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Matrices C with $C^n \to 0$*

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1. INTRODUCTION

Matrices with $C^n \to 0$ are of interest in many iteration processes. They have also been studied in connection with topological algebra. Here a "canonical form" is derived for them by using a characterization obtained by P. Stein. Such matrices are closely connected with stable matrices A, i.e., matrices for which the real parts of the characteristic roots are negative. A theorem by Lyapunov characterizing stable matrices is closely linked with Stein's theorem. This link is studied here and used to derive the "canonical form". We also study classes of matrices K such that if $C^n \to 0$ then $(KC)^n \to 0$.

A. S. Householder and R. S. Varga had previously noticed some connection between Lyapunov's and Stein's theorems and kindly communicated this to me.

2. THE CONNECTION BETWEEN LYAPUNOV'S AND P. STEIN'S THEOREMS

We begin by stating these theorems as Theorems 1 and 4. All matrices considered are $n \times n$ matrices with complex elements. The class of positive definite hermitian matrices will be denoted by Π. The class of negative definite hermitian matrices by N.

THEOREM 1 (Lyapunov's theorem; see [*1, 2*]). *A matrix A is stable if and only if a $G \in \Pi$ can be found such that*

$$AG + GA^* = -I. \tag{1}$$

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Proofs for this theorem can be found in [*3*, *4*, *8*]. A slightly weaker form of the theorem is as follows:

THEOREM 2. *A matrix A is stable if for some $G_1 \in N$ a $G \in \Pi$ can be found such that*

$$AG + GA^* = G_1 . \tag{2}$$

Proof. We may put $G_1 = -RR^*$ where R is a nonsingular matrix. This implies $R^{-1}AR \cdot R^{-1}GR^{*-1} + R^{-1}GR^{*-1} \cdot R^*A^*R^{*-1} = -I.$ We observe that

(α) $R^{-1}AR$ has the same characteristic roots as A;

(β) $R^{-1}GR^{*-1} \in \Pi$;

(γ) $R^*A^*R^{*-1}$ is the $*$-map of $R^{-1}AR$.

It follows from Theorem 1 that $R^{-1}AR$ and hence also A is stable.

THEOREM 3. *If A is stable then (2) can be solved with $G \in \Pi$ for every $G_1 \in N$.*

Proof. Again put $G_1 = -RR^*$ with R nonsingular. Since $R^{-1}AR$ is stable the equation

$$R^{-1}AR \cdot G + G \cdot R^*A^*R^{*-1} = -I$$

can be solved with $G \in \Pi$ by Theorem 1. Put

$$G_2 = RGR^*.$$

We then have

$$R^{-1}AR \cdot R^{-1}G_2R^{*-1} + R^{-1}G_2R^{*-1} \cdot R^*A^*R^{*-1} = -I$$

which is equivalent to

$$AG_2 + G_2A^* = -RR^* = G_1 .$$

THEOREM 4 (P. Stein [*5*]). *The matrix C satisfies* $\lim C^n = 0$ *if and only if a matrix $H \in \Pi$ exists such that $H - CHC^* \in \Pi$.*

We shall now discuss the connection between Theorem 2 and Theorem 4.

LEMMA. *Let C satisfy $C^n \to 0$. Put*

$$A = (C + I)^{-1}(C - I). \tag{3}$$

Let $H \in \Pi$ be such that

$$AH + HA^* \in N. \tag{4}$$

Then for any such H we have

$$H - CHC^* \in \Pi \tag{5}$$

Conversely, let A be stable. Put

$$C = (I - A)^{-1}(I + A). \tag{6}$$

Let $H \in \Pi$ be such that

$$H - CHC^* \in \Pi.$$

Then for any such H we have

$$AH + HA^* \in N.$$

Proof. Since $C^n \to 0$, $C + I$ is nonsingular and the transformation (3) is meaningful. Substituting from (3) in (4) we get

$$(C + I)^{-1}(C - I)H + H(C^* - I)(C^* + I)^{-1} \in N$$
$$(C - I)H(C^* + I) + (C + I)H(C^* - I) \in N$$

i.e.

$$2CHC^* - 2H \in N$$

giving (5).

Conversely, if A is stable, (6) is meaningful. Substituting from (6) in (5) we get

$$H - (I - A)^{-1}(I + A)H(I + A^*)(I - A^*)^{-1} \in \Pi$$

which implies

$$(I - A)H(I - A^*) - (I + A)H(I + A^*) \in \Pi$$

i.e.,

$$-2AH - 2HA^* \in \Pi$$

which gives (4).

THEOREM 5. *Let C satisfy $C^n \to 0$. Then*

$$H - CHC^* = P_1$$

can be solved with $H \in \Pi$ for every $P_1 \in \Pi$.

Proof. From the proof of the lemma it follows that

$$AH + HA^* = Q \tag{7}$$

goes over into

$$2(H - CHC^*) = -(C + I)Q(C^* + I)$$

under the transformation $A = (C - I)(C + I)^{-1}$. From Theorem 3 it follows that for a suitable H any $Q \in N$ can be represented by $AH + HA^*$. Hence we first construct

$$Q = -2(C + I)^{-1} P_1(C^* + I)^{-1}$$

so that

$$-(C + I)Q(C^* + I) = 2P_1$$

and then solve (7) for H with this Q.

