# Interlacing Inequalities and Control Theory

Ion Zaballa
*Departamento de Matemáticas*
*Escuela Universitaria de Magisterio*
*Universidad del Pais Vasco*
*01006. Vitoria-Gasteiz, Spain*

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ABSTRACT

Given a $p$ by $p$ matrix $A$, we solve the problem of the existence of a $p$ by $q$ matrix $B$ such that $(A,B)$ has prescribed controllability indices and $[\lambda I_p - A, -B]$ has prescribed invariant polynomials. The solution of this problem, together with an earlier theorem of the author's, is used to provide a new proof of the Sà-Thompson interlacing inequalities for invariant polynomials.

## 1. INTRODUCTION

Let $\mathbb{F}$ be an arbitrary field, $\mathbb{F}[\lambda]$ the ring of polynomials with coefficients in $\mathbb{F}$, and $\mathbb{F}^{m\times n}$ and $\mathbb{F}[\lambda]^{m\times n}$ the vector spaces over $\mathbb{F}$ of the $m$ by $n$ matrices with coefficients in $\mathbb{F}$ and $\mathbb{F}[\lambda]$, respectively. $\mathrm{Gl}_n(\mathbb{F})$ denotes the linear group of order $n$ over $\mathbb{F}$, and $I_n$ the identity matrix of order $n$. Throughout this paper $A\in\mathbb{F}^{p\times p}$, $B\in\mathbb{F}^{p\times q}$, $n=p+q$, $[A,B]\in\mathbb{F}^{p\times n}$, and if $[A_1,B_1]\in\mathbb{F}^{r\times(r+s)}$, then we have to understand that $A_1\in\mathbb{F}^{r\times r}$ and $B_1\in\mathbb{F}^{r\times s}$. The elements of $\mathbb{F}[\lambda]$ will be denoted by Greek letters; all the polynomials will be considered monic; $\alpha :> \beta$ means "$\alpha$ divides $\beta$," and $d(\alpha)$ will be the degree of the polynomial $\alpha$.

In this paper (Theorem 3.1) we solve the following problem: Let $A\in\mathbb{F}^{p\times p}$; let $\mu_1 :> \cdots :> \mu_p$ be $p$ monic polynomials, and $k_1\geqslant\cdots\geqslant k_r$ positive integers. Find a necessary and sufficient condition for the existence of a matrix $B\in\mathbb{F}^{p\times q}$ such that rank $B=r$, $\mu_1,\dots,\mu_p$ are the invariant factors of




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$[\lambda I_p - A, \ -B]$, and $k_1, \ldots, k_r$ are the controllability indices of the pair $(A, B)$.

We will use the solution of this problem and Theorem 5.1 of [14] to obtain a new proof of the interlacing inequalities for the invariant factors of a matrix with elements in $\mathbb{F}$ and those of a prescribed principal submatrix [9–11].

## 2. AUXILIARY RESULTS

In this section we collect some lemmas which will be needed for the proof of our main result: Theorem 3.1. Some of the lemmas are known. In this case we just state them for the reader's convenience, indicating where the proof can be found.

DEFINITION. Let $A_1, A_2 \in \mathbb{F}^{p\times p}$, $B_1, B_2 \in \mathbb{F}^{p\times q}$, and $n = p + q$. If there exist matrices $P \in \mathrm{Gl}_p(\mathbb{F})$ and $Q \in \mathrm{Gl}_q(\mathbb{F})$ such that $A_1 = PA_2P^{-1}$ and $B_1 = PB_2Q$, then we will say that $[A_1, B_1]$ and $[A_2, B_2]$ are *$P, Q$-equivalent* matrices. The $P, I_q$-equivalent matrices will be said to be *P-equivalent*.

We observe that if $[A_1, B_1]$ and $[A_2, B_2]$ are $P, Q$-equivalent, then $A_1$ and $A_2$ are similar matrices, and, $B_1$ and $B_2$ are equivalent in $\mathbb{F}^{p\times q}$.

