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Hereditarily h -complete groups[☆]

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Abstract

A topological group G is *h-complete* if every continuous homomorphic image of G is (Raïkov-) complete; we say that G is *hereditarily h-complete* if every closed subgroup of G is *h-complete*. In this paper, we establish open-map properties of hereditarily *h-complete* groups with respect to large classes of groups, and prove a theorem on the (total) minimality of subdirectly represented groups. Numerous applications are presented, among them: (1) Every hereditarily *h-complete* group with quasi-invariant basis is the projective limit of its metrizable quotients; (2) If every countable discrete hereditarily *h-complete* group is finite, then every locally compact hereditarily *h-complete* group that has small invariant neighborhoods is compact. In the sequel, several open problems are formulated. © 2004 Elsevier B.V. All rights reserved.

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Introduction

By the well-known Kuratowski–Mrówka theorem, a (Hausdorff) topological space X is compact if and only if for any (Hausdorff) topological space Y the projection

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$p_Y : X \times Y \rightarrow Y$ is closed. Inspired by this theorem, one says that a topological group G is *categorically compact* (or briefly, *c-compact*) if for any topological group H the image of every closed subgroup of $G \times H$ under the projection $\pi_H : G \times H \rightarrow H$ is closed in H . (All topological groups in this paper are assumed to be Hausdorff.) The problem of whether every *c-compact* topological group is compact has been an open question for more than ten years.

The most extensive study of *c-compact* topological groups was done by Dikranjan and Uspenskij in [6], which has also been a source of inspiration for part of the author's Ph.D. Dissertation [11].

Categorical compactness (in the general setting of structured sets) was introduced in 1974 by Manes [13] and studied by Dikranjan and Giuli [3], and by Clementino et al. [1] who consider in particular categorically compact groups. In [2] Clementino and Tholen proved (independently of Dikranjan and Uspenskij [6], but in greater generality) the Tychonoff theorem for *c-compact* groups.

The notion of hereditary *h-completeness* we introduce in Section 1 is motivated by the observation that all known results relating *c-compactness* to any compactness-like property remain valid if *c-compactness* is replaced by hereditary *h-completeness*. Our main result is Theorem 2.3, stating that hereditarily *h-complete* groups satisfy open-map properties with respect to large classes of groups (groups of countable tightness and *k-groups*); its immediate corollary generalizes [6, 3.2]. As an application, we examine groups admitting a quasi-invariant basis (see Section 3 for definition), and prove a structure theorem for hereditarily *h-complete* groups with this property (Theorem 3.1). In the course of this investigation, an important result on (total) minimality of groups subdirectly represented by (totally) minimal groups is established (Theorem 3.3), which turns out to have several consequences. We conclude with a reduction theorem (Theorem 4.5), relating the compactness of locally compact *c-compact* groups with small invariant neighborhoods to finiteness of countable discrete *c-compact* groups. In the sequel, several open problems are formulated.

1. Preliminaries

A topological group is *minimal* if it does not admit a coarser (Hausdorff) group topology; a group G is *totally minimal* if every continuous surjective homomorphism $\varphi : G \rightarrow H$ is open, or equivalently, if every quotient of G is minimal. It is an important result that *every closed separable subgroup of a c-compact group is totally minimal* [6, 3.6]. This is obtained as a consequence of [6, 3.4], which requires only the closed normal subgroups of the separable subgroup to be *h-complete*, and thus the condition of *c-compactness* can be slightly weakened, as we explain below.

A topological group G is said to be *h-complete* if for any continuous homomorphism $\varphi : G \rightarrow H$ into a topological group, $\varphi(G)$ is closed in H ([4]). Equivalently, G is *h-complete* if every continuous homomorphic image of G is Raikov-complete (i.e., complete in the uniformity given by the join of the left and the right uniformity of the group). Every *c-compact* group is *h-complete*; moreover, since *c-compactness* is a closed-hereditary property, every closed subgroup of a *c-compact* group is *h-complete*. Motivated by this, the author introduces the concept of hereditary *h-completeness* in the obvious way:

G is *hereditarily h -complete* if every closed subgroup of G is h -complete. The following immediate consequence of [6, 3.4] will play an important role in establishing the results of this paper.

