A note on unitary similarity preserving linear mappings on $B(H)^\diamond$

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Abstract

Let $H$ be an infinite-dimensional complex Hilbert space. We give the characterization of surjective mappings on $B(H)$ that preserve unitary similarity in both directions.

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Mappings preserving similarity on several operator spaces were treated recently in a series of papers. This topic is a part of a broad field of linear preserver problems. Many results of this kind can be found in the survey papers [1,4,10].

In 1987 Hiai characterized all linear mappings $\phi$ on $M_n(\mathbb{C})$, the algebra of all complex $n \times n$ matrices, that preserve similarity. This means that if matrices $A$ and $B$ are similar ($B = S^{-1}AS$ for some invertible matrix $S$) then $\phi(A)$ and $\phi(B)$ are similar as well. Later Lim, Li and Tsing improved and extended his result [5,6,10,11].

Similarity preserving mappings on infinite-dimensional operator spaces were studied by Ji, Du, Hou and the present author [9,7,8,2,3,13]. Beside linear mappings also

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additive ones were studied and the similarity preserving property was also replaced by a weaker assumption of asymptotic similarity preserving.

Hiai, Li and Tsing studied not only similarity-preserving mappings but also unitary similarity-preserving ones on finite-dimensional spaces. Our aim is to extend their result to $B(H)$, the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space $H$.

We say that operators $A, B \in B(H)$ are similar if $A = S^{-1}BS$ for some $S \in B(H)$ and theya re unitary similar if $A = U^*BU$ for some unitary operator $U$ on $H$. By $\mathcal{S}_u(A)$ we denote the unitary similarity orbit of $A$, i.e. the set of all operators that are unitary similar to $A$. We denote by $A \sim_u B$ the relation of unitary similarity.

Linear mapping $\phi : B(H) \to B(H)$ preserves unitary similarity in both directions if $A$ is unitary similar to $B$ if and only if $\phi(A)$ is unitary similar to $\phi(B)$. By $\mathcal{F}(H)$ we denote all finite rank operators on $B(H)$ and by $\mathcal{F}_0(H)$ the subspace of all finite rank operators $X$ with $\text{tr} X = 0$, where $\text{tr}$ denotes the trace. We use the notation $x \otimes y$ for a rank-one operator $u \mapsto \langle u, y \rangle x$. It is well known that $x \otimes y$ is a rank-one nilpotent if both $x$ and $y$ are non-zero and $\langle x, y \rangle = 0$. Let us note that $\mathcal{S}_u(x \otimes y) = \{re \otimes f; \langle e, f \rangle = 0, \|e\| = \|f\| = 1\}$ and $r = \|x\|\|y\|$. We say that a rank-one nilpotent $N$ is written in the normalized form if $N = rx \otimes y$, $\|x\| = \|y\| = 1$, $\langle x, y \rangle = 0$ and $r > 0$. Note that rank-one nilpotents $e \otimes f$ and $x \otimes y$ are unitary similar if and only if $\|e\|\|f\| = \|x\|\|y\|$. Let us state our main result.

**Main Theorem.** Let $\phi : B(H) \to B(H)$ be a linear surjective mapping preserving unitary similarity in both directions. Then there exist a non-zero constant $c$ and a unitary operator $U \in B(H)$ such that either

(a) $\phi(X) = cU^*XU$ ($X \in B(H)$) or

(b) $\phi(X) = cU^*X^tU$ ($X \in B(H)$),

where $X^t$ denotes the transpose of $X$ relative to a fixed but arbitrary orthonormal basis of $H$.