LEMMA 2.1 [14, Proposition 2.4]. *If $[A_1, B_1]$, $[A_2, B_2] \in \mathbb{F}^{p\times n}$ are $P, Q$-equivalent matrices, then $[\lambda I_p - A_1, \ -B_1]$ and $[\lambda I_p - A_2, \ -B_2]$ are equivalent in $\mathbb{F}[\lambda]^{p\times n}$.*

LEMMA 2.2. *Let $C_1$ be the companion matrix of the polynomial $\lambda^s - a_1\lambda^{s-1} - \cdots - a_{s-1}\lambda - a_s$. Let $C_2 \in \mathbb{F}^{s\times t}$ and $C_3 \in \mathbb{F}^{t\times t}$. Then there exist matrices $T \in \mathbb{F}^{s\times t}$ and $d \in \mathbb{F}^{1\times t}$ such that*

$$\begin{bmatrix} I_s & T \\ 0 & I_t \end{bmatrix}\begin{bmatrix} C_1 & C_2 \\ 0 & C_3 \end{bmatrix}\begin{bmatrix} I_s & -T \\ 0 & I_t \end{bmatrix} = \begin{bmatrix} C_1 & D \\ 0 & C_3 \end{bmatrix},$$

*where*

$$D = \begin{bmatrix} 0 \\ d \end{bmatrix} \in \mathbb{F}^{s\times t}.$$

*Proof.* Let $t_1, \dots, t_s$ and $c_1, \dots, c_s$ be the rows of $T$ and $C_2$ respectively. Take

$$t_1 = 0, \qquad t_k = \sum_{j=1}^{k-1} c_j C_3^{k-j-1} \quad (k = 2, \dots, s), \qquad d = \sum_{j=1}^{s} \big( c_j C_3^{s-j} - a_j t_{s-j+}$$

and check. ■

Note that

$$\begin{bmatrix} C_1 & C_2 \\ 0 & C_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_1 & D \\ 0 & C_3 \end{bmatrix}$$

are similar matrices.

By defining the controllability matrix and the controllability indices of $(A, B)$ as in [14] we can give the following result:

LEMMA 2.3. *Let* $[A, B] \in \mathbb{F}^{p \times n}$, *and let* $S(A, B)$ *be the controllability matrix of* $(A, B)$. *If* $\operatorname{rank} B = r$, $\operatorname{rank} S(A, B) = s$, *and* $k_1 \geqslant \cdots \geqslant k_r$ $(> 0)$ *are the nonzero controllability indices of* $(A, B)$ *then* $[A, B]$ *is P,Q-equivalent to a matrix* $[A_c, B_c]$ *verifying*:

(i)
$$A_c = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad B_c = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

*where* $(A_1, B_1)$ *is a completely controllable pair and* $k_1 + \cdots + k_r = s$.

(ii)
$$A_1 \in \mathbb{F}^{s \times s}, \qquad B_1 \in \mathbb{F}^{s \times q}.$$

(iii)
$$A_1 = \big[ A_{ij} \big]_{1 \leqslant i, j, \leqslant r},$$

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 1 \\ a_{ii}^{(1)} & a_{ii}^{(2)} & a_{ii}^{(3)} & \cdots & a_{ii}^{(k_i)} \end{bmatrix} \in \mathbb{F}^{k_i \times k_i},$$

*and for $i \neq j$,*

$$A_{ij} = \begin{bmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \\ a_{ij}^{(1)} & \cdots & a_{ij}^{(k_j)} \end{bmatrix} \in \mathbb{F}^{k_i \times k_j}, \qquad i, j = 1, \ldots, r.$$

(iv) $$A_2 = \begin{bmatrix} C_{21} \\ \vdots \\ C_{2r} \end{bmatrix}, \qquad C_{2i} = \begin{bmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \\ c_{i1} & \cdots & c_{ip-s} \end{bmatrix} \in \mathbb{F}^{k_i \times (p-s)}.$$

(v) $$B_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_r \end{bmatrix}, \qquad E_i = \begin{bmatrix} 0 \\ e_i \end{bmatrix} \in \mathbb{F}^{k_i \times q},$$

*and $e_i$ is the $i$th row of $I_q$, $i = 1, \ldots, r$.*

*Proof.* By making suitable changes in [13, p. 80–86] we can see that $[A, B]$ is $P$-equivalent to a matrix $[\bar{A}, \bar{B}]$ with the following form:

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & \bar{A}_3 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix},$$

where $\bar{A}_1 = [\bar{A}_{ij}]_{1 \leqslant i, j \leqslant r}$, $\bar{A}_{ij} \in \mathbb{F}^{m_i \times m_j}$ are matrices with the required form, $m_1, \ldots, m_r$ being a certain reordering of $k_1, \ldots, k_r$. Moreover

$$\bar{B}_1 = \begin{bmatrix} B_{11} \\ \vdots \\ B_{1r} \end{bmatrix}, \qquad B_{1i} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & * & \cdots & * \end{bmatrix} \in \mathbb{F}^{m_i \times q},$$

with the 1 in the $i$th column, and $\bar{A}_2$ is a matrix which perhaps has not the required form and the stars denote unspecified elements. Now, it is clear that there exists a permutation matrix $P_1$ of order $p$ and a matrix $Q \in \mathrm{Gl}_q(\mathbb{F})$ such that

$$\tilde{A} = P_1 \bar{A} P_1^{-1} = \begin{bmatrix} A_1 & \tilde{A}_2 \\ 0 & A_3 \end{bmatrix}, \qquad B_c = P_1 \bar{B} Q.$$

We consider $\tilde{A}_2$ partitioned as follows:

$$\tilde{A}_2 = \begin{bmatrix} C_1 \\ \vdots \\ C_r \end{bmatrix},$$

where $C_i \in \mathbb{F}^{k_i \times (p-s)}$, $i = 1, \ldots, r$. By Lemma 2.2 for each $i = 1, \ldots, r$ there exist matrices $T_i \in \mathbb{F}^{k_i \times (p-s)}$ and $d_i \in \mathbb{F}^{1 \times (p-s)}$ such that

$$\begin{bmatrix} I_{k_i} & T_i \\ 0 & I_{p-s} \end{bmatrix} \begin{bmatrix} A_{ii} & C_i \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} I_{k_i} & -T_i \\ 0 & I_{p-s} \end{bmatrix} = \begin{bmatrix} A_{ii} & D_i \\ 0 & A_3 \end{bmatrix},$$

where

$$D_i = \begin{bmatrix} 0 \\ d_i \end{bmatrix} \in \mathbb{F}^{k_i \times (p-s)}.$$

Setting

$$P_2 = \begin{bmatrix} I_{k_1} & 0 & \cdots & 0 & T_1 \\ 0 & I_{k_2} & \cdots & 0 & T_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{k_r} & T_r \\ 0 & 0 & \cdots & 0 & I_{p-s} \end{bmatrix},$$

we have that $A_c = P_2 \tilde{A} P_2^{-1}$ and $B_c = P_2 B_c$. Now, $[A_c, B_c]$ is $P, Q$-equivalent to $[A, B]$ and has the required form. ■

We remark that if $(A, B)$ is a completely controllable pair, i.e. $\operatorname{rank} S(A, B) = p$, then $A_c = A_1$ and $B_c = B_1$. Moreover, in Remark 2.1 of [8] it is shown that the numbers $k_i$ are invariant for the $P, Q$-equivalence.

Lemma 2.4. *Let $[A, B] \in \mathbb{F}^{p \times n}$, and let $[A_c, B_c] \in \mathbb{F}^{p \times n}$ be a matrix with the form of Lemma 2.3 above and $P, Q$-equivalent to $[A, B]$. Then the nontrivial invariant factors (i.e., those distinct from 1) of the polynomial matrix $[\lambda I_p - A, -B]$ are those of $\lambda I_{p-s} - A_3$.*

*Proof.* Since $(A_1, B_1)$ is a controllable pair, it follows (Corollary I of Theorem 2.11 of [14]) that the Smith normal form of $[\lambda I_s - A_1, -B_1]$ is

$[I_s,0]$. Therefore, $[\lambda I_s - A_1, -A_2, -B_1]$ has $[I_s,0,0]$ as its Smith normal form. So there exist unimodular matrices $P(\lambda) \in \mathbb{F}[\lambda]^{s\times s}$ and $Q(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ such that

$$P(\lambda)[\lambda I_s - A_1, -A_2, -B_1]Q(\lambda) = [I_s,0,0].$$

Since

$$\begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{p-s} \end{bmatrix}\begin{bmatrix} \lambda I_s - A_1 & -A_2 & -B_1 \\ 0 & \lambda I_{p-s} - A_3 & 0 \end{bmatrix} Q(\lambda)$$
$$= \begin{bmatrix} I_s \quad 0 \quad 0 \\ [0, \lambda I_{p-s} - A_3, 0]\, Q(\lambda) \end{bmatrix},$$

we conclude that the matrix on the right hand side is equivalent to $[\lambda I_p - A_c, -B_c]$. The invariant factors of $\lambda I_{p-s} - A_3$ are of course the same as those of $[0, \lambda I_{p-s} - A_3, 0]$. Therefore $[\lambda I_p - A_c, -B_c]$ has as invariant factors those of $\lambda I_{p-s} - A_3$ and $s$ invariant factors equal to 1. ■

Just as in [14] we define a column degree dominant matrix as a matrix whose element in the position $(i,i)$ is a monic polynomial whose degree is greater than that of any other in the $i$th column. In some of the following lemmas we are going to use this concept.