Observation 1.1. *Every closed separable subgroup of a hereditarily h -complete group is totally minimal.*

In fact, only the “minimal” part of Observation 1.1 is used, and the condition could be further weakened, as it was pointed out above. The reason for choosing hereditary h -completeness as the condition, beyond its relatively simple definition, is its feature that every continuous homomorphic image of a hereditarily h -complete group is hereditarily h -complete. On the other hand, this property is not that far from c -compactness, because by [6, 2.16], every hereditarily h -complete SIN group is c -compact. (G is SIN if its left and right uniformities coincide.) A second illustration for the phenomenon described in the Introduction is [6, 5.1], stating that every locally compact c -compact group is compact. It is proved using Iwasawa’s [9] theorem: A connected locally compact group which has no closed subgroups (topologically) isomorphic to \mathbb{R} is compact. Dikranjan and Uspenskij’s argument can be adjusted to hereditary h -completeness: since \mathbb{R} is Abelian and non-compact, it is not h -complete (see [6, 3.7]), so no hereditarily h -complete group can contain it as a closed subgroup. Thus, one concludes:

Observation 1.2. *Every connected locally compact hereditarily h -complete group is compact.*

An obvious example is [6, 3.10], where instead of c -compactness, the condition of “all closed subgroups of G are h -complete” is imposed. As a corollary, one can observe that every hereditarily h -complete soluble group is compact, which is a slight generalization of [6, 3.12]. We conclude this series of observations with one related to a result of the author. A group is called *maximally almost periodic* (or briefly, MAP) if it admits a continuous monomorphism $m : G \rightarrow K$ into a compact group K . According to [12, Corollary 7], every c -compact MAP group is compact, but from [12, Theorem 6] one can also derive that every hereditarily h -complete MAP group is compact.

The examples above show that so far no one seems to have exploited the “extra” that c -compactness appears to have, compared to hereditary h -completeness. This raises the following three problems:

Problem I. Is every hereditarily h -complete group c -compact?

Problem II. Is there a class, wider than that of the SIN groups, in which the notions of hereditary h -completeness and c -compactness coincide?

A universal algebraic approach might shed some more light on Problem I: for a class \mathcal{W} of topological groups, put $\mathbf{P}(\mathcal{W})$ for the class of their (arbitrary) products, $\mathbf{H}(\mathcal{W})$ for the class of their continuous homomorphic images, and $\bar{\mathbf{S}}(\mathcal{W})$ for the class of the closed subgroups of groups from \mathcal{W} . A group G is h -complete if $\mathbf{H}(G)$ consists of (Raïkov-)complete

groups. Thus, G is hereditarily h -complete if each group in $\bar{\mathbf{S}}(G)$ is h -complete, in other words, if every group in $\mathbf{HS}(G)$ is complete. Since the product of any family of h -complete groups is h -complete (see [6, 2.13]), if each group in $\bar{\mathbf{S}}(G)$ is h -complete, then so is every group in $\mathbf{PS}(G)$, and therefore every group in $\mathbf{HPS}(G)$ is complete. On the other hand, the product of any family of c -compact groups is c -compact (see [6, 2.8] and [2, 4.4]), and c -compactness is a closed-hereditary property. Thus, if G is c -compact, then every group in $\bar{\mathbf{SP}}(G)$ is c -compact, and in particular h -complete; therefore, the groups in $\mathbf{HSP}(G)$ are complete. In most cases, one has $\mathbf{HPS}(G) \subsetneq \mathbf{HSP}(G)$, which leads to a third problem:

Problem III. Is hereditary h -completeness preserved under the formation of arbitrary products?

Denote by I, II and III, respectively, the statements that the answers to Problems I, II and III are affirmative. Implications $\text{I} \Rightarrow \text{II}$, III , and $\text{I} \Rightarrow \text{III}$ are trivial, but we are unable to say anything further. It might be tempting to try to prove $\text{III} \Rightarrow \text{I}$ (or even a stronger statement, that if every group in $\mathbf{HSP}(G)$ is complete, then G is c -compact), but we have a certain doubt about its truth. In any case, it would be very interesting to characterize the structure of the groups G such that every group in $\mathbf{HSP}(G)$ is complete.

