d = 2$, was a third-order differential equation satisfied by these polynomials given [19,33].

Our purpose here is to extend these results to $d \geq 1$. From [19, Eq.(2.10)], we have

$$\begin{cases} \gamma_{n+1}^0 = b, \\ \tilde{\gamma}_{n+1}^0 = \dfrac{(n+2d+2)(n+1)}{(n+d+1)(n+d+2)}b. \end{cases} \tag{40}$$

Then by virtue of Theorem 4.4, they satisfy the following $(d + 1)$-order differential equation:

$$\left[D^{d+1} + \frac{1}{(d+1)^{d+1}b}(n - xD)\prod_{j=0}^{d-1}\{\mathrm{d}xD + n + (d+1)(j+1)\}\right] y = 0. \tag{41}$$

In particular we have
- For $d = 1$ and $b = 1$, we obtain the differential equation satisfied by the Chebyshev polynomials of the second kind.
- For $d = 2$, we get the third-order differential equation satisfied by the Chebyshev 2-OPS of the second kind [19,21].

4.3. About the derivatives

It is well known that the derivatives of classical orthogonal polynomials are also classical orthogonal. In this subsection, we derive a similar property for the d-symmetric classical d-OPSs. We improve Proposition 2.5 by the following.

Theorem 4.6. *Let $\{P_n\}_{n\geq0}$ be a d-symmetric classical d-OPS. Then $\{Q_n\}_{n\geq0}$ is also a d-symmetric classical d-OPS.*

This result follows from Theorem 3.1 and the fact that the generating function of $\{Q_n\}_{n\geq0}$ is of type (12).
The iterations of Theorem 4.6 lead to the following.

Corollary 4.7. *Let $\{P_n\}_{n\geq0}$ be a d-symmetric classical d-OPS. Then, for each nonnegative integer r, $\{P_n^{(r)}\}_{n\geq r}$ is also a d-symmetric classical d-OPS.*

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