1. **Proof of the Main Theorem**

Let $\phi$ be as in the Main Theorem. We easily observe [13] that $\phi$ is injective and that $\phi(I)$ is a scalar operator. So, we can assume without loss of generality that $\phi(I) = I$. We will show that $\phi$ preserve $\mathcal{F}_0(H)$. In order to do this, we introduce a minimal u-similarity-invariant subspace. We call a subspace $\mathcal{V} \subseteq B(H)$ u-similarity-invariant if it contains $\mathcal{S}_u(A)$ for every $A \in \mathcal{V}$. Moreover, a subspace $\mathcal{V} \subseteq B(H)$ is minimal u-similarity-invariant subspace if

1. $\mathcal{V}$ is a u-similarity-invariant subspace of $B(H)$,
2. If $\mathcal{W} \subseteq \mathcal{V}$ is a u-similarity-invariant subspace of $B(H)$ then $\mathcal{W} = [0]$ or $\mathcal{W} = \mathcal{V}$.
It is an elementary exercise to see that \( \phi \) maps a minimal u-similarity-invariant subspace to one having the same property.

**Lemma 1.** The non-trivial subspace \( \mathcal{V} \subseteq B(H), I \notin \mathcal{V}, \) is a minimal u-similarity-invariant subspace of \( B(H) \) if and only if \( \mathcal{V} = \mathcal{F}_0(H). \)

**Proof.** The “if” statement is obvious. In order to prove the “only if” it is enough to show that \( \mathcal{V} \) has a rank-one nilpotent operator. As \( I \notin \mathcal{V} \) there exists a non-scalar operator \( A \in \mathcal{V}. \) We can choose an \( x \in H \) such that \( x \) and \( Ax \) are linearly independent. Let \( X = \text{span}\{x, Ax\} \) and \( H = X \oplus X^\perp. \) According to this decomposition we represent \( A \) in an operator matrix form

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

where \( A_{11} \) is not a scalar matrix. Let \( U_1 \) be a unitary operator on \( X \) satisfying \( A_{11} - U_1^*A_{11}U_1 \neq 0. \) Let \( U = U_1 \oplus I_{X^\perp}. \) Then

\[
B = A - U^*AU = \begin{bmatrix}
A_{11} - U_1^*A_{11}U_1 & * \\
* & 0
\end{bmatrix} \in \mathcal{V},
\]

it is of at most rank four and also an element of \( \mathcal{F}_0(H). \)

By splitting \( H = Y \oplus Y^\perp \) where \( Y = \text{Im } B, \) the operator \( B \) can be written in the form

\[
B = \begin{bmatrix}
B_{11} & B_{12} \\
0 & 0
\end{bmatrix}
\]

where the space \( Y \) is \( k \)-dimensional and \( 1 \leq k \leq 4. \) If rank \( B = 1, \) we are done, so from now on, we assume \( 1 < k \leq 4 \) and rank \( B > 1. \) We can assume without loss of generality that \( B_{11} = 0 \) or \( B_{12} = 0. \) Indeed, if \( B_{11} \neq 0 \) we observe that

\[
(-I_Y \oplus I_{Y^\perp})^*B(-I_Y \oplus I_{Y^\perp}) + B = \begin{bmatrix}
2B_{11} & 0 \\
0 & 0
\end{bmatrix} \in \mathcal{V}.
\]

Let us first consider the case \( B_{11} = 0. \) Let \( \{e_1, e_2, \ldots, e_k\} \) be an orthonormal basis of \( Y. \) According to decomposition \( H = \text{span}\{e_1\} \oplus \text{span}\{e_2, \ldots, e_k\} \oplus Y^\perp \) we have

\[
B = \begin{bmatrix}
0 & 0 & B_{13} \\
0 & 0 & B_{23} \\
0 & 0 & 0
\end{bmatrix}, \quad B_{13}, \ B_{23} \neq 0.
\]

