Extension of the Leray–Schauder degree for abstract Hammerstein type mappings

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Abstract

We introduce a new extension of the classical Leray–Schauder topological degree in a real separable reflexive Banach space. The new class of mappings for which the degree will be constructed is obtained essentially by replacing the compact perturbation by a composition of mappings of monotone type. It turns out that the class contains the Leray–Schauder type maps as a proper subclass. The new class is not convex thus preventing the free application of affine homotopies. However, there exists a large class of admissible homotopies including subclass of affine ones so that the degree can be effectively used. We shall construct the degree and prove that it is unique. We shall generalize the Borsuk theorem of the degree for odd mappings and show that the ‘principle of omitted rays’ remains valid. To illuminate the use of the new degree we shall briefly consider the solvability of abstract Hammerstein type equations and variational inequalities. © 2006 Elsevier Inc. All rights reserved.

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1. Introduction

Topological degree theory is an effective, widely used standard tool in the study of nonlinear equations. The mapping degree and the fixed-point index can be effectively used to obtain existence theorems and fixed points. Moreover, via the topological degree one obtains results concerning eigenvalues, existence of multiple solutions, continuum of the fixed points, local and global bifurcation. The mapping degree also provides a method to prove open mapping theorems and generalizations of the Jordan curve theorem. An extensive overview may be found in [32] (see also [21,26]).
The notion of the topological degree is closely connected to 0-epi maps, see [23] and the references therein. For other closely related topics and equivalent concepts, like essential mappings, A-proper mappings, rotation theory of vector fields, degree theory for equivariant vector fields, coincidence degree, index theories on manifolds, winding numbers and discussion on the validity of the Hopf’s theorem we refer to [21,26,31,32] and references therein.

The topological degree was first introduced by Brouwer in 1912 [17] for continuous functions in \( \mathbb{R}^n \). Leray and Schauder generalized in 1934 [25] the degree theory for compact perturbations of identity in infinite-dimensional Banach spaces. For mappings \( F = I + C : X \to X \) the compactness condition for \( C \) used by Leray and Schauder can be considerably relaxed, see [26,32], for instance. Since 1934 various generalizations of degree theory have been defined. We refer here to the books [20,21,26,32]. For more generalizations and further properties of the degree we refer to the articles [4–6,8,12–14,16,18,24,27–30], where also various applications can be found.

In 1972 and 1973 Führer [22] and Amann and Weiss [3] proved the Brouwer degree to be uniquely determined by only a few conditions. These conditions provide a natural basis for the formal definition of a classical topological degree. Here we use the word ‘classical’ to refer to the fact that some degree has similar basic properties as the Brouwer and Leray–Schauder degrees. The abstract formulation of classical topological degree as defined by Browder in [18] enables one to ask, whether or not a degree theory exists for a given class of mappings. It is well known that there is no relevant degree theory for continuous mappings in infinite-dimensional space, since there exist continuous mappings from the closed unit ball into itself without fixed points, see [26].

Following the terminology of [33] the abstract Hammerstein equation in a reflexive Banach space \( X \) can be written in the form

\[
    u + ST(u) = 0, \quad u \in X, \quad (1.1)
\]

where \( S : X^* \to X \) and \( T : X \to X^* \). Standard assumptions are that \( S \) and \( T \) are mappings of monotone type, for instance, monotone, accretive or pseudomonotone. In applications the formulation of a boundary value problem often gives an abstract Hammerstein equation, where the solution is in the dual space \( Y^* \) of some Banach space \( Y \) with \( S : Y \to Y^* \) and \( T : Y^* \to Y \). Typically \( S \) is a linear solution operator, which in some cases can presented as an integral operator by means of the classical Green function. This leads to Hammerstein integral equations, which are widely used and studied, see [33]. Note however, that we shall deal with reflexive spaces only and hence we can identify \( Y \) with \( Y^{**} \) by the reflexivity (it is easy to see that this procedure does not affect to the definitions of the classes of mappings). Hence we can denote \( X = Y^* \) and \( X^* = Y \) to obtain (1.1).

An example with nonlinear solution operator for a second order partial differential equation in Sobolev space \( W_{0}^{1,p}(\Omega) \) is presented in Example 2.3. Another example of abstract Hammerstein equation is provided by a formulation of certain variational inequality (Example 8.3). We shall allow both \( S \) and \( T \) to be nonlinear and assume that they satisfy some monotonicity type condition like \( (S_+) \), pseudomonotonicity or quasimonotonicity. In fact, it is easier to define a much wider abstract class which contains mappings of the form \( I + ST \) as a proper subclass. We shall then construct the degree theory for this abstract class.

The paper is organized as follows. In Sections 2 and 3 we define the classes of mappings for which the degree will be constructed. Sections 4–6 are devoted to the approximation of mappings, construction of the degree and the basic properties of the degree. In Section 7 we prove that the degree is unique. In Section 8 we prove a generalization of the Borsuk theorem and apply it.
to solve a variational inequality in Hilbert space. We close this paper by a generalization of some classical results related to Rothe’s and Altmann’s conditions and to the Leray–Schauder principle.

2. Classes of mappings

Let $X$ be a real separable reflexive Banach space with dual space $X^*$ and with continuous pairing $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We can assume without loss of generality that $X$ and its dual space $X^*$ are locally uniformly convex. Recall that for a real reflexive Banach space $X$ there exists an equivalent norm such that $X$ equipped with that norm is locally uniformly convex. As a consequence, also the induced dual norm in $X^*$ is locally uniformly convex. The norm convergence in $X$ and $X^*$ is denoted by $\to$ and the weak convergence by $\rightharpoonup$, respectively. Note that by the reflexivity the weak convergence and weak* convergence in $X^*$ coincide. For any mapping $F$ we denote its domain by $DF$ or $D$. We recall that a mapping $F$ is

- bounded, if it takes any bounded set into a bounded set;
- demicontinuous, if $u_j \to u$ in $DF$ implies $F(u_j) \to F(u)$;
- compact, if it is continuous and the image of any bounded set is relatively compact;
- of Leray–Schauder type, if $F : D_F \subset X \to X$ is of the form $I + C$, where $C$ is compact;
- monotone, if $F : D_F \subset X \to X^*$ satisfies $\langle F(u) - F(v), u - v \rangle \geq 0$ for all $u, v \in D_F$;
- strictly monotone if $F : D_F \subset X \to X^*$ satisfies $\langle F(u) - F(v), u - v \rangle > 0$ for all $u, v \in D_F$, $u \neq v$;
- strongly monotone if $F : D_F \subset X \to X^*$ and there exists a continuous strictly increasing function $g : [0, \infty[ \to [0, \infty[$ with $g(0) = 0$ such that $\langle F(u) - F(v), u - v \rangle \geq g(\|u - v\|) \times \|u - v\|$ for all $u, v \in D_F$. Typically $g(s) = s$.

Mappings of monotone type, e.g. pseudomonotone mappings and mappings of class $(S_+)$, are widely used in the study of partial differential operators of generalized divergence form. These classes were introduced in 1970s by Minty, Skrypnik, Browder, Brezis, Hess and others. The use of mappings of monotone type allows one to treat problems with certain lack of monotonicity and compactness. The basic definitions are the following. A mapping $F : D_F \subset X \to X^*$ is

- of class $(S_+)$, if for any $(u_k) \subset D_F$ with $u_k \rightharpoonup u$, $\limsup \langle F(u_k), u_k - u \rangle \leq 0$, it follows that $u_k \to u$;
- pseudomonotone, if for any $(u_k) \subset D_F$, $u_k \rightharpoonup u$, $\limsup \langle F(u_k), u_k - u \rangle \leq 0$, it follows that $\langle F(u_k), u_k - u \rangle \to 0$ and if $u \in D_F$, then $F(u_k) \rightharpoonup F(u)$;
- quasimonotone, if for any $(u_k) \subset D_F$ with $u_k \rightharpoonup u$, it follows that $\limsup \langle F(u_k), u_k - u \rangle \geq 0$.

In case $D_F$ is weakly closed, we can replace the expression $\langle F(u_k), u_k - u \rangle$ by $\langle F(u_k) - F(u), u_k - u \rangle$ in the definitions above. With this convention one can see how the definitions of classes of monotone type indicate the deviation from the monotonicity. It is easy to see that any demicontinuous monotone map is pseudomonotone and any strongly monotone map is of class $(S_+)$. Moreover, any demicontinuous map of class $(S_+)$ is pseudomonotone and any pseudomonotone map is quasimonotone. Note also that any compact map is quasimonotone and for quasimonotone mapping $\liminf \langle F(u_k), u_k - u \rangle \geq 0$ for any sequence $(u_k) \subset D_F$ with $u_k \rightharpoonup u$. 
The class \((S_+)\) is stable under quasimonotone perturbations, thus for instance pseudomonotone or compact perturbations are allowed. In Hilbert space any Leray–Schauder type map \(I + C\) is of class \((S_+)\), since \(I\) is of class \((S_+)\) and \(C\) is compact thus being quasimonotone.