Solving the following problem is likely to be a key step towards understanding c -compactness and hereditary h -completeness:

Problem IV. Is there a hereditarily h -complete group that is not compact?

If such a group E exists, then either E is c -compact, in which case it is a negative solution to the problem of c -compactness, or E is not c -compact, and thus it provides a negative solution to Problem I. On the other hand, if no such E is found, then every hereditarily h -complete group is compact, and in particular every c -compact group is compact. In the latter case, hereditary h -completeness, c -compactness and compactness coincide.

2. Open map properties

A subset F of a topological space X is ω -closed if $\bar{C} \subseteq F$ for every countable subset $C \subseteq F$. A topological space X has *countable tightness* if every ω -closed subset is closed in X .

Proposition 2.1. *Let (G, \mathcal{T}) be a hereditarily h -complete topological group, and let \mathcal{T}' be a coarser group topology on G . Then:*

- (a) *every ω -closed subset in \mathcal{T} is ω -closed in \mathcal{T}' ;*
- (b) *\mathcal{T}' and \mathcal{T} have the same compact subspaces.*

Proof. Let $\iota: (G, \mathcal{T}) \rightarrow (G, \mathcal{T}')$ be the identity map. For a countable subset D , let $S = \overline{\langle D \rangle}$ be the closed separable subgroup generated by D . By Observation 1.1, the group S is totally

minimal, so $\iota|_S$ is a homeomorphism. In particular, $\iota(\overline{D}) = \overline{\iota(D)}$ and $\iota|_{\overline{D}}$ is a homeomorphism, because both S and $\iota(S)$ are closed in the respective topologies.

(a) If D is a countable subset of $\iota(F)$, then $\overline{D} = \iota(\overline{\iota^{-1}(D)})$. Thus, if F is ω -closed (in \mathcal{T}), then $\overline{\iota^{-1}(D)} \subseteq F$, and therefore $\overline{D} = \iota(\overline{\iota^{-1}(D)}) \subseteq \iota(F)$.

(b) Let K be a \mathcal{T}' -compact subspace, and let D be a countably infinite subset of $\iota^{-1}(K)$. Since K is countably compact, it contains a limit point y_0 of $\iota(D)$. By the foregoing discussion, $\iota|_{\overline{D}}$ is a homeomorphism, and thus $\iota^{-1}(y_0)$ is a limit point of D . Therefore, $\iota^{-1}(K)$ is countably compact. Clearly, $\iota^{-1}(K)$ is also closed; hence, to complete the proof, we recall that a countably compact complete uniform space is compact [19, Example 21, p. 218]. \square

Before proving a more general result, we turn to what promises to be the most simple case: discrete groups. Recall that a discrete group is c -compact if and only if it is hereditarily h -complete [6, 5.3.2.16].

Corollary 2.2. *Let G be a discrete c -compact group. Then for every group topology \mathcal{T} on G :*

- (a) \mathcal{T} is anticompact (i.e., every compact subset is finite);
- (b) countably infinite subspaces are discrete in \mathcal{T} ;
- (c) every subset is ω -closed in \mathcal{T} .

A map $f : X \rightarrow Y$ between Hausdorff spaces is k -continuous if its restriction $f|_K$ to every compact subspace K of X is continuous. Following Noble [15], a topological group G is called a k -group if every k -continuous homomorphism $\varphi : G \rightarrow L$ into a topological group is continuous. Every locally compact or sequential group is a k -group (they are even k -spaces, see [8, p. 152]). Noble showed in his papers that the class of k -groups is closed under the formation of quotients (in fact, continuous homomorphic images), arbitrary direct products, and open subgroups (see [15, 1.2, 1.8] and [14, 5.7]; the latter's statement is actually more general than what we quote here). This shows that the class of k -groups is a quite large one, and the only property it is missing to qualify for a variety is closedness under the formation of (arbitrary) subgroups.

