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Resolving the Singularities of Differential Equations

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0. INTRODUCTION

In this paper we utilize sheaf theoretic methods to present a theory for resolving the singularities of an implicit analytic differential equation which is based upon the corresponding theory of resolving singularities in analytic geometry. There is some arbitrariness in the construction of a resolution. Our choice of construction was dictated largely by our desire to have the construction be natural, and to subsume the theory of singular solutions which must be confronted in any serious study of algebraic differential equations.

In its simplest form, the main theorem shows how to associate with any differential equation of the form:

$$G(x(t),\ y(t),\ q(t),\ p(t)) = 0, \tag{*}$$

$q(t) = dx/dt$, $p(t) = dy/dt$, G analytic in x, y, q, p and homogeneous in q and p, another differential equation (**) defined on a two-dimensional analytic manifold N in such a way that:

(1) There is an analytic mapping $\alpha: N \to E^2$, E^2 being the real or complex 2-space.

(2) The equation (**) is locally of the form:

$$B(x(t),\ y(t))(dy/dt) = A(x(t),\ y(t))(dx/dt), \tag{**}$$

with A and B analytic in x and y.

(3) α maps the solutions of (**) onto the solution of (*) about as well as it can be expected to do so.

The ideas presented here in the setting of complex analytic spaces are actually applicable in each category for which a theory for resolving

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singularities has been developed. In particular, the real case is included in our discussion. In essence, the "resolution" (**) of (*) is just a simpler differential equation arrived at from (*) by means of a sequence of substitutions, e.g., quadratic transformations having certain desirable properties. The well-known Frommer transformation is a classical example of this. Further examples occur in [10] and [1], where certain differential equations of the type (*) are "displayed on" ("separated on" or "lifted to") surfaces such as the torus or Mobius band. In this paper we develop the separating and lifting techniques which are necessary and sufficient to bring the theory of the general substitution to a point where it can be wielded to obtain concrete results. As applications of the theory we include:

(1) A definition of the index a certain class of singularities of (*)—the persistent ones—in a way which makes applying the Poincaré–Hopf theory to counting these singularities tenable.

(2) A finiteness theorem for the number of directions along which infinitely many solutions of (*) may tend to a certain point.

(3) The setting of the notion of singular solution in proper perspective.

1. Notation, Definitions, and Related Results

1.0. *Notation*

Let V or V_i denote pure dimensional complex analytic spaces. Given V, let $T(V)$ be its tangent space, and let $P(V)$ be the projective tangent space of V whose elements are tangent lines in $T(V)$, cf. [14] or [15]. One then has the natural projections $\psi: T(V) \to V$ and $\phi: P(V) \to V$, and a natural map $\gamma: T(V) \to P(V)$ which is not defined on zero tangent vectors but for which $\phi \circ \gamma = \psi$ at nonzero tangent vectors. To indicate the dependence of ϕ on V we shall write ϕ_V in place of ϕ. An analytic map $f: V_1 \to V_2$ induces the differential map $df: T(V_1) \to T(V_2)$. If f does not collapse any open sets into points, df induces its meromorphic projective counterpart $Pf: P(V_1) \to P(V_2)$. If $x \in V$, let $T(V, x)$ denote the vector space $\psi^{-1}(x)$. Then an element of $\phi^{-1}(x)$ is a one-dimensional subspace of $T(V, x)$.

Throughout, we shall use V and W to denote pure two-dimensional complex analytic spaces. N will denote such a space which in addition is normal or nonsingular. We shall also use D, $D_{c,i}$, or D_i only for domains in the complex number system C which contain zero, and η, η_i, $\eta_{c,i}$, $\hat{\eta}$ or ρ for analytic maps defined on such domains.

If (x, y) denote coordinates on an open set U in a two-dimensional manifold, then $P(U) = U \times P^1$, where P^1 is the projective line whose points may be represented by homogeneous coordinates $\{q, p\}$, $|q| + |p| \neq 0$. Thus

$P(U)$ is covered by two coordinate patches having coordinates $(x, y, \{1, p\})$ or $(x, y, \{q, 1\})$ which we will often simply write as (x, y, p) or (x, y, q).

DEFINITION 1.1. An implicit analytic differential equation (IADE) on V is a triple (W, U, V), where U is an open subset of $P(V)$ and W is a closed analytic subvariety of U. An analytic solution of (W, U, V) is a map $\eta: D \to V$ from some D into V for which $P\eta(P(D)) \subseteq W$. (A different but often handier definition of a solution is given in Convention 2.4. We shall normally require η to be nonconstant in which case note that $P\eta$ is defined on all of $P(D)$.)

Let O_V as usual denote the sheaf of germs of holomorphic functions on V.