Due to the reflexivity we can identify \(X^{**}\) with \(X\). To simplify the notation we denote \((y, x) = (x, y)\) for all \(x \in X\) and \(y \in X^*\) and hope that the reader will always be aware of in which space his/her vectors are. Hence it is relevant to define that \(F : D_F \subset X^* \to X\) is of class \((S_+)\), if for any sequence \((w_k) \subset D_F\) with \(w_k \rightharpoonup w\) in \(X^*\) and \(\limsup \langle F(w_k), w_k - w \rangle \leq 0\), it follows that \(w_k \to w\). Similar modification of definitions is relevant for monotone, strictly monotone, strongly monotone, pseudomonotone and quasimonotone mappings.

The basic abstract example of mappings of monotone type is the duality map from a real reflexive Banach space into its dual space. The construction of the duality map shows that mappings of monotone type arise naturally from the structure of the Banach space. The duality map \(J : X \to X^*\) is determined via the Hahn–Banach theorem by the conditions

\[ \|J(u)\| = \|u\|, \quad \langle J(u), u \rangle = \|u\|^2 \quad \text{for all } u \in X. \quad (2.1) \]

By the strict convexity of \(X^*\) the map \(J\) is single valued. The duality map has many important properties listed in the following lemma (see [21,33]).

**Lemma 2.1.** Let \(X\) be a real reflexive Banach space such that \(X\) and \(X^*\) are locally uniformly convex. Then the duality map \(J\) defined by (2.1) satisfies:

1. \(J\) is strictly monotone,
2. \(J\) is of class \((S_+)\),
3. \(J\) is homogeneous,
4. \(J\) is homeomorphism,
5. \(J^{-1}\) is the duality map from \(X^*\) to \(X \cong X^{**}\), and thus of class \((S_+)\),
6. \(J\) is linear if and only if \(X\) is a Hilbert space.

We define the following classes of mappings:

\[ \mathcal{F}_0(D) = \{ F : D \subset X \to X \mid F = I + C, \text{ where } C \text{ is compact} \} \]

and

\[ \mathcal{F}_1(D) = \{ F : D \subset X \to X^* \mid F \text{ is bounded, demicontinuous and of class } (S_+) \}. \]

Let \(\mathcal{O}\) be the collection of all open bounded sets in \(X\). Define

\[ \mathcal{F}_{LS}(X) = \bigcup_{G \in \mathcal{O}} \mathcal{F}_0(\bar{G}) \quad \text{and} \quad \mathcal{F}_{S+}(X) = \bigcup_{G \in \mathcal{O}} \mathcal{F}_1(\bar{G}). \]

For any \(T \in \mathcal{F}_1(D_T)\) and for any \(F : D_F \subset X \to X\) such that \(D_F \subset D_T\) we say that \(F\) satisfies condition \((S_+)T\), if for any sequences \((u_k) \subset D_F\), \((y_k) = (T(u_k)) \subset X^*\) with

\[ u_k \rightharpoonup u, \quad y_k \rightharpoonup y \quad \text{and} \quad \limsup \langle F(u_k), y_k - y \rangle \leq 0, \quad \text{it follows that } u_k \to u. \]
For any $T \in \mathcal{F}_1(D_T)$ and for any $D \subset D_T$ we define

$$\mathcal{F}_T(D) = \{ F : D \subset X \to X \mid F \text{ is bounded demicontinuous and satisfies condition } (S_+)_T \}.$$  

If $T \in \mathcal{F}_1(D_T)$ and $\bar{G} \subset D_T$, we can always replace $T$ by its restriction $T|_{\bar{G}}$ and write $T \in \mathcal{F}_1(\bar{G})$. Hence we finally define our main class

$$\mathcal{F}_2(X) = \{ F : \bar{G} \to X \mid F \in \mathcal{F}_T(\bar{G}) \text{ for some continuous } T \in \mathcal{F}_1(\bar{G}), G \in \mathcal{O} \}.$$  

The continuity of $T$ is not necessary for the construction of the degree but needed to obtain the uniqueness. We shall define a topological degree for the class $\mathcal{F}_2(X)$. For any $F \in \mathcal{F}_T(\bar{G})$ we call $T \in \mathcal{F}_1(\bar{G})$ the essential inner mapping of $F$. For a given $F$ the essential inner mapping is not unique. It should be noted that if $T : X \to X^*$ linear and bounded, then $F \in \mathcal{F}_T(X)$ means that $F$ is of class $(S_+)$ with respect to $T$, i.e., conditions

$$u_k \to u, \quad \limsup(F(u_k), T(u_k - u)) \leq 0$$

imply $u_k \to u$. Such mappings in Hilbert space are studied and used in [8,9]. One of the main motivations of this article is that certain abstract Hammerstein operators can be included into $\mathcal{F}_2(X)$. To be more precise, we have the following result. For brevity we shall frequently use the notation $ST$ for $S \circ T$.

**Lemma 2.2.** Let $T \in \mathcal{F}_1(\bar{G})$, $G \in \mathcal{O}$, be continuous and $S : D_S \subset X^* \to X$ a bounded demicontinuous quasimonotone mapping such that $T(\bar{G}) \subset D_S$. Then $F := I + ST \in \mathcal{F}_T(\bar{G})$.

**Proof.** We have to verify that condition $(S_+)_T$ holds. Assume that $(u_k) \subset \bar{G}$, $(y_k) = (T(u_k)) \subset X^*$ with

$$u_k \to u, \quad y_k \to y \quad \text{and} \quad \limsup(F(u_k), y_k - y) \leq 0.$$  

To prove that $u_k \to u$ we assume the contrary. Then we can find a subsequence $(u_j) \subset (u_k)$ and $\epsilon > 0$ such that always $\|u_j - u\| \geq \epsilon$. Moreover, we can assume that for this subsequence

$$\lim(u_j, y_j - y) \equiv \lim(T(u_j), u_j - u) = q > 0.$$  

Hence

$$\limsup(S(y_j), y_j - y) = \limsup(u_j + S(T(u_j)), y_j - y) - q \leq -q < 0,$$

which is a contradiction, since $S$ is quasimonotone. \qed

As an example of an abstract Hammerstein operator $I + ST \in \mathcal{F}_2(X)$ we consider a boundary value problem with nonlinear solution operator. We shall use some well-known facts concerning divergence operators, for details see [1,11,15].
Example 2.3. We consider a second order divergence operator. Let
\[ A_p u(x) = - \sum_{i=1}^{N} D_i \left( |D_i u(x)|^{p-2} D_i u(x) \right), \quad x \in \Omega, \]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary and \( N > p \geq 2 \). Note that \( A_2 \) is the linear Laplacian. For \( p \neq 2 \) the divergence operator \( A_p \) is nonlinear but positively homogeneous of order \( p - 1 \) and odd. We consider the equation
\[
\begin{cases}
- \sum_{i=1}^{N} D_i \left( |D_i u(x)|^{p-2} D_i u(x) \right) = \lambda u(x) + f(x, u, \nabla u) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(2.2)
where \( \lambda \in \mathbb{R} \) and \( f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) satisfies the Carathéodory condition, i.e., \( f(x, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is continuous for almost all \( x \in \Omega \) and \( f(\cdot, \eta, \xi) : \Omega \rightarrow \mathbb{R} \) is measurable for all \( (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N \). Moreover, we assume the growth condition
\[ |f(x, \eta, \xi)| \leq c (|\eta|^{p-1} + |\xi|^{p-1} + k(x)), \]
for almost all \( x \in \Omega \), all \( \xi = (\eta, \zeta) \in \mathbb{R}^{N+1} \), where \( c > 0 \) is constant and \( k \in L^{p'}(\Omega) \). Denote \( Y = W^{1,p}_0(\Omega) \). Then \( u \in Y \) is a weak solution of Eq. (2.2), if
\[
\sum_{i=1}^{N} \int_{\Omega} |D_i u(x)|^{p-2} D_i u(x) D_i \psi(x) \, dx = \int_{\Omega} \left( \lambda u(x) + f(x, u(x), \nabla u(x)) \right) \psi(x) \, dx
\]
for all \( \psi \in Y \). Define the mappings \( F_p : Y \rightarrow Y^* \) and \( S_\lambda : Y \rightarrow Y^* \) by
\[
\langle F_p(u), v \rangle = \sum_{i=1}^{N} \int_{\Omega} |D_i u(x)|^{p-2} D_i u(x) D_i v(x) \, dx, \quad u, v \in X,
\]
and
\[
\langle S_\lambda(u), v \rangle = - \int_{\Omega} \left( \lambda u(x) + f(x, u(x), \nabla u(x)) \right) v(x) \, dx, \quad u, v \in X.
\]
Then \( u \in Y \) is a weak solution of Eq. (2.2) if and only if
\[ F_p(u) = - S_\lambda(u). \]
It is well known that \( F_p \) and \( S_\lambda \) are bounded continuous mappings from \( Y \) into \( Y^* \). Moreover, due to the compact embedding \( Y = W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega) \) one can show that \( F_p : Y \rightarrow Y^* \) is of class \( (S_+^\lambda) \) and \( S_\lambda : Y \rightarrow Y^* \) is quasimonotone for any \( \lambda \in \mathbb{R} \). The equation
\[
\begin{cases}
- \sum_{i=1}^{N} D_i \left( |D_i u(x)|^{p-2} D_i u(x) \right) = v(x) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
admits a unique weak solution for each \( v \in Y^* \). Hence we define the nonlinear solution operator \( T_p : Y^* \to Y \) by setting \( T_p(v) = u \) if and only if \( v = F_p(u) \). It is not hard to see that the solution operator \( T_p : Y^* \to Y \) is bounded continuous and of class \((S_+)^*\) (see Lemma 5.2). Consequently, \( u \in Y \) is a weak solution of Eq. (2.2) if and only if \( u = T_p(v) \) and

\[
v + S_\lambda T_p(v) = 0.
\]