Theorem 2.3. *Let G be a hereditarily h -complete group, and let $f : G \rightarrow H$ be a continuous homomorphism onto a topological group H . If either*

- (a) H has countable tightness, or
- (b) H is a k -group,

then f is open.

Proof. Since hereditary h -completeness is preserved by quotients, by replacing G with $G/\ker f$ and f with the induced homomorphism, we may assume that f is bijective. Set $g = f^{-1}$.

(a) To show the continuity of g , let F be a closed subset of G . Then F is ω -closed, and by Proposition 2.1(a), $g^{-1}(F) = f(F)$ is ω -closed in H . Since H has countable tightness, this implies that $g^{-1}(F)$ is closed in H , and therefore g is continuous.

(b) Let $K \subseteq H$ be a compact subspace of H . By Proposition 2.1(b), the subset $g(K) = f^{-1}(K)$ is compact, so if F is a closed subset of $g(K)$, then it is compact too. Thus, $g^{-1}(F) = f(F)$ is compact, and in particular, $g^{-1}(F)$ is closed; therefore, g is k -continuous. Hence, g is continuous, because H is a k -group. \square

The next corollary significantly generalizes [6, 3.2].

Corollary 2.4. *Let G be a hereditarily h -complete topological group. Then every continuous homomorphism $f : G \rightarrow H$ onto a metrizable group H is open.*

Remark. Corollary 2.4 might give the false impression of an Open Map theorem that is free of Baire category arguments. However, this is not the case, because Proposition 2.1 relies on Observation 1.1, which in turn is a consequence of the classical Banach's Open Map theorem for complete second countable topological groups.

3. Groups admitting a quasi-invariant basis

Following Kac [10], a group G has a *quasi-invariant basis* if for every neighborhood U of the identity there exists a countable family \mathcal{V} of neighborhoods of the identity such that for any $g \in G$ there exists $V \in \mathcal{V}$ such that $gVg^{-1} \subset U$. In his paper, Kac proved the following two fundamental results on groups admitting a quasi-invariant basis.

Fact A. ([10]) A topological group can be embedded as a subgroup into a direct product of metrizable groups if and only if it has a quasi-invariant basis.

Fact B. ([10]) A topological group G with a quasi-invariant basis admits a coarser metrizable group topology if and only if it has countable pseudocharacter.

The result below is a structure theorem, which generalizes [12, Theorem 5] to the greatest possible extent, because every subgroup of a product of metrizable groups admits a quasi-invariant basis (Fact A).

Theorem 3.1. *Let G be a hereditarily h -complete group admitting a quasi-invariant basis. Then G is the projective limit of its metrizable quotients.*

Proof. Since G has a quasi-invariant basis, by Fact A, it embeds into a product $M = \prod_{\alpha \in I} M_\alpha$ of metrizable groups. For $\pi_\alpha : G \rightarrow M_\alpha$ the restrictions of the canonical projections, we may assume that π_α is surjective, and thus by Corollary 2.4, the π_α are open; in particular, each M_α is a (metrizable) quotient of G . Therefore, G embeds into a product of its metrizable quotients, and since by adding additional metrizable quotients to

the product we cannot ruin the embedding property, we may assume that M is the product of *all* the metrizable quotients of G .

If G/N_1 and G/N_2 are metrizable, then by Corollary 2.4, the continuous monomorphism $G/N_1 \cap N_2 \rightarrow G/N_1 \times G/N_2$ is an embedding, so $G/N_1 \cap N_2$ is metrizable too. Thus, metrizable quotients of G form a projective system. The image of G is certainly contained in the projective limit of its metrizable quotients, and it is dense there. Since G is h -complete, the statement follows. \square

Dikranjan and Uspenskij [6, 3.3] showed that *every h -complete group with a countable network is totally minimal and metrizable*. The statement concerning metrizability can easily be developed further:

Proposition 3.2. *Every hereditarily h -complete group with a quasi-invariant basis that has countable pseudocharacter is metrizable.*

Proof. Let (G, \mathcal{T}) be a group with the stated properties. By Fact B, G admits a coarser metrizable group topology \mathcal{T}' . Applying Corollary 2.4 to the identity map

$$\iota: (G, \mathcal{T}) \rightarrow (G, \mathcal{T}'),$$

one gets $\mathcal{T} = \mathcal{T}'$, and thus \mathcal{T} is metrizable.