DEFINITION 1.2. An explicit meromorphic differential equation (EMDE) on V is a coherent sub-O_V-module $\mathscr{M}$ of the sheaf Ω'_V of germs of holomorphic differential 1-forms on V, which is locally of rank one, i.e., for every point $m \in V$, the fiber $\mathscr{M}_m$ of $\mathscr{M}$ is generated (as a module over $O_{V,m}$) by a single element ω_m of the fiber $\Omega'_{V,m}$ of Ω'_V: For each $m \in V$ there exists an element $\omega_m \in \Omega'_{V,m}$ such that $\mathscr{M}_m = O_{V,m} \cdot \omega_m$. An analytic solution of $\mathscr{M}$ is a map $\rho: D \to V$ with $\rho^*(\mathscr{M}) = 0$, i.e., the sub-O_D-module of Ω'_D defined by $(\rho^*(\mathscr{M}))_x = O_{D,x} \cdot \rho^*(\omega_y)$, where $y = \rho(x)$, $\omega_y \in \Omega'_{V,y}$ and $\mathscr{M}_y = O_{V,y} \cdot \omega_y$, is zero at each point.

Notation 1.3. If S is a sheaf on Y and $f: X \to Y$ is a map let $f^*(S)$ denote the sheaf on X induced from S by f.

Observe that if S is coherent and locally free of rank one, cf. Definition 1.2 then so is $f^*(S)$. Observe also that the operation of inducing sheaves is natural: if $g: Z \to X$, then $(f \circ g)^*(S) = g^*(f^*(S))$.

DEFINITION 1.4. Let V be an analytic manifold. Let (x, y) denote local coordinates on V, and let (x, y, q) or (x, y, p) denote the associated coordinales on $P(V)$, cf. Sect. 1.0. Then the canonical contact structure on $P(V)$, defined locally by the forms $dx - q\,dy$ or $p\,dx - dy$, induces a coherent $O_{P(V)}$-module which is locally of rank one and which we denote by $\mathscr{C}$.

Remark 1.5. To verify that $\mathscr{C}$ in Definition 1.4 is well-defined note that if $\bar{x}, \bar{y}$ is another chart on V and $\bar{x}, \bar{y}, \bar{q}$ or $\bar{p}$ is the associated chart in $P(V)$, then $dx - q\,dy = e(\bar{x}, \bar{y})(d\bar{x} - \bar{q}\,d\bar{y})$, where

$$e(\bar{x}, \bar{y}) = \left(\begin{vmatrix} \dfrac{\partial x}{\partial \bar{x}} & \dfrac{\partial x}{\partial \bar{y}} \\ \dfrac{\partial y}{\partial \bar{x}} & \dfrac{\partial y}{\partial \bar{y}} \end{vmatrix} \Bigg/ \left(\bar{q} \frac{\partial y}{\partial \bar{x}} + \frac{\partial y}{\partial \bar{y}} \right) \right) \neq 0.$$

EXAMPLE 1.6a. The sheaf $\mathscr{C}$ of Definition 1.4 may be viewed as an EMDE on $P(V)$, although in Definition 1.2 we were concerned with an EMDE on a two-dimensional space.

EXAMPLE 1.6b. Let $\tau: N \to P(V)$, N as in Section 1.0, be given. Then $\tau^*(\mathscr{C})$ is an EMDE on N, see the observation following Notation 1.3.

DEFINITION 1.7. Let N be normal, $m \in N$ and $\mathscr{M}$ be an EMDE on N. Consider ω_m in Definition 1.2 as a linear map $\omega_m: T(N, m) \to C$. Define the singular set $H(\mathscr{M})$ of $\mathscr{M}$ by $H(\mathscr{M}) = \{m \in N \mid \text{dimension kernel } \omega_m > 1\}$.

Note 1.8. Because of the coherence of $\mathscr{M}$, the same ω_m may be used to define $H(\mathscr{M})$ in a neighborhood of m. Thus if $\omega_m \not\equiv 0$, $H(\mathscr{M})$ is analytic and has codimension at least one in N. Since dimension $T(N, m) > 1$ if m belongs to the singular set N^x of N, cf. [15], it follows that $H(\mathscr{M}) \supseteq N^x$.

DEFINITION 1.9. A map $\rho: D \to H(\mathscr{M})$ will be called a singular solution of $\mathscr{M}$. Clearly a singular solution is a solution.