Moreover, taking \( U = (1 \oplus (-I_{k-1}) \oplus I_{Y^\perp}) \) we observe that

\[
U^*BU + B = \begin{bmatrix}
0 & 0 & B_{13} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

is a rank-one nilpotent in \( \mathcal{V}. \) If \( B_{11} \neq 0 \) we may consider \( B \) as a \( k \times k \) matrix and without loss of generality assume that it is not diagonal. By permutation similarity
we can achieve that $B$ has at least one non-zero non-diagonal entry in the first row. Using a block matrix form

$$B = \begin{bmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad C_{12} \neq 0, \quad c_{11} \in \mathbb{C}, \quad C_{22} \in \mathbb{C}^{(k-1) \times (k-1)},$$

and taking $V = -1 \oplus I_{k-1}$, we arrive at

$$D = B - V^* BV = \begin{bmatrix} 0 & 2C_{12} \\ 2C_{21} & 0 \end{bmatrix} \in \mathcal{V},$$

which is of at most rank 2. If rank $D$ is 1, we have found a nilpotent of rank one, otherwise assume rank $D = 2$. As $\text{tr} \, D = 0$, we have two possibilities: $D$ is a nilpotent, or it has non-zero eigenvalues $\alpha$ and $-\alpha$ of multiplicity one. By a finite series of transformations $D \mapsto D + W^* DW$, where the diagonal unitary matrix $W$ has one diagonal entry $-1$ and all the others equal to 1, we either find a nilpotent of rank one in $\mathcal{V}$, or find out that $\begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{V}$, $D_{11} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Finally, $D_{11}$ is unitary similar to $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$i(D_{11} \oplus 0) - (E \oplus 0) = \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix} \oplus 0$$

is a rank-one nilpotent. □

**Lemma 2.** If an operator $A \in \mathcal{F}(H)$ satisfies $A \sim_u \alpha A$ for all $|\alpha| = 1$, then $A$ is nilpotent.

**Proof.** The proof is left to the reader. □

Many linear preserver problems can be reduced to rank-one preservers. We will show that $\phi$ preserves rank-one nilpotents in both directions. This observation is based on the following characterization of rank-one nilpotents by unitary similarity.

**Proposition 3.** An operator $N \in \mathcal{F}_0(H)$ is a rank-one nilpotent if and only if

(i) $N \sim_u \alpha N$ for all $|\alpha| = 1$, and

(ii) for every $M \in \mathcal{S}_u(N)$, which is not a multiple of $N$

the following implication holds true:

If $N + M \sim_u \gamma N$ for some $\gamma \neq 0$

then for every $\beta$, there exists a non-zero $\delta$

such that $N + \beta M \sim_u \delta N$. 


Proof. Let $N = re \otimes f$ and $M = ru \otimes v$ be unitary similar rank-one nilpotents written in the normalized form. Unitary similarity of $N + M$ and $\gamma N$ implies that $N + M$ must be of rank one, so either $e$ and $u$ are linearly dependent, or, $f$ and $v$ are linearly dependent. Suppose $u = te$ for some $t$ with $|t| = 1$. Compute

$$N + \beta M = re \otimes (f + \beta tv)$$

$$= \|f + \beta tv\| re \otimes \frac{f + \beta tv}{\|f + \beta tv\|} \sim_u \delta N$$

where $\delta = \|f + \beta tv\|$. The case when $f$ and $v$ are linearly dependent can be treated similarly.

In order to show that conditions (i) and (ii) are also sufficient for $N \in \mathcal{F}_0(H)$ to be a rank-one nilpotent we assume that $N$ is not a rank-one nilpotent. Then, if $N$ fulfills (i), (ii) should be false. So, we will find an operator $M$ such that $N + M \sim_u \gamma N$ for some $\gamma \neq 0$ while there will be impossible to choose a $\delta$ such that $N - iM \sim_u \delta N$.