Let now \( X = Y^* \) and identify \( X^* \) with \( Y \) by the reflexivity. Then by the above considerations \( I + S_\lambda T_p \in \mathcal{F}_2(X) \) for any \( \lambda \in \mathbb{R} \) and \( 2 \leq p < N \).

The solvability of (2.3) can be proved for instance in the following case. Assume that \( p > 2 \), \( \lambda \in \mathbb{R} \) is fixed and \( f \) satisfies the growth condition

\[
|f(x, \eta, \zeta)| \leq c(|\eta|^q-1 + |\zeta|^q-1 + k(x)),
\]

where \( c > 0 \) is constant, \( k \in L^{p'}(\Omega) \) and \( q < p \). Due to the Poincaré inequality we can use the norm

\[
\|u\| = \left( \sum_{i=1}^{N} \|D_i u\|_p^p \right)^{1/p}, \quad u \in W_0^{1,p}(\Omega),
\]

where \( \| \cdot \|_p \) stands for the \( L^p \)-norm. According to Theorem 9.2 it suffices to show that the solution set of the homotopy equation

\[
v + tS_\lambda T_p(v) = 0, \quad v \in X, \ 0 \leq t \leq 1,
\]

is bounded. Indeed, assume that \( v + tS_\lambda T_p(v) = 0 \) for some \( v \in X \) and \( 0 \leq t \leq 1 \). Denote \( u = T_p(v) \). By the growth condition one get easily the estimate

\[
\langle T_p(v), v \rangle = \|T_p(v)\|^p = -t[S_\lambda(T_p(v)), T_p(v)]
\]

\[
= t \int_\Omega (\lambda u(x) + f(x, u(x), \nabla u(x))) u(x) \, dx
\]

\[
\leq c_1(\|T_p(v)\|_p^2 + \|T_p(v)\|^q + \|T_p(v)\|^p),
\]

where \( c_1 > 0 \) is a constant. Hence it is easy to see that \( T_p(v) \) remains bounded and consequently, there exists \( R > 0 \) such that \( \|v\| < R \).

The following results follow directly from the definitions.

**Lemma 2.4.** Let \( T \in \mathcal{F}_1(\mathcal{G}) \), \( G \in \mathcal{O} \), be continuous. Then

1. \( \mathcal{F}_0(\mathcal{G}) \subset \mathcal{F}_T(\mathcal{G}) \),
2. \( ST \in \mathcal{F}_T(\mathcal{G}) \) for any \( S : X^* \to X \), which is bounded demicontinuous and of class \((S_+)^*\),
3. \( \mathcal{F}_{T+C}(\mathcal{G}) = \mathcal{F}_T(\mathcal{G}) \) for any compact \( C : \mathcal{G} \to X^* \).
By Lemma 2.4(1) the Leray–Schauder type maps belong to \( F_2(X) \). The structure of the classes \( F_T(X) \) and \( F_2(X) \) will be analyzed more closely in Section 3. Note that any \( F \) satisfying condition \((S_+)_T\) is proper on bounded subsets, that is, for any compact set \( K \subset X \) and any bounded closed set \( A \subset X \) the preimage \( F^{-1}(K) \cap A \) is compact. It seems that a necessary condition for the existence of a degree theory is that the mappings involved are proper on bounded subsets.

Crucial for the effective use of any degree theory is the class of admissible homotopies. Let \( G \in \mathcal{O} \) be given. Recall that an admissible Leray–Schauder type homotopy in \( X \) is of the form \( H = I + C \), where \( C : [0,1] \times \overline{G} \to X \) is compact. An admissible \((S_+)-\)homotopy from \( G \) into \( X^* \) is a demicontinuous bounded mapping \( H : [0,1] \times \overline{G} \to X^* \) satisfying the condition \((S_+):\) for any sequences \((u_k) \subset \overline{G} \) with \( u_k \rightharpoonup u \) and \((t_k) \subset [0,1], t_k \to t\), such that \( \lim \sup \langle H(t_k, u_k), y_k - y \rangle \leq 0 \), it follows that \( u_k \to u \).

We start with a limited class of affine homotopies

\[
\mathcal{H}_{aff} = \{ H : [0,1] \times \overline{G} \to X \mid H(t, \cdot) = (1-t)F + tS, \ 0 \leq t \leq 1, \ F, S \in F_T(G) \text{ with continuous } T \in F_1(G), \ G \in \mathcal{O} \},
\]

where \( F \) and \( S \) have a common essential inner mapping. The class of admissible homotopies can be extended considerably to cover variations in \( T \), as we will see in Section 9 (see also Theorem 6.1). However, \( \mathcal{H}_{aff} \) is sufficiently large to make the construction of the unique topological degree possible. By definition it is easy to prove the following property, which will be the starting point of the general definition of the class of admissible homotopies in Section 9.

**Lemma 2.5.** Let \( H \in \mathcal{H}_{aff} \) with the continuous essential inner mapping \( T \in F_1(\overline{G}) \). Assume that \( t_k \to t \in [0,1], \ (u_k) \subset \overline{G} \) with \( u_k \rightharpoonup u \) in \( X \) and \( y_k = T(u_k) \rightharpoonup y \) in \( X^* \) are sequences such that

\[
\lim \sup \langle H(t_k, u_k), y_k - y \rangle \leq 0.
\]

Then \( u_k \to u \).

From now on we call any mapping \( F : \overline{G} \to X \ F_2\)-admissible, if \( F \in F_2(X) \). Similarly, a homotopy \( H : [0,1] \times \overline{G} \to X \) is \( \mathcal{H}_2\)-admissible, provided \( H \in \mathcal{H}_{aff} \). Note that by Lemma 2.5 (taking \( t_k \equiv t \)) for any \( H \in \mathcal{H}_{aff} \) with \( t \) fixed we have \( H(t, \cdot) \in F_2(X) \). The existence and the properties of the Leray–Schauder degree and the degree theory for mappings of class \( (S_+)_T \) are well known, see [18] for instance. Denoting by \( \mathcal{H}_{LS}(X) \) and \( \mathcal{H}_{S+}(X) \) the admissible Leray–Schauder and \( (S_+)-\)homotopies, respectively, we can state

**Proposition 2.6.**

(a) \((\text{Leray–Schauder degree})\) There exists a unique degree function \( d_0 \) for classes \( F_{LS}(X) \) and \( \mathcal{H}_{LS}(X) \) normalized by \( I \).

(b) \((\text{}(S_+)-\text{degree})\) There exists a unique degree function \( d_1 \) for classes \( F_{S+}(X) \) and \( \mathcal{H}_{S+}(X) \) normalized by \( J \).
3. On the structure of the classes

The properties of classes \( \mathcal{F}_{LS}(X) \) and \( \mathcal{F}_{S+}(X) \) are well known. Any subclass \( \mathcal{F}_0(\overline{G}) \subset \mathcal{F}_{LS}(X) \), \( G \in \mathcal{O} \), of Leray–Schauder maps is invariant under compact perturbations and it is convex. Any subclass \( \mathcal{F}_1(\overline{G}) \subset \mathcal{F}_{S+}(X) \) of mappings of class \((S_+)\) has conical structure in the sense that for all \( \alpha > 0 \), \( F_0, F_1 \in \mathcal{F}_1(\overline{G}) \) we have \( \alpha F_0 \in \mathcal{F}_1(\overline{G}) \) and \( F_0 + F_1 \in \mathcal{F}_1(\overline{G}) \). Hence \( \mathcal{F}_1(\overline{G}) \) is convex. Moreover, the class \( \mathcal{F}_1(\overline{G}) \) is invariant under bounded demicontinuous quasi-monotone perturbations. Recall that any compact mapping \( C : \overline{G} \to X^* \) is quasimonotone. The convexity of classes \( \mathcal{F}_0(\overline{G}) \) and \( \mathcal{F}_1(\overline{G}) \) is one of the features, which allow one to construct an effective degree with affine homotopies of the form

\[
H_a(t, \cdot) = (1-t)F_0 + tF_1, \quad 0 \leq t \leq 1,
\]

where \( F_0 \) and \( F_1 \) are in the same class. Indeed, if \( F_0, F_1 \in \mathcal{F}_0(\overline{G}) \), then \( H_a \in \mathcal{H}_{LS}(X) \) and if \( F_0, F_1 \in \mathcal{F}_1(\overline{G}) \), then \( H_a \in \mathcal{H}_{S+}(X) \). Unfortunately, all affine homotopies are not available in class \( \mathcal{F}_2(X) \), as we will see.