DISCUSSION 1.10. Giving an EMDE $\mathscr{M}$ on an open set U in C^2 is equivalent to giving a holomorphic differential 1-form $\omega(x, y) = a(x, y)dx + b(x, y)dy$, defined up to a multiplicative unit e (i.e., $e(x, y) \neq 0$ for any $(x, y) \in U$). Then $H = H(\mathscr{M})$ is the set of zeros common to $a(x, y)$ and $b(x, y)$. In general, assuming $\mathscr{M}$ to be nontrivial, ω determines a meromorphic section $v: N \to P(N)$ given by $v(m) = \text{kernel } \omega_n$, cf. Definition 1.7. In terms of local coordinates x, y, p or q in $P(U)$, v is given by $v(x, y) = (x, y, q = -b(x, y)/a(x, y)$ or $p = -a(x, y)/b(x, y))$. One may think of v as assigning a slope dy/dx at (x, y) given by $dy/dx = p = -a(x, y)/b(x, y)$. A solution $\rho: D \to U$ is a map for which $(\dot{x}, \dot{y})$ is parallel to $(b(x, y), -a(x, y))$, i.e., $\dot{x}(t)\, a(x(t), y(t)) + \dot{y}(t)\, b(x(t), y(t)) = 0$. Giving an EMDE $\mathscr{M}$ on any N is equivalent to giving a meromorphic section $v: N \to P(N)$ and an analytic subset H of N of codimension at least one, outside of which v is analytic. The fact that this equivalence holds with U being replaced by a normal space N follows from Levi's continuation theorem, [12, p. 133], according to which it suffices to describe a meromorphic function on the complement of a subset of N of codimension two. When we wish to think of $\mathscr{M}$ in this form, we shall write it as (v, H, N).

Two meromorphic functions on V are normally identified if they coincide on an open dense subset of V. Since two different forms with different singular sets (and hence two EMDEs with different singular sets) may give rise to the same section $v: V \to P(V)$, it is important when expressing an EMDE on V in the form (v, H, V) to keep track of the singular set H. We shall see in Section 3 that H is the key to the notion of a singular solution of an IADE.

2. The Normalization and Resolution of an IADE on a Manifold $\mathscr{V}$

2.0

In the rest of this paper let (W, U, V) be an IADE in which V is a manifold, and let (N, τ), $\tau: N \to W$, be either a normalization of W (cf. [9 or 15]) or a resolution of W (cf. [4 or 9]). In particular τ is proper, analytic and onto, $\tau: \tau^{-1}$ (the set W^- of nonsingular points of W) $\to W^-$ is an analytic equivalence, $\tau^{-1}(W^-)$ is dense in N, and in the case of a normalization N is normal and τ is finite, or in the case of a resolution N is nonsingular. (In the case of a normalization the singular set N^x of N has codimension not less than two in N).

Definition 2.1. Let (W, U, V) be an IADE, let $\mathscr{C}$ be as in Definition 1.4, and suppose that $\tau: N \to W$ is a normalization (or a resolution, or a minimal resolution; cf. [2 or 9]) of W. Consider τ as a map into $P(V)$. The EMDE τ^* given by the O_N-module $\tau^*(\mathscr{C})$ is called a normalization (resp. a resolution or a minimal resolution) of (W, U, V).

Let $\tau^*(\mathscr{C}) = (v, H, N)$ as in Discussion 1.10. The next proposition describes $\tau^*(\mathscr{C})$ when viewed in the form (v, H, N).

Proposition 2.2. *Let* (W, U, V), $\tau: N \to W$, $\tau^*(\mathscr{C}) = (v, H, N)$ *be as above. Then* (v, H, N) *is uniquely characterized by conditions* A_1 *and* A_2 *or equivalently by condition* B:

(A_1) $m \in H$ *if and only if* m *is a singular point of* N *or* $d(\phi \circ \tau)(T(N, m)) \subseteq \tau(m)$.

(A_2) *if* $m \notin H$, $(d(\phi \circ \tau))(v(m)) \subseteq \tau(m)$.

(B) $v(m) = T(N, m) \cap (d(\phi \circ \tau))^{-1}(\tau(m))$ *if the latter is one dimensional; and* $m \in H$ *otherwise.*

Proof. The case of a singular point m of N follows from Note 1.8. So assume m to be a regular point of N, and let τ_1, τ_2, τ_3 be the components of τ in the chart x, y, q near $\tau(m)$, $\tau(m)$ being considered as a point in $P(V)$, cf. Section 1.0. One has for $\omega = dx - q\,dy$ that $\tau^*(\omega) = d\tau_1 - \tau_3\, d\tau_2$; and for $X \in T(N, m)$ that $\tau^*(\omega)(m)(X) = 0 \leftrightarrow \tau_3(m) = d\tau_1(m)(X)/d\tau_2(m)(X) \leftrightarrow d(\phi \circ \tau)(X) \subseteq \tau(m)$. Since when it is defined $v(m) \subseteq \{X \in T(M, m) \mid \tau^*(\omega)(m)(X) = 0\}$, and since $m \in H$ if and only if this set is not one dimensional, the proposition follows easily from the preceding relations and the definitions.