By (i) and the previous Lemma $N \in \mathcal{F}_0(H)$ is nilpotent. Suppose rank $N = r > 1$. There exists an $0 \neq y \in \text{Ker} N \cap \text{Im} N$. According to the decomposition $H = \text{span}\{y\} \oplus \{y\}^\perp$ represent

$$N = \begin{bmatrix} 0 & N_1 \\ 0 & N_2 \end{bmatrix}, \quad \text{rank} N_1 = 1, \quad \text{rank} N_2 \geq 1.$$ 

By taking $U = \mathbb{I} \oplus I$ for any $|u| = 1$ and computing $U^* N U = \begin{bmatrix} 0 & uN_1 \\ 0 & uN_2 \end{bmatrix}$ we see that $N \sim_u \begin{bmatrix} 0 & uN_1 \\ 0 & uN_2 \end{bmatrix}$ for every $|u| = 1$. Combining it with (1) we get that

$$N \sim_u \begin{bmatrix} 0 & uN_1 \\ 0 & vN_2 \end{bmatrix}, \quad \text{for all} \quad u, v \quad \text{with modulus one.}$$

So, $M = \begin{bmatrix} 0 & iN_1 \\ 0 & -iN_2 \end{bmatrix} \in \mathcal{S}_u(N)$, it is not a multiple of $N$,

$$N + M = \sqrt{2} \begin{bmatrix} 0 & \frac{1+i}{\sqrt{2}} N_1 \\ 0 & \frac{1-i}{\sqrt{2}} N_2 \end{bmatrix} \sim_u \sqrt{2} N,$$

however

$$N - iM = \begin{bmatrix} 0 & 2N_1 \\ 0 & 0 \end{bmatrix}$$

is of rank one and therefore, it cannot be similar to any multiple of $N$. \qed

Using this characterization of rank-one nilpotents it is easy to see that $\phi$ preserves rank-one nilpotents in both directions. Standard methods give us the existence of a non-zero constant $c$ and an invertible operator $A \in B(H)$ such that either

$$\phi(x \otimes y) = cA x \otimes A^{-1} y$$

$(x, y \in H$, $\langle x, y \rangle = 0)$ (1)
or
\[
\phi(x \otimes y) = c A y \otimes x A^{-1} \quad (x, y \in H, \langle x, y \rangle = 0).
\] (2)

It is now our aim to show that \( A \) must be a unitary operator. We may assume that \( \phi \) has the property (1). Let us fix an orthogonal pair of unit vectors \( e \) and \( f \). As \( \phi(e \otimes f) \) is a rank-one nilpotent, we will assume (by changing \( \phi \), if necessary) that
\[
\phi(e \otimes f) = e \otimes f, \quad c > 0 \quad \text{and} \quad e \otimes f = c A e \otimes f A^{-1} = c A e \otimes (A^*)^{-1} f.
\]

There is no loss of generality in assuming \( A e = e \) and \( c (A^*)^{-1} f = f \). The latter is equivalent to \( A^* f = c f \). Consequently, \( \{ e \}^\perp \) is an \( A^* \)-invariant subspace and \( \{ f \}^\perp \) is invariant for \( A \). The relation
\[
e \otimes h \sim_u u \| h \| e \otimes f, \quad h \in \{ e \}^\perp,
\]
gives us that
\[
\phi(e \otimes h) = c e \otimes (A^*)^{-1} h \sim_u u \| h \| e \otimes f
\]
and consequently,
\[
\| c (A^*)^{-1} h \| = \| h \| \quad \text{for all} \quad h \in \{ e \}^\perp.
\] (3)

Similarly, we take any \( h' \in \{ f \}^\perp \), apply \( \phi(h' \otimes f) = A h' \otimes f \sim_u u \| h' \| e \otimes f \), and obtain
\[
\| A h' \| = \| h' \| \quad \text{for all} \quad h' \in \{ f \}^\perp.
\] (4)

So, the restrictions of \( c (A^*)^{-1} \) to \( \{ e \}^\perp \) and \( A \) to \( \{ f \}^\perp \) are linear isometries and therefore, unitary operators. Recalling \( A^* f = c f \) and \( A e = e \) operator \( A^* \) can be presented according to the decomposition \( H = \text{span}[e] \oplus \text{span}[f] \oplus \{ e, f \}^\perp \) as
\[
A^* = \begin{bmatrix}
1 & 0 & 0 \\
? & c & 0 \\
? & 0 & c U^*
\end{bmatrix},
\]
where \( U \) is a unitary operator acting on \( \{ e, f \}^\perp \). Then, by unitarity of \( A \) on \( \{ f \}^\perp \) we observe that for any \( g \in \{ e, f \}^\perp \)
\[
\langle Ag, e \rangle = \langle Ag, Ae \rangle = \langle g, e \rangle = 0
\]
and \( c = 1 \), so,
\[
A = \begin{bmatrix}
1 & ? & 0 \\
0 & 1 & 0 \\
0 & 0 & U
\end{bmatrix}.
\]