Consider first the subclass \( \mathcal{F}_T(\overline{G}) \subset \mathcal{F}_2(X) \), \( G \in \mathcal{O} \), of mappings with common essential inner mapping \( T \in \mathcal{F}_1(\overline{G}) \). It is easy to see that \( \mathcal{F}_T(\overline{G}) \) has conical structure, i.e., for all \( \alpha > 0 \), \( F_0, F_1 \in \mathcal{F}_T(\overline{G}) \) we have \( \alpha F_0 \in \mathcal{F}_T(\overline{G}) \) and \( F_0 + F_1 \in \mathcal{F}_T(\overline{G}) \). Especially \( \mathcal{F}_T(\overline{G}) \) is convex. To deal with the perturbations it is convenient to define the property \((QM)_T\) for a mapping \( F : \overline{G} \to X \) by the condition: if \( (u_k) \subset \overline{G}, (y_k) = (T(u_k)) \subset X^* \) with \( u_k \to u, y_k \to y \), then \( \lim\sup(F(u_k), y_k - y) \geq 0 \). Clearly any compact map \( C : \overline{G} \to X \) has the property \((QM)_T\).

The ‘quasimonotone class’ of bounded demicontinuous mappings having property \((QM)_T\) has a conical structure and it is in a sense optimal class of perturbations according to the following lemma.

**Lemma 3.1.** Let \( T \in \mathcal{F}_1(\overline{G}), G \in \mathcal{O} \), be continuous.

1. If \( F \in \mathcal{F}_T(\overline{G}) \), then \( F \) has the property \((QM)_T\).
2. Let \( S : \overline{G} \to X \) be a bounded demicontinuous map. Then \( S \) has the property \((QM)_T\) if and only if \( F + S \in \mathcal{F}_T(\overline{G}) \) for all \( F \in \mathcal{F}_T(\overline{G}) \).

**Example 3.2.** Let \( T \in \mathcal{F}_1(\overline{G}), G \in \mathcal{O} \), be continuous and \( F_i : X^* \to X, i = 1, 2, \ldots, l \), be bounded demicontinuous and quasimonotone. Consider a bounded demicontinuous mapping \( F : \overline{G} \to X \) of the form

\[
F = \alpha I + C + \sum_{i=1}^l F_i \circ (T + C_i),
\]

where \( \alpha \geq 0 \), and \( C_i : \overline{G} \to X^*, C : \overline{G} \to X \) are compact, \( i = 1, 2, \ldots, l \). In case \( \alpha > 0 \), we have \( F \in \mathcal{F}_T(\overline{G}) \). If \( \alpha = 0 \), then \( F \in \mathcal{F}_T(\overline{G}) \) if in addition at least one \( F_i \), say \( F_1 \), is of class \((S_+)\).

If \( X \) is an Hilbert space and we identify \( X^* \) with \( X \), then we have \( \mathcal{F}_{LS}(X) \subset \mathcal{F}_{S+}(X) \) and any continuous \( F \in \mathcal{F}_{S+}(X) \) belongs to \( \mathcal{F}_2(X) \). In the general case we have \( J \circ F \in \mathcal{F}_{S+}(X) \) for all \( F \in \mathcal{F}_{LS}(X) \) and for any continuous \( F \in \mathcal{F}_1(\overline{G}) \) we have \( J^{-1} \circ F \in \mathcal{F}_F(\overline{G}) \). Thus \( \mathcal{F}_{LS}(X) \) can be embedded into \( \mathcal{F}_{S+}(X) \) and continuous mappings of \( \mathcal{F}_{S+}(X) \) into \( \mathcal{F}_2(X) \) via the duality
mapping. We will see in Section 6 (Theorem 6.3) that in view of degree theory this kind of inclusion is relevant.

The following example shows that \( \mathcal{F}_2(X) \) is ‘larger’ than \( \mathcal{F}_{S+}(X) \) and \( \mathcal{F}_{T_1}(\overline{G}) \cup \mathcal{F}_{T_2}(\overline{G}) \) is not always convex thus preventing the free use of affine homotopies in \( \mathcal{F}_2(X) \).

**Example 3.3.** Let \( H \) be a real infinite-dimensional Hilbert space with orthonormal basis \( \{e_n\}_{n=1}^{\infty} \). We identify \( H^* \) with \( H \) and replace the pairing by the inner product \( \langle \cdot, \cdot \rangle_H \). Define the linear bounded operators \( F: H \to H \) and \( T: H \to H \) by setting

\[
F(e_n) = e_n + (-1)^{n+1} e_{n+(-1)^n+1}, \quad n = 1, 2, 3, \ldots,
\]

and

\[
T(e_n) = e_n + (-1)^n e_{n+(-1)^n+1}, \quad n = 1, 2, 3, \ldots.
\]

By a direct calculation \( \langle F(u), u \rangle_H = \langle T(u), u \rangle_H = \|u\|^2_H \) for all \( u \in H \), where \( \|u\|^2_H = \langle u, u \rangle_H \). Consequently, both \( F \) and \( T \) are strongly monotone and thus of class \( (S_+) \). It is easy to see that \( FT \) and \( TF \) are of class \( (S_+) \). However, \( FF \) and \( TT \) do not satisfy condition \( (S_+) \). This can be seen by considering a weakly convergent sequence \( (u_k) \), where \( u_k = e_{2k} \). Then \( u_k \rightharpoonup 0 \),

\[
\limsup_{n \to \infty} \langle F(F(u_k)), u_k - 0 \rangle_H = 0,
\]

but \( u_k \to 0 \). The same counterexample applies for \( TT \). Hence \( FF \in \mathcal{F}_F(\overline{G}) \subset \mathcal{F}_2(H) \) for any \( G \in \mathcal{O} \) but \( FF \notin \mathcal{F}_{S+}(H) \) with a similar conclusion for \( TT \). Moreover, \( (1 - \frac{1}{2}) FF + \frac{1}{2} TT \equiv 0 \), from which we conclude that \( \mathcal{F}_F(\overline{G}) \cup \mathcal{F}_T(\overline{G}) \) is not convex, since the zero-map is not proper and thus \( 0 \notin \mathcal{F}_2(H) \). Thus the affine homotopy between \( FF \in \mathcal{F}_2(H) \) and \( TT \in \mathcal{F}_2(H) \) cannot be \( \mathcal{H}_2 \)-admissible.

4. Approximation of mappings

First we recall the following embedding theorem due to Browder and Ton [19], see also [7].

**Theorem 4.1.** Let \( Y \) be a real separable Banach space. Then there exist a real separable Hilbert space \( W \) and a compact linear injection \( \phi: W \to Y \) such that \( \phi(W) \) is dense in \( Y \).

Let \( X \) be a real separable reflexive Banach space with dual space \( X^* \) such that \( X \) and its dual space \( X^* \) are locally uniformly convex. Taking \( Y = X^* \) in Theorem 4.1 we conclude the existence of a real separable Hilbert space \( W \) and a compact linear injection \( \phi: W \to X^* \) such that \( \phi(W) \) is dense in \( X^* \). We define a further ‘adjoint’ mapping \( \hat{\phi}: X \to W \) by setting

\[
(\hat{\phi}(u)|v)_W = \langle u, \phi(v) \rangle \quad \text{for all } v \in W \text{ and } u \in X,
\]

where \( (\cdot|\cdot)_W \) stands for the inner product of \( W \). Then \( \hat{\phi}: X \to W \) is compact linear mapping. Since \( \phi(W) \) is dense in \( X^* \), it is easily seen that also \( \hat{\phi} \) is injective.

We shall approximate any \( F \in \mathcal{F}_2(X) \) by a mapping in \( \mathcal{F}_{S+}(X) \). Let \( F \in \mathcal{F}_T(\overline{G}) \) with some \( G \in \mathcal{O} \) and \( T \in \mathcal{F}_1(\overline{G}) \). We associate to \( F \) a family of mappings \( \{F_\lambda \mid \lambda > 0\} \) defined by

\[
F_\lambda = T + \lambda \phi \hat{\phi} F \quad \text{for any } \lambda > 0.
\]
Clearly $F_\lambda$ maps $\tilde{G}$ into $X^*$ and $F_\lambda \in \mathcal{F}_1(\tilde{G}) \subset \mathcal{F}_{S^+}(X)$.