Before stating the next lemma note that since D is one-dimensional $P(D)$ is essentially the same as D, and when there is no likelihood of confusion will be identified with it. The next lemma shows that specifying a solution η of an IADE (W, U, V) is equivalent to specifying a map $\gamma: D \to W$ with $\gamma^*(\mathscr{C}) \equiv 0$, where $\mathscr{C}$ and $\gamma^*(\mathscr{C})$ are as in Example 1.6., and γ is considered as a map into $P(V)$.

LEMMA 2.3. *A map $\eta: D \to V$ is a solution of (W, U, V) if and only if $\eta = \phi \circ \gamma$ where $\gamma: D \to W$ is such that $\gamma^*(\mathscr{C})$ is identically zero. Conversely, a curve $\gamma(t): D \to W$ is such that $\phi \circ \gamma: D \to V$ is a solution of (W, U, V) if and only if $\gamma^*(\mathscr{C})$ is identically zero.*

Proof. Let (x, y) and (x, y, q) be local coordinates in V and $P(V)$ respectively as in Section 1.0. Write $\gamma(t) = (x(t), y(t), q(t))$. The $\gamma^*(\mathscr{C}) \equiv 0$ means $(dx - q\,dy)(\dot{x}, \dot{y}, \dot{q})) = 0$ which says $q(t) = (dx/dt)/(dy/dt)$, i.e., $\gamma = P(\phi \circ \gamma)$, or more precisely $\gamma = P(\phi \circ \gamma) \circ \phi_D^{-1}$.

For the first statement, if η is a solution then $P\eta$ considered as a map from D into $P(V)$ maps D into W, and if we take $\gamma = P\eta$, then clearly $\gamma = P(\phi \circ \gamma)$. Hence $\gamma^*(\mathscr{C}) \equiv 0$. Conversely if $\gamma: D \to W$ and $\gamma^*(\mathscr{C}) \equiv 0$, then $P(\phi \circ \gamma) = \gamma$. Set $\eta = \phi \circ \gamma$. Then image $P\eta =$ image $\gamma \subseteq W$. Hence η is a solution.

For the second statement, if $\gamma: D \to W$ is such that $\phi \circ \gamma$ is a solution, then $\gamma = P(\phi \circ \gamma)$, and hence $\gamma^*(\mathscr{C}) \equiv 0$. Conversely, if $\gamma^*(\mathscr{C}) \equiv 0$, then $\gamma = P(\phi \circ \gamma)$. Thus image $P(\phi \circ \gamma) =$ image $\gamma \subseteq W$, and hence $\phi \circ \gamma$ is a solution.

Convention 2.4. By Lemma 2.3 we will mean by a solution of (W, U, V) either a map $\eta: D \to V$ with $P\eta \subseteq W$ or a map $\gamma: D \to W$ with $\gamma^*(\mathscr{C}) \equiv 0$. Of course η and γ are related by $\eta = \phi \circ \gamma$ and $\gamma = P\eta$.

LEMMA 2.5. *Let (W, U, V) and $\tau: N \to W$ be given, and let $\mathscr{C}$ and $\tau^*(\mathscr{C})$ be as usual. Then $\eta: D \to N$ is a solution of $\tau^*(\mathscr{C})$ if and only if $(\tau \circ \eta)^*(\mathscr{C}) \equiv 0$. By Convention 2.4., this means that η is a solution of $\tau^*(\mathscr{C})$ if and only if $\tau \circ \eta$ is a solution of (W, U, V).*

Proof. By the definition of a solution of an EMDE and the functoriality of the operation of inducing sheaves, η is a solution of $\tau^*(\mathscr{C}) \leftrightarrow \eta^*(\tau^*(\mathscr{C})) \equiv 0 \leftrightarrow (\tau \circ \eta)^*(\mathscr{C}) \equiv 0$.

The next theorem roughly states that $\eta \leftrightarrow \tau \circ \eta$ is a one–one correspondence between the solutions of (W, U, V) and its resolution.

THEOREM 2.6. *Let (W, U, V) be an* IADE, *let $\tau: N \to W$ be a resolution (or normalization), and let $\tau^* = \tau^*(\mathscr{C})$ be the corresponding resolution (or normalization) of (W, U, V). Then a curve $\eta: D \to N$ is a solution of τ^* if and only if $\tau \circ \eta$ is a solution of (W, U, V). Further, if $\eta: D \to W$ is a solution of*

(W, U, V), then there is a solution $\eta_1: D_1 \to N$ with $\eta_1(0) = \eta(0)$ of τ^* and some $D_2 \subseteq D$ for which $\eta(D_2)$ and $\eta_1(D_1)$ define the same set germ at $\eta(0)$.