Hereby we have proved that \( \text{span}[e, f] \) is a reducing subspace of \( A \). We now assume without loss of generality that \( U \) is identity. Finally, let us take any unit vector \( g \in \{ e, f \}^\perp \) and compute \( \phi(f \otimes g) = Af \otimes (A^*)^{-1} g = Af \otimes U g = Af \otimes g \sim_u e \otimes f \). The property \( \| Af \| = \| f \| \) finally yields unitarity of \( A \).

From now on, we can assume that \( \phi(N) = N \) for any finite rank nilpotent \( N \) and, consequently, \( \phi(F) = F \) for any \( F \in \mathcal{F}_0(H) \). The following proposition brings us the control over projections.

**Proposition 4.** Let \( P \in B(H) \) be a non-trivial (\( \neq 0, I \)) projection (\( P = P^*, P^2 = P \)). If for \( B \in B(H) \) the following holds true:
\[
P + F \sim_u P \iff B + F \sim_u B \quad \text{for every} \quad F \in \mathcal{F}_0(H),
\] (5)
then there exist a non-zero \( \alpha \) and \( \mu \in \mathbb{C} \) such that \( B = \alpha P + \mu I \).
Proof. Let us choose a unit vector $e \in \text{Im} P$ and write $P$ and $B$ in operator matrix form according to the decomposition $H = \text{span} \{e\} \oplus \{e\}^\perp$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}, \quad Q = Q^*, \quad Q^2 = Q, \quad B = \begin{bmatrix} b_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. $$

Let $U_{\beta} = \text{diag}(\beta, 1)$, $|\beta| = 1$, $\beta \neq 1$, be a unitary operator. Simple computation shows

$$U_{\beta} B U_{\beta}^* = B + \begin{bmatrix} 0 & (\beta - 1)B_{12} \\ (\beta - 1)B_{21} & 0 \end{bmatrix} = B + F, \quad F \in \mathcal{F}_0(H).$$

By (5) we have $P + F \sim_u P$, so, the operator $P + F$ must be a self-adjoint idempotent. This implies $F = 0$ and

$$B = \begin{bmatrix} b_{11} & 0 \\ 0 & B_{22} \end{bmatrix}. $$

So, $Be = b_e e$, $b_e \in \mathbb{C}$, for every $e \in \text{Im} P$, and the restriction of $B$ to $\text{Im} P$ is thus a scalar operator. The verification that the restriction of $B$ to $\text{Ker} P$ is also scalar is analogous and will be omitted. \hfill \Box

We close the proof of the Main Theorem similarly to the one in [13] by observing first that $\phi(P) = P + \mu P I$ for some $\mu P \in \mathbb{C}$. By the result of Pearcy and Topping [12] every operator in $B(H)$ is a linear combination of a finite number of projections. Therefore, $\phi(A) = A + f(A) I$ for some linear functional $f$ on $B(H)$. It remains to show that $f(A) = 0$ for all $A \in B(H)$. As every operator on $B(H)$ is a finite sum of square zero operators it is enough to observe that $f(N) = 0$ for every $N$ with $N^2 = 0$. Clearly, as $N \sim_u \alpha N$ for all $|\alpha| = 1$, $\phi(N)$ and $\phi(\alpha N)$ have only one point in the spectrum ($f(N)$ and $\alpha f(N)$, respectively) and the spectrum of both coincides for every $|\alpha| = 1$. Therefore, $f(N) = 0$ and consequently, $\phi(A) = A$ for all $A \in B(H)$.

References