Note that $F_\lambda$ depends explicitly on $T \in \mathcal{F}_{S^+}(X)$, which is not unique and not, a priori, part of $F$. Instead, $T$ is a crucial part of the monotonicity condition defining the class $\mathcal{F}_T(\tilde{G})$. The approximation given above is a variant of so-called 'elliptic super-regularization' used in [4,10]. It can be viewed as an abstract version of the method of elliptic regularization used by Lions, Nirenberg and others; see [19] and the references therein.

The basic result in view of our construction is the following.

**Lemma 4.2.** Let $F \in \mathcal{F}_T(\tilde{G}) \subset \mathcal{F}_2(X)$, $G \in \mathcal{O}$, and assume that $0 \notin F(A)$, where $A \subset \tilde{G}$ is closed. Then there exists $\lambda_0 > 0$ such that $0 \notin F_\lambda(A)$ for all $\lambda > \lambda_0$.

**Proof.** We shall argue by contradiction. Assume that the assertion is false. Then we can find sequences $(\lambda_k)$ and $(u_k) \subset A$ such that $\lambda_k \to \infty$ and $F_\lambda(u_k) = 0$ for all $k \in \mathbb{Z}_+$. At least for subsequences we can write

$$u_k \rightharpoonup u \in X, \quad F(u_k) \rightharpoonup w \in X, \quad y_k := T(u_k) \rightharpoonup y \in X^*.$$  

Clearly $\hat{\phi}_k F(u_k) = -\lambda_k^{-1} T(u_k) \to 0 = \hat{\phi}_k w$ and hence $w = 0$. Thus

$$\limsup \langle F(u_k), y_k - y \rangle = \limsup \langle F(u_k), y_k \rangle = \limsup \langle F(u_k), -\lambda_k \hat{\phi} F(u_k) \rangle = \limsup \left\{ -\lambda_k \left\| \hat{\phi} F(u_k) \right\|_W^2 \right\} \leq 0,$$

where $\| \cdot \|_W$ stands for the norm of $W$. Consequently $u_k \rightharpoonup u \in A$ implying $F(u_k) \rightharpoonup F(u) = 0$; a contradiction completing the proof. □

To prove the homotopy invariance property, we need the following generalization of Lemma 4.2 with $A = \partial G$, which can proved analogously. We consider any affine homotopy $H \in \mathcal{H}_{aff}$, $H(t, \cdot) = (1 - t) F + t S$, where $F, S \in \mathcal{F}_T(\tilde{G})$. We denote

$$H_\lambda(t, \cdot) = T + \lambda \hat{\phi} H(t, \cdot), \quad \lambda > 0.$$  

Then $H_\lambda(t, \cdot)$ defines a bounded homotopy of class $(S_+)$ for any fixed $\lambda > 0$.

**Lemma 4.3.** Let $H \in \mathcal{H}_{aff}$ and assume that $0 \notin H(t, \partial G)$ for all $0 \leq t \leq 1$, where $G \in \mathcal{O}$. Then there exists $\lambda_0 > 0$ such that $0 \notin H_\lambda(t, \partial G)$ for all $0 \leq t \leq 1$ and $\lambda > \lambda_0$.

**5. Protodegree for $\mathcal{F}_2(X)$**

Let $F \in \mathcal{F}_T(\tilde{G}) \subset \mathcal{F}_2(X)$ and $F_\lambda \in \mathcal{F}_1(\tilde{G}) \subset \mathcal{F}_{S^+}(X)$ be as defined in Section 4. The $(S_+)$-degree $d_1(F_\lambda, G, 0)$ is well defined as soon as $0 \notin F_\lambda(\partial G)$. We shall apply Lemma 4.2 with $A = \partial G$.

**Lemma 5.1.** Let $F \in \mathcal{F}_T(\tilde{G})$ with $G \in \mathcal{O}$ such that $0 \notin F(\partial G)$. Then there exists $\lambda_0 > 0$ such that $0 \notin F_\lambda(\partial G)$ and the value of $d_1(F_\lambda, G, 0)$ is constant for all $\lambda > \lambda_0$. 
Proof. By Lemma 4.2 there exists $\lambda_0 > 0$ such that $0 \notin F_\lambda(\partial G)$ for all $\lambda > \lambda_0$. As a consequence, the $(S_+)$-degree $d_1(F_\lambda, G, 0)$ is well defined for all $\lambda > \lambda_0$. Let $\lambda_2 > \lambda_1 > \lambda_0$ be fixed. Then $F_{\lambda_2}, \lambda_1 \leq \lambda \leq \lambda_2$, defines an admissible $(S_+)$-homotopy and $0 \notin F_\lambda(\partial G)$ for all $\lambda_1 \leq \lambda \leq \lambda_2$. Hence $d_1(F_{\lambda_1}, G, 0) = d_1(F_{\lambda_2}, G, 0)$. Consequently, $d_1(F_\lambda, G, 0)$ remains constant for all $\lambda > \lambda_0$ completing the proof. □

Since $F_\lambda$ depend on $F$ as well as on the essential inner mapping $T$, it is relevant by the previous lemma to define a ‘protodegree’ $D_2$ by setting

$$D_2(F, T, G, 0) = \lim_{\lambda \to \infty} d_1(F_\lambda, G, 0)$$

whenever $0 \notin F(\partial G)$. For any $h \notin F(\partial G)$ we set

$$D_2(F, T, G, h) = D_2(F - h, T, G, 0).$$

The dependence on $T$ is explicitly shown in our notation above.

First we shall verify that the integer-valued function $D_2$ defined for classes $\mathcal{F}_2(X)$ and $\mathcal{H}_{\text{aff}}$ satisfies the properties of the classical topological degree:

(a) If $D_2(F, T, G, h) \neq 0$, then there exists a solution for the equation $F(u) = h$ in $G$.

(b) (Additivity) Let $G_1$ and $G_2$ be a pair of disjoint open subsets of $G$ such that $h \notin F(G \setminus (G_1 \cup G_2))$. Then

$$D_2(F, T, G, h) = D_2(F, T, G_1, h) + D_2(F, T, G_2, h).$$

(c) (Invariance under homotopies) If $H \in \mathcal{H}_{\text{aff}}$ with the essential inner mapping $T \in \mathcal{F}_1(\overline{G})$ and $\{h_t: 0 \leq t \leq 1\}$ is a continuous curve in $X$ such that $h_t \notin H(t, \partial G)$ for all $t \in [0, 1]$, then the value $D_2(H(t, \cdot), T, G, h_t)$ is constant in $t$ on $[0, 1]$.

(d) (Normalizing map) The identity map $I = J^{-1} \circ J \in \mathcal{F}_J(\overline{G})$ for any $G \in \mathcal{O}$. For any $h \in G$ we have $D_2(I, J, G, h) = +1$.

Proof of (a), (b) and (c): Using the known properties of the $(S_+)$-degree $d_1$ the conclusions of (a) and (b) follow from Lemma 4.2 by taking $A = \overline{G}$, $A = \overline{G \setminus (G_1 \cup G_2)}$, respectively. The homotopy invariance property (c) is easily proved by using Lemma 4.3 together with the corresponding property of $d_1$.

We shall indicate in Section 9 how the property (c), invariance under homotopies, can be extended to cover variations in $T$.

Proof of (d): Assume that $h \in G$. Take some $R > \|h\|$. By the additivity property (b) we get

$$D_2(I, J, G, h) = D_2(I, J, B_R(0), h).$$

Since $th \notin I(\partial B_R(0))$ for all $0 \leq t \leq 1$, we conclude by (c) that

$$D_2(I, J, B_R(0), h) = D_2(I, J, B_R(0), 0).$$
By the definition of $D_2$ together with the homotopy invariance property of $d_1$ we finally obtain for some fixed $\lambda' > 0$

$$D_2(I, J, B_R(0), 0) = d_1(J + \lambda' \phi \hat{\phi} I, B_R(0), 0) = d_1(J, B_R(0), 0) = +1,$$

where we have used the $(S_+)$-homotopy $S_t = J + t \lambda' \phi \hat{\phi} I$, $0 \leq t \leq 1$, which has no zeros on the boundary of $B_R(0)$ due to the fact that $\langle J(u) + t \lambda' \phi \hat{\phi} u, u \rangle \geq \|u\|^2$ for all $u \in X$ and $0 \leq t \leq 1$.

At first sight one can get the impression that we have constructed for each fixed essential inner mapping $T$ a separate degree function $d_T(\cdot, \cdot, \cdot, \cdot) := D(\cdot, T, \cdot, \cdot, \cdot)$ for the class $\bigcup_{G \in \mathcal{O}} \mathcal{F}_T(G)$. Note however, that the normalizing condition (d) is not necessarily valid in case $T$ is fixed. Since the properties from (a) to (c) of the topological degree hold with fixed $T$, it is clear that $d_T$ is a degree function with $I$ as a normalizing map provided $I \in \mathcal{F}_T(G)$ for all $G \in \mathcal{O}$. This is true in the following case.