The difficulty with extending the first part of [6, 3.3] (concerning minimality) is that given a hereditarily h -complete metrizable group G , the best we can say (beyond Proposition 2.1) is that each separable subgroup of G is metrizable in every coarser topology.

Problem V. Is every metrizable hereditarily h -complete group totally minimal?

Since we do not know the answer to Problem V, we present a result relating it to the possibility of weakening the conditions of [6, 3.3] in some sense. However, in order to do that, an auxiliary result is required, which turns out to be interesting on its own. Its proof is modeled on the proof of [6, 3.4].

Theorem 3.3. *Let G be a topological group and let \mathcal{N} be a filter-base of h -complete normal subgroups of G . Suppose that G naturally embeds into the product*

$$P = \prod_{N \in \mathcal{N}} G/N.$$

If each quotient G/N is (totally) minimal, then G is (totally) minimal too.

Remark. The situation where G is naturally embedded into a product of its quotients is called a *subdirect representation*.

Proof. Let $f: G \rightarrow H$ be a continuous surjective homomorphism, and let U be an open neighborhood of the identity element in G . Since G embeds into P , there exists a finite collection $\{N_1, \dots, N_k\} \subset \mathcal{N}$ such that $q^{-1}(V) \subseteq U$ for an open neighborhood V of the

identity in the product $G/N_1 \times \cdots \times G/N_k$ (where q is the diagonal of the respective projections). Since \mathcal{N} is a filter base, there exists $N \in \mathcal{N}$ such that $N \subseteq N_1 \cap \cdots \cap N_k$, and therefore without loss of generality we may assume that $U = p^{-1}(V)$ for an open neighborhood V of the identity in G/N , where $p: G \rightarrow G/N$ is the canonical projection.

Since N is h -complete, $f(N)$ is closed in H , so f induces a continuous surjective homomorphism $\tilde{f}_N: G/N \rightarrow H/f(N)$. We have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ p \downarrow & & \downarrow \pi \\ G/N & \xrightarrow{\tilde{f}_N} & H/f(N) \end{array}$$

If \tilde{f}_N is open, then so is the composite $\tilde{f}_N \circ p$, and thus $\tilde{f}(p(U)) = \pi(f(U))$ is open in $H/f(N)$. Therefore, $f(U)f(N) = f(UN) = f(U)$ is open in H (one has $UN = U$, because $U = p^{-1}(V)$). Hence, f is open if and only if \tilde{f}_N is open for every $N \in \mathcal{N}$.

One concludes that if each G/N is totally minimal, then each \tilde{f}_N , being a continuous surjective homomorphism, is open, and therefore G is totally minimal. The argument for minimal groups is similar, because each \tilde{f}_N is bijective when f is so. \square

Before returning to groups with quasi-invariant basis, we show how a weaker version of [5, 7.3.9(b)] follows from Theorem 3.3.

Corollary 3.4. *Let $\{G_i\}_{i \in I}$ be a family of h -complete groups and put $G = \prod_{i \in I} G_i$.*

- (a) *If each G_i is minimal, then G is minimal too;*
- (b) *If each G_i is totally minimal, then G is totally minimal too.*

Part (a) of the corollary is a new result, because [5, 7.3.9(b)] deals only with products of totally minimal groups (and not minimal ones); on the other hand, part (b) is weaker, because in [5, 7.3.9(b)] it suffices for the factors G_i to have complete quotients, and they are not required to have complete homomorphic images. In order to prove Corollary 3.4, a weak version of [7, (3)] is needed:

Fact C. The product of two (totally) minimal topological groups G_1 and G_2 is (totally) minimal if one of them is h -complete.