Proof. The first part of the theorem follows from the preceding lemma. The second part follows from the first, provided we show that, roughly speaking, any $\eta: D \to W$ is (in the sense of the theorem) an image under τ of some $\eta_1: D_1 \to N$. The details follow from the purely analytic path lifting property which is given in the appendix.

Note 2.7. Due to the failure of the proper and open mapping theorems for real varieties, the last part of the theorem must in the real case be modified to read as follows: There are finitely many solutions $\eta_i: D_1 \to N$ of τ^* and some $D_2 \subseteq D$ for which $\eta(D_2)$ and $\bigcup_i \eta_i(D_i)$ define the same set germ at $\eta(0)$.

3. Applications

For the applications we shall need the following lemma. We adhere to our previous notation.

Lemma 3.1. *Given an* EMDE $\mathscr{M}$ *on* N, *there is another* EMDE $\tilde{\mathscr{M}}$ *on* N *which is characterized by the two properties*:

(1) *If the differential form germ ω_m generates the fiber $\mathscr{M}_m$ of $\mathscr{M}$ above m (over $O_{N,m}$), cf. Definition* 1.2, *while $\tilde{\omega}_m$ generates the fiber $\tilde{\mathscr{M}}_m$ of $\tilde{\mathscr{M}}$ above m, then ω_m is a multiple of $\tilde{\omega}_m$, i.e., $\mathscr{M}_m \subseteq \tilde{\mathscr{M}}_m$, and*

(2) *the singular set $\tilde{H}$ of $\tilde{\mathscr{M}}$ consists of isolated points and is a minimal such set.*

Proof. When $\mathscr{M}$ is viewed in the form (v, H, N) of Discussion 1.10, it is easy to see that the lemma is equivalent to the statement that the set of nonremovable singularities of a meromorphic function on N has codimension at least two. A proof in the case in which N is nonsingular may be found in [3, p. 247] or [14, pp. 92, 57, 136]. The proof in the case in which N is normal is accomplished by taking a resolution $\sigma: N_1 \to N$ of N and applying the preceding case to $v \circ \sigma$. The minimality part of the lemma is clear.

To gain insight into the lemma, consider $\mathscr{M}$ near a nonsingular point m of N and let (x, y) be coordinates valid near m. Let $\omega_m = a(x, y)\,dx + b(x, y)\,dy$. We may suppose that $a(x, y) = C(x, y)\,a_1(x, y)$ and $b(x, y) = C(x, y)\,b_1(x, y)$, where C, a_1, and b_1 are analytic near m and where m is the only zero common to a_1 and b_1. Clearly the lemma is satisfied if we let $\tilde{\omega}_m = a_1(x, y)\,dx + b_1(x, y)\,dy$. For $m \in N^x$ one uses an extension of this argument or simply applies Levi's continuation theorem (cf. [12, p. 133]).

DEFINITIONS 3.2. We shall call the EMDE $\tilde{\mathcal{M}}$ of Lemma 3.1 the reduced equation of $\mathcal{M}$, and if H denotes the singular set of $\mathcal{M}$ we let $\tilde{H}$ denote that of $\tilde{\mathcal{M}}$. A point of $\tilde{H}$ is called an essential singular point of $\mathcal{M}$ (as well as of $\tilde{\mathcal{M}}$). The points of H which are not in $\tilde{H}$ are the removable singular points of $\mathcal{M}$. The points of N which are not in H are called regular points of $\mathcal{M}$. A solution of $\tilde{\mathcal{M}}$ is called a regular solution of $\mathcal{M}$ (as well as of $\tilde{\mathcal{M}}$). A solution $\rho: D \to N$ is called singular if $\rho(D) \subseteq H$.

Note 3.3. A solution may be both regular and singular. Note also that $\rho: D \to N$ is a solution of $\mathcal{M}$ if and only if either ρ is a solution of $\tilde{\mathcal{M}}$ or $\rho(D) \subseteq H$.

In what follows we suppose that (W, U, V) and $\tau: N \to W$ are given and let the corresponding resolution or normalization be τ^*. We also set $\alpha = \phi_v \circ \tau$. Recall the following:

Note 3.4. Let $\mathcal{M}$ be an EMDE on N. According to Discussion 1.10, if $m \in N^-$ one may view $\tilde{\mathcal{M}}$ on a neighborhood U of m as a differential form $a(x, y)\,dx + b(x, y)\,dy$ which vanishes at most at m. This form gives rise to $S: U \to T(U)$, $S(x, y) = (x, y, (-b(x, y), a(x, y)))$. The line element field $v = \gamma \circ S$ is then orientable near m in the sense of [11], and the index of v near m is defined in [11] as the degree of v considered as a map from a deleted neighborhood of m into P^1.