**Lemma 5.2.** Assume that $T : X \to X^*$ is bounded, continuous mapping of class $(S_+)$, which is bijective and satisfies the condition

$$\|T(u)\| \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Then $T^{-1} : X^* \to X$ is bounded, continuous and of class $(S_+)$. Hence $I = T^{-1} T \in \mathcal{F}_T(G)$ for all $G \in \mathcal{O}$ and $d_T$ is a classical degree function.

If $X = H$ is a real separable Hilbert space and $T : H \to H$ linear, then the degree $d_T$ is a special case of the degree introduced in [8].

6. Degree for $\mathcal{F}_2(X)$

Let $G \in \mathcal{O}$ and $F \in \mathcal{F}_T(G) \subset \mathcal{F}_2(X)$ with $T \in \mathcal{F}_1(G)$. The next result shows that the value of $D_2(F, T, G, h)$ depends only on the values of $F$ on the boundary $\partial G$. This is a very characteristic feature of any relevant degree theory, usually proved using a simple affine homotopy—a method which is not available in our case. As a consequence we will see that the value of $D_2(F, T, G, h)$ is independent of the particular essential inner mapping $T$.

**Theorem 6.1 (Boundary dependence).** Let $F, \tilde{F} \in \mathcal{F}_2(X)$ with the essential inner mappings $T \in \mathcal{F}_1(G)$ and $\tilde{T} \in \mathcal{F}_1(G)$, respectively. Assume that $F$ and $\tilde{F}$ coincide on $\partial G$ and $h \notin F(\partial G)$. Then

$$D_2(F, T, G, h) = D_2(\tilde{F}, \tilde{T}, G, h).$$

**Proof.** We can assume that $h = 0$. By definition there exists $\lambda' > 0$ such that

$$D_2(F, T, G, 0) = d_1(T + \lambda' \phi \hat{\phi} F, G, 0)$$

and

$$D_2(\tilde{F}, \tilde{T}, G, 0) = d_1(\tilde{T} + \lambda' \phi \hat{\phi} \tilde{F}, G, 0).$$
Taking $\lambda'$ large enough we have

\[ [(1 - t)(T + \lambda' \phi \hat{\phi} F) + t(\tilde{T} + \lambda' \phi \hat{\phi} \tilde{F})] (u) \neq 0, \quad 0 \leq t \leq 1, \ u \in \partial G. \quad (6.1) \]

To prove (6.1) we assume the contrary. Then we can find sequences $(\lambda_k, t_k) \in [0, 1]$ and $(u_k) \subset \partial G$ such that $\lambda_k \to \infty$ and

\[(1 - t_k)T(u_k) + t_k \tilde{T}(u_k) + \lambda_k \phi \hat{\phi} F(u_k) = 0 \]

for all $k \in \mathbb{Z}_+$, where we used the fact that $F(u) = \tilde{F}(u)$ for all $u \in \partial G$. At least for subsequences we have $u_k \rightharpoonup u \in X$ and $t_k \to t \in [0, 1]$. Without loss of generality we can also write

\[ F(u_k) \rightharpoonup w \in X, \quad y_k := T(u_k) \rightharpoonup y \in X^*, \quad \tilde{y}_k := \tilde{T}(u_k) \rightharpoonup \tilde{y} \in X^*. \]

Clearly $\phi \hat{\phi} F(u_k) \to 0 = \phi \hat{\phi} w$ and hence $w = 0$. Thus

\[(1 - t) \limsup \{ F(u_k), y_k - y \} = \limsup \{ F(u_k), (1 - t_k) y_k \} \]
\[ = \limsup \{ F(u_k), -t_k \tilde{y}_k - \lambda_k \phi \hat{\phi} F(u_k) \} \]
\[ = \limsup \{ -t_k \tilde{F}(u_k), \tilde{y}_k - \tilde{y} \} - \lambda_k \| \tilde{\phi} F(u_k) \|^2_W \]
\[ \leq -t \liminf \{ \tilde{F}(u_k), \tilde{y}_k - \tilde{y} \} \leq 0, \]

where the last inequality follows from the fact that $\tilde{F}$ has the property $(QM)_{\tilde{T}}$. If $t \neq 1$, then $u_k \to u \in \partial G$ and $F(u) = 0$, a contradiction. By symmetry, using the property $(QM)_T$ of $F$, the conclusion is valid also in case $t = 1$. Hence (6.1) is valid for some sufficiently large $\lambda'$.

Hence by the homotopy invariance property of $d_1$ we have

\[ d_1(T + \lambda' \phi \hat{\phi} F, G, 0) = d_1(\tilde{T} + \lambda' \phi \hat{\phi} \tilde{F}, G, 0), \]

which completes the proof. \(\square\)

**Corollary 6.2.** Let $F \in \mathcal{F}_T(G) \cap \mathcal{F}_{\tilde{T}}(\tilde{G})$ with $T, \tilde{T} \in \mathcal{F}_1(G)$. Assume that $h \notin F(\partial G)$. Then

\[ D_2(F, T, G, h) = D_2(F, \tilde{T}, G, h). \]

**Proof.** The conclusion follows from Theorem 6.1 with $F = \tilde{F}$. \(\square\)

Hence we can define a degree function $d_2$ for class $\mathcal{F}_2(X)$, such that the value of the degree is independent of the particular $T$. For any $F \in \mathcal{F}_2(X)$ with essential inner mapping $T$ and for any $h \notin F(\partial G)$, where $G \subset X$ is an open bounded set, we define

\[ d_2(F, G, h) = D_2(F, T, G, h). \]

Hence we have established the existence of a topological degree function $d_2$ for classes $\mathcal{F}_2(X)$ and $\mathcal{H}_{aff}$, which is normalized by $I$. The following result justifies that $d_2$ is an extension of $d_0$ and, in some sense, also extension of $d_1$. The first part (i) follows from the uniqueness of the Leray–Schauder degree $d_0$ and from the definition of $d_2$ with suitable homotopy. The second part
(ii) is a consequence of the construction of \( d_2 \) with suitable homotopy (the known uniqueness of \( d_1 \) is not needed here).

**Theorem 6.3.**

(i) If \( I + C \in \mathcal{F}_0(\overline{G}) \subset \mathcal{F}_{LS}(X) \) and \( 0 \notin (I + C)(\partial G) \), then

\[
d_0(I + C, G, 0) = d_1(J \circ (I + C), G, 0) = d_2(I + C, G, 0).
\]

(ii) If \( T \in \mathcal{F}_1(\overline{G}) \subset \mathcal{F}_{S^+}(X) \), \( T \) is continuous and \( 0 \notin T(\partial G) \), then

\[
d_1(T, G, 0) = d_2(J^{-1} \circ T, G, 0).
\]

**7. Uniqueness of the degree \( d_2 \)**

The existence of the degree is already established. In this section we shall prove the uniqueness of the degree function \( d_2 \) constructed in Section 6. The uniqueness holds, provided any integer-valued function \( \hat{d}_2 \) for classes \( \mathcal{F}_2(X) \) and \( \mathcal{H}_{aff} \) satisfying the properties (a) to (d) coincides with \( d_2 \). The uniqueness follows from the uniqueness of the \((S^+)\)-degree \( d_1 \). Note that here we need the assumed continuity of the essential inner mapping \( T \). We shall state the result more explicitly as follows.