Proof. For each finite subset $F \subseteq I$, put $N_F = \prod_{i \in I \setminus F} G_i \times \prod_{i \in F} \{e\}$. By [6, 2.13], the product of any family of h -complete groups is h -complete, so each N_F is h -complete. The quotient G/N_F is topologically isomorphic to the finite product $\prod_{i \in F} G_i$, which is (totally) minimal by Fact C, and the N_F certainly form a filter-base. Therefore, the conditions of Theorem 3.3 are fulfilled, and G is (totally) minimal. \square

A group G is *perfectly totally minimal* if $G \times H$ is totally minimal for every totally minimal group H ([17]).

Theorem 3.5. *The following statements are equivalent:*

- (i) *every metrizable hereditarily h -complete group is minimal;*
- (ii) *every hereditarily h -complete group with a quasi-invariant basis is perfectly totally minimal.*

Proof. (i) \Rightarrow (ii): Since hereditarily h -completeness and metrizability are preserved by quotients, (i) implies that every metrizable hereditarily h -complete group is totally minimal. By Theorem 3.1, every hereditarily h -complete group G with a quasi-invariant basis embeds into the product of its metrizable quotients, that are totally minimal by (i). Since G is hereditarily h -complete, the kernel of each metrizable quotient is h -complete, and therefore, by Theorem 3.3, G is totally minimal. Hence, by Fact C, G is perfectly totally minimal, because it is h -complete. \square

4. Locally compact SIN groups

We recall that a group G has *small invariant neighborhoods* (or briefly, G is *SIN*), if any neighborhood U of $e \in G$ contains an invariant neighborhood V of e , i.e., a neighborhood V such that $g^{-1}Vg = V$ for all $g \in G$. Equivalently, G is SIN if its left and right uniformities coincide. The interest in the class of SIN group in the context of this paper arises from an aforesaid theorem of Dikranjan and Uspenskij's [6, 2.16], stating that in this class, c -compactness and hereditary h -completeness coincide. It turns out that locally compact SIN groups are closely related to discrete ones, and as an illustration we start with two observations.

Observation 4.1. *If every closed subgroup of a locally compact SIN group G is totally minimal, then G is c -compact.*

This statement was made concerning discrete groups in [6, 5.4].

Proof. Let S be a closed subgroup of G , and let $f : S \rightarrow H$ be a continuous surjective homomorphism. Since S is totally minimal, f is open, and H is a quotient of S . The group S , being a closed subgroup of G , is locally compact, and thus so is H ; in particular, H is complete. Therefore, S is h -complete, and hence G is hereditarily h -complete, which coincides with c -compactness, because G is SIN. \square

Observation 4.2. *A σ -compact locally compact SIN group G is c -compact if and only if every closed subgroup of G is totally minimal.*

This mimics [6, 5.5], and it is an immediate consequence of the previous observation and [6, 3.5]. The condition of countability in [6, 5.5] was traded for σ -compactness, which is the “right” concept for the context of LC groups.

Since locally compact connected c -compact groups are known to be compact [6, 5.1], and compactness has the three-space property (i.e., if G/N and N are compact for a normal subgroup N , then G is compact too), we turn first to totally disconnected groups.

Proposition 4.3. *Every locally compact totally disconnected SIN group is the projective limit of its discrete quotients with compact kernel.*

Remark. In [11, 6.2, p. 68], a different and more detailed proof of the proposition is also available.

Proof. In every locally compact totally disconnected group, the compact-open subgroups form a base at the identity. Since G is also SIN, each compact-open subgroup contains a compact-open normal subgroup, so if we let \mathcal{N} be the set of such subgroups, then \mathcal{N} is a base at e . Thus, G admits a continuous monomorphism ν into $D = \prod_{N \in \mathcal{N}} G/N$. The statement regarding the limit follows from a result established by Weil in [18, p. 25]. \square

We are now ready to formulate a result of the same kind as Theorem 3.5.