Application 1: The Index of a Singularity of an IADE $(W, P(V), V)$

DEFINITION 3.5. For definiteness let τ^* be a minimal resolution of $(W, P(V), V)$, and consider the associated reduced equation $\tilde{\tau}^*$ with singular locus $\tilde{H}$. Call a point v of V a higher order singular point of $(W, P(V), V)$ if $v \in \alpha(\tilde{H})$, $\alpha = \phi \circ \tau$. Define the index of v to be the sum of the usual indices—see [11] and Note 3.4—of the singularities of $\tilde{\tau}^*$ which lie in $\alpha^{-1}(v)$.

Remark 3.6. The compactness of P^1 ensures that the sum in Definition 3.5 is finite. The index of a higher order singularity of an IADE of the form $F(x, y, p) = 0$, $p = \dot{y}/\dot{x}$, f being a polynomial of degree k in p, in which $W_1 = \{(x, y, p) \mid F(x, y, p) = 0\}$, $\phi: W_1 \to C^2$ and $\phi^{-1}(x, y)$ is not compact, may be obtained from the above by first homogenizing the equation by multiplying it by $(\dot{x})^k$. This way one obtains an equation of the form $(W, P(C^2), C^2)$, $W \supset W_1$, containing (W, C^3, C^2) as a "subequation." In calculating the index of a singularity of (W_1, C^3, C^2) one sums only over the singularities of $\tilde{\tau}^*$, $\tau: N \to W$, which lie in W_1. In the real case in which W_1 is compact, this procedure is of course redundant.

PROPOSITION 3.7. *Consider a* (*real*) IADE $(W, P(V), V)$ *in which* W *is* *compact and nonsingular. Then the sum of the indices of the higher order singularities of* $(W, P(V), V)$ *equals twice the Euler characteristic of* W.

Proof. This follows from Theorem 2 of [11] applied to $\tilde{\tau}^*$, τ being the identity map, upon using Note 3.4 to verify that N in that theorem is zero.

In the complex case of Definition 3.5, $\tilde{\tau}^*$ can be viewed as a divisor or line bundle on N. The Chern class of this line bundle is an invariant of $(W, P(V), V)$ which is natural with respect to morphisms of differential equations.

Application 2

In our next application we show that there are at most finitely many directions (slopes) along which infinitely many distinct solutions of an IADE may pass through a given point. We leave it to the reader to verify that the same is true of the number of directions along which infinitely many solutions "approach" a given point.

The next proposition applies to any differential equation of the form $G(x, y, q, p) = 0$, $q = \dot{x}$, $p = \dot{y}$, where G is analytic on C^4 and is homogeneous in q and p, by letting $V = C^2$, $U = P(C^2)$ and $W = \{(x, y, \{q, p\}) \in C^2 \times P^1 \mid G(x, y, q, p) = 0\}$. Equations of the form $dy/dx = A(x, y)/B(x, y)$ may be included by writing them in the form $G = \dot{y}B - \dot{x}A = 0$. We use the notation of Section 1.0. The reader is invited to attempt proving Proposition 3.8 without using the preceding theory.

PROPOSITION 3.8. *There are at most finitely may directions* $c = \{c_1, c_2\} \in P^1$ *for which the* IADE $(W, P(C^2), C^2)$ *has infinitely many solutions* $\eta_{c,i} = (x_{c,i}, y_{c,i})\colon D_{c,i} \to C^2$, $i = 1, 2, 3, \ldots$, *with*

$$\lim_{t \to 0} (x_{c,i}(t), y_{c,i}(t), \{\dot{x}_{c,i}(t), \dot{y}_{c,i}(t)\}) = (0, 0, c),$$

distinct in the sense that $\eta_{c,i}(D_{c,i}) \cap \eta_{c,j}(D_{c,j}) = (0, 0)$ *for each* i *and* j *with* $i \neq j$.