**Theorem 7.1.** Let \( \hat{d}_2 \) be a topological degree function for classes \( \mathcal{F}_2(X) \) and \( \mathcal{H}_{aff} \) normalized by \( I \). For any \( F \in \mathcal{F}_T(\overline{G}) \subset \mathcal{F}_2(X) \) and any \( h \notin F(\partial G) \) we have

\[
d_2(F, G, h) = \hat{d}_2(F, G, h).
\]

**Proof.** Without loss of generality we can assume that \( h = 0 \). Hence we start to prove that

\[
d_2(F, G, 0) = \hat{d}_2(F, G, 0)
\]

for any \( F \in \mathcal{F}_T(\overline{G}) \subset \mathcal{F}_2(X) \) such that \( 0 \notin F(\partial G) \). By definition there exists \( \lambda' > 0 \) such that

\[
d_2(F, G, 0) = d_1(F_{\lambda'}, G, 0).
\]

For any continuous bounded \((S^+)\)-mapping \( S \in \mathcal{F}_1(\overline{D}) \subset \mathcal{F}_{S^+}(X) \), \( D \in \mathcal{O} \), and any \( y \in X^* \setminus S(\partial D) \) we define a new function \( \hat{d}_1 \) by setting

\[
\hat{d}_1(S, D, y) = \hat{d}_2(J^{-1}(S - y), D, 0).
\]

It is easy to see that \( \hat{d}_1 \) is a degree function for continuous mappings of \( \mathcal{F}_{S^+}(X) \) and \( \mathcal{H}_{S^+}(X) \) normalized by \( J \). The \((S^+)\)-degree is known to be unique (even its restriction to continuous mappings, cf. [4,18]) and thus \( \hat{d}_1 = d_1 \). Hence we obtain

\[
d_2(F, G, 0) = d_1(F_{\lambda'}, G, 0) = \hat{d}_1(F_{\lambda'}, G, 0) = \hat{d}_2(J^{-1}F_{\lambda'}, G, 0).
\]
The proof is complete as soon as we obtain the equality
\[
\hat{d}_2(J^{-1}F_{\lambda'}, G, 0) = \hat{d}_2(F, G, 0). \tag{7.3}
\]

First we notice that \(F\) and \(J^{-1}F_{\lambda'}\) have the common essential inner mapping \(T\). Indeed, if \(F \in F_T(\overline{G})\) and \(T \in F_1(\overline{G})\) is continuous, then a direct verification of the condition \((S_+)\) shows that \(J^{-1}F_{\lambda'} \in F_T(\overline{G})\). Hence the affine homotopy between \(F\) and \(J^{-1}F_{\lambda'}\) is \(\mathcal{H}_2\)-admissible. It is sufficient to show that taking \(\lambda' > 0\) large enough
\[
[(1 - t)F + tJ^{-1}F_{\lambda'}](u) \neq 0, \quad 0 \leq t \leq 1, \ u \in \partial G.
\]
We shall argue by contradiction. Then we can find sequences \((\lambda_k), t_k \in ]0, 1[\) and \((u_k) \subset \partial G\) such that \(\lambda_k \to \infty\) and
\[
(1 - t_k)F(u_k) + t_kJ^{-1}\left(T(u_k) + \lambda_k\phi\hat{\phi}F(u_k)\right) = 0
\]
for all \(k \in \mathbb{N}^+\). At least for subsequences we have \(u_k \to u \in X, t_k \to t \in [0, 1], F(u_k) \to w \in X,\) and \(y_k := T(u_k) \to y \in X^*\). Using the properties of the duality map we obtain the key relation
\[
-t_k\langle F(u_k), y_k \rangle = (1 - t_k)\|F(u_k)\|^2 + t_k\lambda_k\|\hat{\phi}F(u_k)\|^2_W. \tag{7.4}
\]
Assume first that \(t_k \to t \neq 0\). Then \(F(u_k) \to 0\). From (7.4) it is easily seen that
\[
\limsup\langle F(u_k), y_k - y \rangle \leq 0
\]
implying \(u_k \to u \in \partial G\) and \(F(u) = 0\), a contradiction. Assume then that \(t_k \to 0\). From equality (7.4) we get the estimate
\[
t_k\|F(u_k)\|\|y_k\| \geq -t_k\langle F(u_k), y_k \rangle \geq (1 - t_k)\|F(u_k)\|^2,
\]
whence \(F(u_k) \to 0\). Consequently by the properness of \(F, u_k \to u \in \partial G\) and thus \(F(u) = 0\) giving a contradiction. Hence we have proved that (7.3) holds and thus the proof is complete. \(\square\)

8. Borsuk’s theorem, variational inequalities

The Borsuk theorem for Leray–Schauder type mappings roughly speaking states that the degree of an odd mapping is odd. The same result holds for mappings of class \((S_+)\) (see [4]). We shall prove that the result is valid also in \(F_2(X)\).

**Theorem 8.1.** Let \(G\) be an open bounded set of \(X\), which is symmetric with respect to the origin and \(0 \in G\). Let \(F \in F_T(\overline{G}) \subset F_2(X)\) be odd on \(\partial G\) and \(0 \notin F(\partial G)\). Then \(d_2(F, G, 0)\) is an odd number and the equation \(F(u) = 0\) has at least one solution in \(G\).

**Proof.** The \((S_+)\)-approximation \(F_{\lambda}\) of \(F \in F_T(\overline{G})\) is not odd on \(\partial G\), unless the essential inner mapping \(T\) is. Thus we cannot use the Borsuk theorem for mappings of class \((S_+)\) directly. By definition there exists \(\lambda_0 > 0\) such that
\[
d_2(F, G, 0) = d_1(F_{\lambda}, G, 0) \quad \text{for all } \lambda > \lambda_0.
\]
Denote \( \hat{T}(u) = -T(-u) \) and \( \hat{F}(u) = -F(-u) \). Note that \( \hat{F}(u) = F(u) \) for all \( u \in \partial G \), since \( F \) is odd on the boundary of \( G \). Clearly \( \hat{T} \in \mathcal{F}_1(\overline{G}) \) and it is easy to see that \( \hat{F} \in \mathcal{F}_1(\overline{G}) \). Define a further mapping

\[
S_\lambda = \frac{1}{2}(T + \hat{T}) + \lambda \phi \hat{\phi} F, \quad \lambda > 0.
\]

Then \( S_\lambda \in \mathcal{F}_1(\overline{G}) \) and \( S_\lambda \) is odd on \( \partial G \). We shall prove that there exists \( \lambda' > \lambda_0 \) such that

\[
(1 - t) F_{\lambda'}(u) + t S_{\lambda'}(u) \neq 0, \quad 0 \leq t \leq 1, \ u \in \partial G.
\]

Otherwise we can find sequences \( (\lambda_k), (t_k) \in [0, 1] \) and \( (u_k) \subset \partial G \) such that \( \lambda_k \to \infty \) and

\[
(1 - t_k) F_{\lambda_k}(u_k) + t_k S_{\lambda_k}(u_k) = 0,
\]

i.e.,

\[
(2 - t_k) T(u_k) + t_k \hat{T}(u_k) + 2\lambda_k \phi \hat{\phi} F(u_k) = 0
\]

for all \( k \in \mathbb{Z}_+ \). For subsequences \( u_k \to u \in X, \ t_k \to t \in [0, 1] \) and

\[
F(u_k) \rightharpoonup w \in X, \quad y_k := T(u_k) \to y \in X^*, \quad \hat{y}_k := \hat{T}(u_k) \to \hat{y} \in X^*.
\]

Consequently, \( w = 0 \) and

\[
(2 - t) \limsup\langle F(u_k), y_k - y \rangle = \limsup\langle F(u_k), (2 - t_k)y_k \rangle = \limsup\langle F(u_k), -t_k \hat{y}_k - 2\lambda_k \phi \hat{\phi} F(u_k) \rangle
\]

\[
= \limsup\{-t_k \langle F(u_k), \hat{y}_k - \hat{y} \rangle - 2\lambda_k \| \hat{\phi} F(u_k) \|_W \} \leq -t \liminf\langle F(u_k), \hat{y}_k - \hat{y} \rangle = -t \liminf\langle \hat{F}(u_k), \hat{y}_k - \hat{y} \rangle \leq 0,
\]

where we have used Lemma 3.1 and the fact that for \( u_k \in \partial G \) we have \( F(u_k) = \hat{F}(u_k) \). Hence \( \limsup\langle F(u_k), y_k - y \rangle \leq 0 \) implying \( u_k \to u \in \partial G \) and \( F(u) = 0 \), a contradiction. Thus

\[
d_2(F, G, 0) = d_1(F_{\lambda'}, G, 0) = d_1(S_{\lambda'}, G, 0).
\]

By the Borsuk theorem for mappings of class \((S_+)\) the value of \( d_1(S_{\lambda'}, G, 0) \) is an odd number completing the proof. \( \square \)

In case the essential inner mapping \( T \) is odd or more generally \( T(u) - T(-u) \) defines a compact map, we get the following variant of the Borsuk theorem.
Corollary 8.2. Let $G$ be an open bounded set of $X$, which is symmetric with respect to the origin and $0 \in G$. Assume that $T \in \mathcal{F}_1(G)$ is continuous and such that $T(u) + T(-u) = C_0(u)$, where $C_0 : G \to X^*$ is compact ($C_0 = 0$ possible). Assume that $F \in \mathcal{F}_T(G) \subset \mathcal{F}_2(X)$, $0 \notin F(\partial G)$ and $F(u) \neq F(-u)$ for all $u \in \partial G$.

Then $d_2(F, G, 0)$ is an odd number and the equation $F(u) = 0$ has at least one solution in $G$.

We close this section by an application to variational inequalities. For simplicity we deal with the problem only in Hilbert space $H$. A more general treatment in reflexive Banach space is possible, if we replace the projection $P_K : H \to K \subset H$ onto closed convex set by the generalized projection $\pi_K : X^* \to K \subset X$ introduced by Y.I. Alber in [2], which is known to be continuous and monotone.