LEMMA 2.5. *Consider the matrix* $[\lambda I_p - A_c, -B_c]$, *using the same notation as in Lemma* 2.3. *This matrix is equivalent to a matrix of the form* $[A(\lambda), F_r]$ *where:*

(i) $A(\lambda)$ *is given by*

$$A(\lambda) = \begin{bmatrix} I_{s-r} & 0 & 0 \\ 0 & \lambda I_{p-s} - A & 0 \\ 0 & C & T(\lambda) \end{bmatrix}$$

*with* $A \in \mathbb{F}^{(p-s)\times(p-s)}$, $C \in \mathbb{F}^{r\times(p-s)}$, *and* $T(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$.

(ii) $F_r$ *is given by*

$$F_r = \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix}.$$

(iii) $T(\lambda)$ *is a column degree dominant matrix, the degree of its* $i$*th column being* $k_i$.

*Moreover,*

(iv) $\lambda I_p - A_c$ *and* $A(\lambda)$ *are equivalent* $\lambda$*-matrices.*

*Proof.* Assume $A_c$ and $B_c$ are in the form (i) of Lemma 2.3. We are going to perform certain elementary operations on the rows and columns of $[\lambda I_p - A_c, -B_c]$ which transform this matrix into $[A(\lambda), F_r]$. This proves (i), (ii), and (iii). Notice that the column elementary operations will all be performed either within the block $\lambda I_p - A_c$ or within the block $-B_c$. This proves (iv).

Let us now describe the elementary operations.

To column $k_1 - 1$ add column $k_1$ multiplied by $\lambda$. To the column $k_1 - 2$ add column $k_1 - 1$ multiplied by $\lambda$. Go on, and stop when the second column multiplied by $\lambda$ has been added to the first column. The block $A_{11}$ of $A_1$ has now the form (assuming $k_1 = 3$ for simplicity)

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ \tau_{11} & \theta_{12} & \theta_{13} \end{bmatrix},$$

where $\tau_{11}, \theta_{12}, \theta_{13}$ are polynomials with $\tau_{11}$ of degree $k_1$.

The block $A_{21}$ of $A_1$ becomes (assuming $k_1 = 3$, $k_2 = 2$ for simplicity)

$$\begin{bmatrix} 0 & 0 & 0 \\ \tau_{21} & \theta_{22} & \theta_{23} \end{bmatrix},$$

where $\tau_{21}, \theta_{22}, \theta_{23}$ are polynomials and $\tau_{21}$ has degree smaller than $k_1$. It is easy to see that $\tau_{11}$ is the polynomial with largest degree in its column.

Now we proceed in the same way for the columns passing through the block $A_{22}$, then for those passing through $A_{33}$, etc., up to $A_{rr}$.

Let us now focus our attention on the elements of each block $A_{ii}$ immediately above the principal ones. We see that they are equal to $-1$ and are the only nonzero in their rows [in the big $p \times (p+q)$ matrix]. For a moment let us call these $-1$'s "relevant elements." It is obvious that with elementary operations on the rows we can annihilate all the elements in the columns of the "relevant elements" except themselves, leaving unchanged the elements that are not in the column of the "relevant elements." Now multiply by $-1$ the last $p - s$ rows, the columns of the "relevant elements," and columns $s+1, s+2, \ldots, p$, $p+1, \ldots, p+r$. At this point we have a matrix with the form (assuming for simplicity $p=8,\ q=3,\ r=2,\ k_1=3,$ $k_2=2$)

$$\left[\begin{array}{ccc|cc|c|c}
0 & 1 & 0 & 0 & 0 & & \\
0 & 0 & 1 & 0 & 0 & & \\
\tau_{11} & 0 & 0 & \tau_{12} & 0 & A_2 & B_1 \\
\cline{1-5}
0 & 0 & 0 & 0 & 1 & & \\
\tau_{21} & 0 & 0 & \tau_{22} & 0 & & \\
\hline
 & 0 & & & 0 & \lambda I_3 - A_3 & 0
\end{array}\right].$$