Proof. We let $\tau^* = (v, H. N)$, $\tau\colon N \to W$, be a normalization of $(W, P(C^2), C^2)$ and apply Theorem 2.6 to curves in sets of the form $\tau^{-1}(P\eta_{c,i}(P(D_{c,i}))$. Since τ is finite, each family of $\eta_{c,i}$'s belonging to the same slope c, gives rise to at least one point $\tilde{c}$ in $\tau^{-1}(0, 0, c) = \tau^{-1}(P\eta_{c,i}(\phi_D^{-1}(0)))$ through which pass infinitely many distinct solutions of τ^*. Since by Note 1.8 $H(\tau^*) = H$ is at most one-dimensional analytic, it follows from Note 3.3 that infinitely many of these solutions must be solutions of $\tilde{\tau}^*$. It then follows from the uniqueness of solutions theorem for ordinary analytic differential equations near a regular point that $\tilde{c}$ is not a regular point of $\tilde{\tau}^*$, cf. Definition 3.2. Thus points of the form $\tilde{c}$ lie in $\tilde{H} \cap \tau^{-1}((0, 0) \times P^1)$. Since τ is proper $\tau^{-1}((0, 0) \times P^1)$ is compact, and since by Lemma 3.1 $\tilde{H}$ consists of isolated points, it follows that the possible set of $\tilde{c}$'s is finite. Hence the possible set of c's in the proposition is finite.

Application 3: Regular and Singular Solutions

Remark 3.9. In the same sense in which a complete function shares certain properties with its function elements, there are certain properties which apply to a solution defined on a domain D if and only if they apply to some restriction of it to a nonempty subdomain of D. The property of a solution being a member of a nondegeneate one-parameter family of solutions is one such. Using the analyticity and the dimensionality of the singular set of an EMDE it is easy to see that also the property of being singular or regular is one such. We may now lay down the following definition.

DEFINITION 3.10. Call a solution $\eta: D \to W$ of (W, U, V) regular (resp. singular) if some restriction of it to a nonempty subdomain of D is of the form $\tau \circ \rho$ for some regular (resp. singular) solution ρ of $\tau^* = \tau^*(W, U, V)$, where $\tau: N \to W$, and τ^* is a normalization or a resolution of (W, U, V).

Since every resolution of W can be factored through the normalization of W, cf. [9] p. 46, the naturality of the induced equations τ^* with respect to composition of maps, cf. Notation 1.3 and the observation following it, may be used to show that this definition is independent of whether τ^* is a normalization or a resolution of (W, U, V).

Note 3.11. It is easy to see that a solution which is not regular is singular (cf. Note 3.3).

The next proposition extends the fact that any regular solution of an EMDE is a member of a nondegenerate one-parameter family of solutions to the case of an IADE. The failure of a solution to be a member of a nondegenerate one-parameter family of solutions is often taken as the defining property of a singular solution (cf. [5, p. 12]). Technically this would be incorrect, since a solution may be both regular and singular depending on the "branch" or "component" or "factor" of the differential equation of which it is viewed to be a solution.

PROPOSITION 3.12. *A solution of (W, U, V) is regular if and only if it is a member of a nondegenerate one-parameter family of solutions of (W, U, V).*

Proof. Let $\tau: N \to W$ be a normalization, and let $\eta: D \to W$ be a regular solution of (W, U, V), cf. Convention 2.4. We may assume that $\eta = \tau \circ \rho$, where $\rho: D_1 \to N$ is a regular solution of τ^* and is nonconstant. Then ρ is a solution of $\tilde{\tau}^*$. Since $H(\tilde{\tau}^*)$ consists of isolated points we may by

Remark 3.9 assume that $\rho(D_1)$ does not meet $H(\tilde{\tau}^*)$. It follows from the fundamental theorem on the existence and uniqueness of solutions of ordinary differential equations near a resular point that ρ is a member of a nondegenerate one-parameter family F or ρ_t of solutions of $\tilde{\tau}^*$. Here the parameter t ranges over some domain E in C, $F: D \times E \to N$, and $\rho_t = F \mid D \times \{t\}$, where $\rho = \rho_0$. Clearly $\tau \circ \rho_t$ is a one-parameter family of solutions of (W, U, V) containing $\tau \circ \rho$ as a member. Non degeneracy merely means that the rank of F at some point of $D \times E$ is two, or equivalently that $F(D \times E)$ is two dimensional. Since τ preserves dimension, $\tau \circ F$ is nondegenerate.

To prove the converse, let $G: D \times E \to W$, $\eta_t = G \mid D \times \{t\}$ be a non-degenerate one-parameter family of solutions of (W, U, V) containing the nonconstant solution $\eta_0 = \eta$. By restricting the domain of G and excluding $(0, 0)$ from this domain if necessary, we may by [15, p. 245] assume that:

(1) $G(0, 0) \in W^-$ and $G(D \times E)$ determines a single irreducible set germ component T of W near $G(0, 0)$, and that

(2) $G(0, 0)$ has a neighborhood K in W for which $\tau^{-1}(K)$ consists of disjoint open sets $N_1, ..., N_s$, one for each point of $\tau^{-1}(G(0, 0))$, or equivalently, one for each irreducible component of the germ of W at $G(0, 0)$.