Example 8.3. Let $H$ be a real separable Hilbert space. We identify $H^*$ with $H$ and the replace the pairing by the inner product $\langle \cdot, \cdot \rangle_H$. Let $K \subset H$ be a closed convex set such that $0 \in K$ and $K$ is symmetric with respect to the origin. Let $T : H \to H$ be a bounded continuous mapping. For any $h \in H$ the variational inequality

$$u \in K : \langle u + T(u), v - u \rangle_H \geq \langle h, v - u \rangle_H$$  \hspace{1em} \text{for all } v \in K \tag{8.1}$$

is equivalent to the fixed point equation

$$u \in H : u = P_K \left(u - \alpha (T(u) + u - h)\right),$$ \hspace{1em} \text{for all } u \in K \tag{8.2}$$

where $\alpha > 0$ is arbitrary but fixed and $P_K : H \to K$ is the (minimum distance) projection, which is continuous bounded and monotone. Moreover, $P_K$ is odd by the symmetry of $K$. Denote

$$F := I + P_K \left(\alpha T + (\alpha - 1)I - \alpha h\right),$$

which is a mapping of abstract Hammerstein type. Then $F(u) = 0$ if and only if $u \in K$ and $u$ is a solution of (8.1). We can proceed with degree theoretic arguments, provided the mapping

$$T_\alpha := \alpha T + (\alpha - 1)I,$$

which is our candidate to be the essential inner mapping, is of class $(S_+)$. This holds, if

(a) $T$ is quasimonotone and we take $\alpha > 1$, or

(b) $T$ is $-q$-monotone, i.e.,

$$\langle T(u) - T(v), u - v \rangle_H \geq -q \|u - v\|_H^2$$ \hspace{1em} \text{for all } u, v \in H,$$

with $q < 1$. In this case we take $\alpha > \frac{1}{1-q}$. 


Assume either (a) or (b). Then $T_\alpha \in \mathcal{F}_{S^+}(H)$ and $F \in \mathcal{F}_{T_\alpha}(\overline{G})$ for all $G \in \mathcal{O}$. Assume that
\[
\liminf_{\|u\| \to \infty} \frac{\langle T(u), u \rangle_H}{\|u\|} > -\infty \tag{8.3}
\]
and there exists $r > 0$ such that
\[
T(-u) = -T(u) \quad \text{for all } \|u\| \geq r. \tag{8.4}
\]
Then the variational inequality (8.1) admits at least one solution $u \in K$ for any $h \in H$. Indeed, by (8.3) and by the fact that $0 \in K$ it is not hard to prove that there exists $R \geq r$ such that
\[
u + P_K(T_\alpha(u) - t\alpha h) \neq 0 \quad \text{for all } 0 \leq t \leq 1, \|u\| = R
\]
implying
\[
d_2(F, B_R(0), 0) \neq d_2(I + P_K T_\alpha, B_R(0), 0) \neq 0,
\]
where the last inequality is true by the Borsuk theorem. Hence the existence of at least one solution $u \in K$ of (8.1) is proved.

9. Extension of the class of homotopies with applications

So far we have considered only the class of $\mathcal{H}_2$-admissible affine homotopies between two mappings in $\mathcal{F}_T(\overline{G}) \subset \mathcal{F}_2(X)$ with a common essential inner mapping $T \in \mathcal{F}_1(\overline{G})$. We shall generalize the concept. Instead of $T \in \mathcal{F}_1(\overline{G}) \subset \mathcal{F}_{S^+}(X)$ we consider ‘essential inner homotopy,’ by which we mean a bounded continuous homotopy $T : [0, 1] \times \overline{G} \to X$ satisfying condition $(S^+)$. Then we denote by $\mathcal{H}_T(\overline{G})$ the class of bounded demicontinuous mappings $H : [0, 1] \times \overline{G} \to X$, $G \in \mathcal{O}$, satisfying the condition.

If $(u_k) \subset \overline{G}$, $(t_k) \subset [0, 1]$, $(y_k) = (T(t_k, u_k)) \subset X^*$ with $u_k \rightharpoonup u$, $t_k \to t$, $y_k \rightharpoonup y$ and $\limsup(H(t_k, u_k), y_k - y) \leq 0$, then $u_k \to u$.

The class of $\mathcal{H}_2$-admissible homotopies, denoted by $\mathcal{H}_2(X)$, is then the union of all classes $\mathcal{H}_T(\overline{G})$, where $G \in \mathcal{O}$ and $T : [0, 1] \times \overline{G} \to X^*$ is a bounded continuous homotopy of class $(S^+)$. As a special case of ‘constant homotopy’ we can have $T \in \mathcal{F}_{S^+}(X)$. Hence it is clear by Lemma 2.5 that $\mathcal{H}_{\text{aff}} \subset \mathcal{H}_2(X)$ and it is not hard to prove the following result.

**Lemma 9.1.** If $H \in \mathcal{H}_T(\overline{G}) \subset \mathcal{H}_2(X)$ and $h_t, 0 \leq t \leq 1$, is a continuous curve in $X$ such that $H(t, u) \neq h_t$ for all $u \in \partial G$, $0 \leq t \leq 1$, then
\[
d_2(H(t, \cdot), G, h_t) \quad \text{is constant in } t \text{ on } [0, 1].
\]

The class $\mathcal{H}_2(X)$ extends the class of $\mathcal{H}_2$-admissible homotopies considerably. To obtain generalizations of some classical results we consider a very special class of simple affine homotopies of the form
\[
(1 - t)I + tF, \quad 0 \leq t \leq 1,
\]
where $F \in \mathcal{F}_2(X)$. It is easy to see that this homotopy is $\mathcal{H}_2$-admissible. Then the next basic geometrical result holds.
Theorem 9.2 (Principle of omitted rays). Let $F \in \mathcal{F}_T(G) \subset \mathcal{F}_2(X)$ and $h_0 \in G$. Assume that

$$u - F(u) \neq h_0 + \eta(u - h_0) \quad \text{for all } u \in \partial G, \eta > 1.$$ 

Then the equation $F(u) = 0$ admits at least one solution in $\overline{G}$. Moreover, $d_2(F, G, h) = +1$ whenever defined.

Proof. If $0 \notin F(\partial G)$, the condition of omitted rays is equivalent to

$$(1 - t)u + tF(u) \neq (1 - t)h_0 \quad \text{for all } u \in \partial G, 0 \leq t \leq 1.$$ 

Consequently, the conclusion follows from the homotopy invariance property of $d_2$. \Box

We can replace $F$ by $F - h$ in Theorem 9.2 to obtain solution for $F(u) = h$. As an implication of the principle of omitted rays we obtain the following results familiar in the context of Leray–Schauder type mappings.

Corollary 9.3. Let $F \in \mathcal{F}_T(G) \subset \mathcal{F}_2(X)$. The equation $F(u) = 0$ has at least one solution in $\overline{G}$ provided one of the following conditions is satisfied.

(i) (Rothes’s condition) The set $G$ is convex and $(I - F)(\partial G) \subset G$.

(ii) (Leray–Schauder principle) $0 \in G$ and

$$t(u - F(u)) \neq u \quad \text{for all } u \in \partial G, 0 < t < 1.$$ 

(iii) (Altmann’s condition) There exists $h_0 \in G$ such that

$$\|F(u)\|^2 \geq \|u - h_0 - F(u)\|^2 - \|u - h_0\|^2 \quad \text{for all } u \in \partial G.$$ 

Proof. Either $F(u) = 0$ for some $u \in \partial G$ or the principle of omitted rays applies. \Box

To illuminate the use of the above theorem we consider an abstract Hammerstein type equation with coercivity conditions.

Example 9.4. We shall apply the Leray–Schauder principle to the equation

$$u + ST(u) = h, \quad u \in X, \quad (9.1)$$

where $h \in X$, $T : X \to X^*$ is bounded continuous mapping of class $(S_+)$ and the mapping $S : X^* \to X$ is bounded demicontinuous and quasimonotone. Then $F := I + ST \in \mathcal{F}_T(G) \subset \mathcal{F}_2(X)$ for all $G \in \mathcal{O}$. We assume that $S$ is coercive, i.e.,

$$\frac{\langle S(y), y \rangle}{\|y\|} \to \infty \quad \text{as } \|y\| \to \infty.$$ 

For $T$ we assume that

$$\|T(u)\| \to \infty \quad \text{as } \|u\| \to \infty.$$
and there exists \( r > 0 \) such that

\[
\langle T(u), u \rangle \geq 0 \quad \text{for all } \|u\| \geq r.
\]

Then Eq. (9.1) admits a solution for all \( h \in X \). Indeed, we first notice that we may assume that \( h = 0 \), since we can always replace \( S \) by \( S - h \). The Leray–Schauder principle applies, which can be seen using the equality

\[
\langle T(u), u + tS(T(u)) \rangle = \langle T(u), u \rangle + t\langle S(T(u)), T(u) \rangle
\]

together with the assumptions on \( T \) and \( S \).

References