Theorem 4.4. *The following statements are equivalent:*

- (i) *every discrete c -compact group is minimal;*
- (ii) *every locally compact c -compact group admitting small invariant neighborhoods is perfectly totally minimal.*

In order to prove the theorem, we need a corollary of Eberhardt, Dierolf and Schwanengel:

Fact D. ([7, (7)]) *If a topological group G contains a compact normal subgroup N such that G/N is totally minimal, then G is totally minimal.*

Proof. (i) \Rightarrow (ii): Let G be a group described in (ii). By Fact C, it suffices to show that G is totally minimal, because it is h -complete. Set N to be the connected component of the identity in G ; as a closed subgroup of G , N is c -compact. Thus, N is compact, because it is locally compact, connected and c -compact (Observation 1.2). The quotient G/N inherits all the aforesaid properties of G ; furthermore, it is totally disconnected. By Fact D, it suffices to show that G/N is totally minimal, so we may assume that G is totally disconnected from the outset.

It follows from Proposition 4.3 that G embeds into the product of its discrete quotients, where the kernels of the quotients are the compact-open subgroups (certainly a filter-base), and in particular they are h -complete. By (i), every discrete c -compact group is minimal, and since quotients of discrete c -compact groups are again discrete c -compact, (i) implies that they are actually totally minimal. Thus, each discrete quotient of G is totally minimal, and therefore, by Theorem 3.3, G is totally minimal. \square

Given a topological group G , the kernel of its Bohr-compactification $\kappa_G : G \rightarrow bG$ is called the *von Neumann radical* of G , and is denoted by $n(G)$. We say that G is *minimally almost periodic* (or briefly, *m.a.p.*) if $n(G) = G$, or equivalently, if it has no non-trivial finite-dimensional unitary representations. Since G is MAP if $n(G)$ is trivial, one can say that m.a.p. is the “opposite” of MAP. We conclude the paper with a result similar in nature to [12, Theorem 9].

Theorem 4.5. *The following statements are equivalent:*

- (i) *every countable discrete c -compact minimally almost periodic group is trivial;*
- (ii) *every countable discrete c -compact group is maximally almost periodic (and thus finite);*
- (iii) *every locally compact c -compact group admitting small invariant neighborhoods is compact.*

Proof. (i) \Rightarrow (ii): The von Neumann radical of every c -compact group is m.a.p. [12, Corollary 8]. Thus, by (i), if G is a countable discrete c -compact group, then $n(G)$ is trivial; in other words, G is MAP. The finiteness of G follows from [12, Corollary 7], but for the sake of completeness we show it explicitly here: G is MAP, so it admits a continuous monomorphism $m : G \rightarrow K$ into a compact group K . The map m is an embedding, because G is a discrete countable c -compact group, and as such it is totally minimal (Observation 4.2). Its image, $m(G)$, is closed in K , and thus compact, because G is h -complete. Therefore, G is topologically isomorphic to the compact group $m(G)$.

(ii) \Rightarrow (iii): Let G be a group as stated in (iii). The connected component N of the identity in G is compact (Observation 1.2), so it suffices to show that G/N , which inherits all the aforesaid properties of G , is compact. Thus, we may assume that G is totally disconnected from the outset.

If S is a closed separable subgroup of G , then S is c -compact, and its discrete quotients are countable. Since, by (ii), every discrete quotient of S is finite, and S is the projective limits of its discrete quotients (Proposition 4.3), S is a *pro-finite* group; in particular, S is compact. So every closed separable subgroup of G is compact; thus, G is precompact, and therefore, being complete, it is compact. \square

It is not known whether (i) or (ii) is true. The only related result known is a negative one, due to Shelah [16]: under CH there exists an infinite discrete h -complete group. It is unknown whether such an example is available under ZFC. Even if such an example were available under ZFC, it would not be, of course, a counterexample for (ii), because it is neither countable nor c -compact.

Characterizing countable discrete MAP groups is a pure algebraic problem that does not involve group topologies at all. On the other hand, a countable discrete group is c -compact if and only if its subgroups are totally minimal. By Theorem 4.5, putting the two ingredients together can give a solution to the problem of c -compactness in the locally compact SIN case.

An alternative approach could be to extensively study countable discrete m.a.p. groups, which is again a purely algebraic task. Once a characterization of such groups is obtained, it should no longer be difficult to check whether statement (i) in the theorem holds.

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