Recalling the special form of $A_2$ and $B_1$, we can see that with suitable permutations of rows and columns we can bring the matrix to the form announced in the lemma. The elements of $T(\lambda)$ will be the $\tau_{ij}$'s, and the elements of $C$ will be the elements of $A_2$ that are not necessarily zero, Namely,

$$C=[c_{ij}], \qquad i=1,\ldots,r;\ j=1,\ldots,p-s,$$

$$T(\lambda)=[\tau_{ij}], \qquad i,j=1,\ldots,r,$$

where

$$\tau_{ii}=\lambda^{k_i}-\sum_{t=1}^{k_i} a_{ii}^{(t)}\lambda^{t-1},$$

$$\tau_{ij}=-\sum_{t=1}^{k_j} a_{ij}^{(t)}\lambda^{t-1} \qquad \text{for} \quad i\neq j.$$

Finally, the principal elements of $I_r$ are the 1's appearing in $B_1$. ■

REMARK. Notice that the successive transformations to bring $[\lambda I_p - A_c,\ -B_c]$ to the form $[A(\lambda), F_r]$ are invertible. In fact, if a matrix with the form $[A(\lambda), F_r]$ of the lemma is given, the matrix $[\lambda I_p - A_c,\ -B_c]$ obtained by performing the inverse transformations of those of the lemma has $k_1,\ldots,k_r$ as controllability indices, since these numbers are invariant for the $P,Q$-equivalence relation.

LEMMA 2.6 (Theorem I.5.1 of [10]). *Let* $A(\lambda) \in \mathbb{F}[\lambda]^{m \times t}$ *and* $G(\lambda) \in \mathbb{F}[\lambda]^{(m+r)\times(t+s)}$, *where r and s are nonnegative integers. Let* $\alpha_1 :> \cdots :> \alpha_m$ *and* $\gamma_1 :> \cdots :> \gamma_{m+r}$ *be the invariant factors of* $A(\lambda)$ *and* $G(\lambda)$, *respectively, and let us agree that* $\alpha_i = 0$ *for* $i > \operatorname{rank} A(\lambda)$ *and* $\gamma_i = 0$ *for* $i > \operatorname{rank} G(\lambda)$. *Then there exist matrices* $B(\lambda) \in \mathbb{F}[\lambda]^{m \times s}$, $C(\lambda) \in \mathbb{F}[\lambda]^{r \times t}$, *and* $D(\lambda) \in \mathbb{F}[\lambda]^{r \times s}$ *such that* $G(\lambda)$ *is equivalent to*

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}$$

*if and only if*

$$\gamma_i :> \alpha_i :> \gamma_{i+r+s}, \qquad i = 1, \ldots, m. \tag{2.1}$$

LEMMA 2.7 (Lemma 3.4 of [14]). *Let* $A(\lambda) = \operatorname{diag}(I_{p-m}, \lambda I_m - N) \in \mathbb{F}[\lambda]^{p \times p}$ $(p \geqslant m)$, *and let* $\alpha_1 :> \cdots :> \alpha_p$ *be its invariant factors. Let* $\gamma_1 :> \cdots :> \gamma_n$ *be n monic polynomials, where* $n = p + q$ $(q \geqslant 0)$. *Then there exist matrices* $C \in \mathbb{F}^{q \times m}$ *and* $T(\lambda) \in \mathbb{F}[\lambda]^{q \times q}$ *such that*

$$\begin{bmatrix} I_{p-m} & 0 & 0 \\ 0 & \lambda I_m - N & 0 \\ 0 & C & T(\lambda) \end{bmatrix}$$

*has* $\gamma_1 :> \cdots :> \gamma_n$ *as invariant factors if and only if*

$$\gamma_i :> \alpha_i :> \gamma_{i+q}, \qquad i = 1, \ldots, p. \tag{2.2}$$

*Moreover, if* (2.2) *is verified, then* $T(\lambda)$ *can be constructed as a column degree dominant lower triangular matrix such that its diagonal elements are the polynomials*

$$\sigma_j = \frac{\beta^j}{\beta^{j-1}}, \qquad j = 1, \ldots, q,$$

*where* $\beta^j = \beta_1^j \cdots \beta_{p+j}^j$ *and* $\beta_i^j = \operatorname{lcm}(\alpha_{i-j}, \gamma_i)$, $i = 1, \ldots, p + j$, $j = 0, 1, \ldots, q$.