Let N_T be the component in (2) which corresponds to T. Making use of the defining properties of a normalization (cf. Sect. 2.0), we may by further shrinking of the domain of G and of K also suppose that $\tau \mid N_T$ is a one-one map, and that $\tau(N_T) \supseteq G(D \times E)$. Hence there is no ambiguity in defining $\rho: D \times E \to N_T$ by $\rho = \tau^{-1} \circ G$. The analyticity of ρ follows from [15, p. 258]. This ρ is just a lifting of an appropriate restriction of G to N. The fact that it is a one-parameter family of solutions of τ^* with the required properties now follows from the first part of Theorem 2.6.

For other aspects of singular solutions we refer the reader to [6], [7], [8] and [13].

4. Appendix

We outline a proof of the following basic property of a normalization or a resolution $\tau: N \to W$ of W, used in the proof of Theorem 2.6.

The Path Lifting Property

Let $\eta_2: D \to W$ be given. Then there exists a map $\hat{\eta}: D_1 \to N$ from some D_1 into N and a sufficiently small neighborhood D_2 of 0 in D (see Fig. 1) for which the following three conditions hold:

$$\begin{tikzcd} & N \arrow[rr, "\tau"] & & W \\ D_1 \subset Q \arrow[ur, "\hat{\eta}"] & & D_2 \subset D \arrow[ur, "\eta_2"'] & \end{tikzcd}$$

FIGURE 1

(1) $\eta_2(0) = (\tau \circ \hat{\eta})(0)$, $(\tau \circ \hat{\eta})(D_1) \subseteq \eta_2(D_2)$, and the sets $\eta_2(D_2)$ and $(\tau \circ \hat{\eta})(D_1)$ define the same set germ at $\eta_2(0) = (\tau \circ \hat{\eta})(0)$.

(2) If D_3 is a neighborhood of 0 in D_2, then $\eta_2(D_3)$ and $\eta_2(D_2)$ define the same set germ at $\eta_2(0)$.

(3) Given $t_1 \in D_1$, there is an element t_2 of D_2 depending on t_1 for which $(\tau \circ \hat{\eta})(t_1) = \eta_2(t_2)$ and $P(\tau \circ \hat{\eta})(T(D_1 t_1)) = P\eta_2(T(D_2, t_2))$.

Outline of the Proof

We may suppose that η_2 is not constant. Then 0 is an isolated point of $\eta_2^{-1}(\eta_2(0))$. By the open mapping theorem [15, p. 122], we may assume (by replacing D by an appropriate neighborhood D_2 of 0 in D if necessary) that $\eta_2(D_2)$ is analytic in W near $\eta_2(0)$ and that $\eta_2(D_2)$ itself is an analytic space. Clearly we may also assume that D_2 is sufficiently small for the set germ of $\eta_2(D_2)$ at $\eta_2(0)$ to be irreducible and for condition (2) to hold. This can be deduced for instance from Theorem 4D, p. 151 of [15]. It follows from the properties of τ (outlined in Sect. 2.0) that $\eta_2(D_2)$ and $\tau^{-1}(\eta_2(D_2))$ are analytic one-dimensional spaces and that $\tau\colon \tau^{-1}(\eta_2(D_2)) \to \eta_2(D_2)$ is proper and is onto. Perhaps the quickest way to proceed from here is to let $\sigma\colon Q \to \tau^{-1}(\eta_2(D_2))$ be a resolution of $\tau^{-1}(\eta_2(D_2))$ (cf. Sect. 2.0). Since $\tau \circ \sigma$ is onto $\eta_2(D_2)$, there exists some point $\bar{0}$ in Q with $(\tau \circ \rho)(\bar{0}) = \eta_2(0)$ and such that no neighborhood of $\bar{0}$ is mapped by $\tau \circ \sigma$ into $\eta_2(0)$. Since the set germ of $\eta_2(D_2)$ at 0 is irreducible, it follows from the proper mapping theorem, cf. [15] or [3], that every sufficiently small neighborhood U of $\bar{0}$ in Q is such that $(\tau \circ \sigma)(U)$ and $\eta_2(D_2)$ have the same set germ at $(\tau \circ \sigma)(\bar{0}) = \eta_2(0)$. Let U_1 be such a neighborhood of $\bar{0}$ which is a domain. For the purposes of the lemma we identify D_1 with U_1 (with 0 in D_1 corresponding to $\bar{0}$ in U_1), and for the $\hat{\eta}$ in the lemma we take the restriction of our σ to $D_1 = U_1$. The proof of the third assertion can be deduced from the fact that $(\tau \circ \hat{\eta})(D_1) \subseteq \eta_2(D_2)$, and that $P(\tau \circ \hat{\eta})$ and $P\eta_2$ are continuous. The details are left to the reader.

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