3. A Canonical Form for Matrices C with $C^n \to 0$

Theorem 6. *Let $C^n \to 0$. Let $H \in \Pi$ satisfy*

$$H - CHC^* = I. \tag{8}$$

Then H has all its characteristic roots ≥ 1.

Proof. By Theorem 5 a matrix $H \in \Pi$ for which (8) is satisfied does exist. Let U be a unitary matrix such that

$$U^{-1}HU = \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix}.$$

Apply the similarity defined by U to (8). The equation

$$\begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix} - E \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix} E^* = I \tag{9}$$

follows where

$$E = U^{-1}CU.$$

Hence

$$\begin{bmatrix} h_1 - 1 & & \\ & \ddots & \\ & & h_n - 1 \end{bmatrix} = E \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix} E^*. \tag{10}$$

Since H is positive definite it follows that

$$h_i \geqslant 1, \qquad i = 1, \cdots, n.$$

THEOREM 7. *Let $C^n \to 0$. Let H be the matrix ($\in \Pi$) of Theorem 6. Then C is unitarily similar to a matrix E such that*

$$E = F \begin{bmatrix} 1/\sqrt{h_1} & & \\ & \ddots & \\ & & 1/\sqrt{h_n} \end{bmatrix}$$

where

$$FF^* = \begin{bmatrix} h_1 - 1 & & \\ & \ddots & \\ & & h_n - 1 \end{bmatrix}.$$

Proof. This follows immediately from the proof of Theorem 6.

4. MATRIX FACTORS WHICH PRESERVE CONVERGENCE

THEOREM 8. *Let $C^n \to 0$. Let H, E be the matrices of Theorems 6 and 7. Let $D = \operatorname{diag}(d_1, \cdots, d_n)$ where $| d_i | < \sqrt{h_i/(h_i - 1)}$. Then DE also has the property that $(DE)^n \to 0$.*

Proof. In Eq. (9) replace E by

$$E \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{bmatrix}$$

where the r_i are to be determined. Then (9) goes over into

$$\begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{bmatrix} - E \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{bmatrix} \times \begin{bmatrix} h_1/| r_1 |^2 & & \\ & \ddots & \\ & & h_n/| r_n |^2 \end{bmatrix} \begin{bmatrix} \bar{r}_1 & & \\ & \ddots & \\ & & \bar{r}_n \end{bmatrix} E^* = I. \qquad (11)$$

Next introduce quantities $k_1, \cdots, k_n$ such that $0 < k_i < 1$ and choose $r_1, \cdots, r_n$ so that

$$l_i = h_i - k_i = h_i/r_i\bar{r}_i, \qquad i = 1, \cdots, n.$$

Then (11) goes over into

$$\begin{bmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{bmatrix} - E \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{bmatrix} \begin{bmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{bmatrix} \begin{bmatrix} \bar{r}_1 & & \\ & \ddots & \\ & & \bar{r}_n \end{bmatrix} E^* = \begin{bmatrix} 1 - k_1 & & \\ & \ddots & \\ & & 1 - k_n \end{bmatrix}. \tag{12}$$

The right hand side of (12) represents a matrix $\in \Pi$. By Stein's theorem (Theorem 4) if follows that

$$E \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{bmatrix}$$

is a matrix whose nth powers converge to zero. Since $k_i < 1$ it follows that

$$r_i \bar{r}_i < h_i/(h_i - 1).$$

Remarks

1. In [*6*] a "canonical form" was obtained for stable matrices via Lyapunov's theorem.

2. In [*4, 6*] a generalization of Lyapunov's theorem to a more general class of matrices was obtained. An analogous result can be found for P. Stein's theorem.

3. The general properties of the linear operator defined by the matrix C on the matrix H in [*5*] are worth studying. For the operator defined by Lyapunov's theorem see [*3*, *4*, *7*, *8*].

References

1. Bellman, R. "Matrix Analysis." McGraw-Hill, New York, 1960.

2. Gantmacher, F. R. "Theory of Matrices," Vol. II. Chelsea, New York, 1959.

3. Givens, W. Elementary divisors and some properties of the Lyapunov mapping $X \to AX + XA^*$. Argonne National Laboratory Report ANL-6456, 1961.

4. Ostrowski, A. and Schneider, H. Some theorems on the inertia of general matrices. *J. Math. Anal. and Appl.* **4** (1962), 72-84.

5. Stein, P. Some general theorems on iterants. *J. Res. Natl. Bur. Std.* **48** (1952), 82-83.

6. Taussky, O. A remark on a theorem of Lyapunov. *J. Math. Anal. and Appl.* **2** (1961), 105-107.

7. Taussky, O. A generalization of a theorem of Lyapunov. *J. Soc. Ind. Appl. Math.* **9** (1961), 640-643.

8. Taussky, O. and Wielandt, H. On the matrix function $AX + X'A'$. *Arch. Rat. Mech. and Anal.* **9** (1962), 93-96.