LEMMA 2.8. *Let* $A \in \mathbb{F}^{p\times p}$, *and let* $\alpha_1 :> \cdots :> \alpha_p$ *be its invariant factors. Let* $\mu_1 :> \cdots :> \mu_p$ *be* $p$ *monic polynomials and* $d(\mu_1 \ldots \mu_p) = m$. *Let* $q$ *and* $r$ *be nonnegative integers such that* $r \leqslant \min(p-m, q)$. *Then*:

(i) *There exists a matrix* $B \in \mathbb{F}^{p\times q}$ *such that* $\operatorname{rank} B = r$ *and* $[\lambda I_p - A, -B]$ *has* $\mu_1, \ldots, \mu_p$ *as invariant factors if and only if*

$$\mu_i :> \alpha_i :> \mu_{i+r}, \quad i = 1, \ldots, p \qquad (\mu_i := 0 \quad \text{if } i > p). \tag{2.3}$$

(ii) *There exists* $B \in \mathbb{F}^{p\times q}$ *such that* $\operatorname{rank} B = \min(p-m, q)$ *and* $\mu_1, \ldots, \mu_p$ *are the invariant factors of* $[\lambda I_p - A, -B]$ *if the following relations hold*:

$$\mu_1 :> \alpha_i :> \mu_{i+q}, \qquad i = 1, \ldots, p. \tag{2.4}$$

(iii) *If* (2.4) *is verified and* $r = \min(p-m, q)$, *then* $\alpha_i :> \mu_{i+r} :> \alpha_{i+r}$, $i = 1, \ldots, p-r$.

*Proof.* (i): Firstly we are going to see that (2.3) is a necessary condition for the existence of $B \in \mathbb{F}^{p\times q}$ verifying the requirements of part (i). Let

$$D_r = \begin{bmatrix} I_r \\ 0 \end{bmatrix} \in \mathbb{F}^{p\times r}, \qquad r \leqslant \min(p-m, q).$$

If $B \in \mathbb{F}^{p\times q}$ and it has rank equal to $r$, then there exist nonsingular matrices $P$ and $Q$ such that

$$P[D_r, 0]Q = B.$$

Let $A_1 = P^{-1}AP$. As $A_1$ and $A$ are similar matrices, they have the same invariant factors. So, if we denote the invariant factors of $[\lambda I_p - A_1, -D_r]$ by $\varepsilon_1 :> \cdots :> \varepsilon_p$, then, by Lemma 2.6,

$$\varepsilon_i :> \alpha_i :> \varepsilon_{i+r}, \qquad i = 1, \ldots, p. \tag{2.5}$$

On the other hand,

$$P[\lambda I_p - A_1, -D_r, 0]\begin{bmatrix} P^{-1} & 0 \\ 0 & Q \end{bmatrix} = [\lambda I_p - A, -B].$$

So $\varepsilon_i = \mu_i$, $i = 1, \ldots, p$, and therefore (2.3) follows from (2.5).

Let us now assume that (2.3) is verified. From Lemma 2.6, there exists $B_1(\lambda) \in \mathbb{F}[\lambda]^{p\times r}$ such that $[\lambda I_p - A, B_1(\lambda)]$ has $\mu_1,\dots,\mu_p$ as invariant factors. As $\lambda I_p - A$ is regular, there exist matrices $S(\lambda) \in \mathbb{F}[\lambda]^{p\times r}$ and $\overline{B} \in \mathbb{F}^{p\times r}$ such that

$$B_1(\lambda) = (\lambda I_p - A)S(\lambda) - \overline{B}.$$

Hence, $\mu_1,\dots,\mu_p$ are the invariant factors of

$$\left[\lambda I_p - A, B_1(\lambda)\right]\begin{bmatrix} I_p & -S(\lambda) \\ 0 & I_r \end{bmatrix} = \left[\lambda I_p - A, -\overline{B}\right].$$

If rank $\overline{B} = r$, setting $B = [\overline{B}, 0] \in \mathbb{F}^{p\times q}$, we have that $[\lambda I_p - A, -B]$ is the required matrix. We assume now that rank $\overline{B} = t < r$. By Lemma 2.3, $[A, \overline{B}]$ is $P,Q$-equivalent to a matrix $[A_c, B_c]$ with the form

$$A_c = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad B_c = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \in \mathbb{F}^{p\times r},$$

where $A_c = PAP^{-1}$, $B_c = P\overline{B}Q$, and $(A_1, B_1)$ is a completely controllable pair. (If $\mu_i = 1$ for all $i$, then by Corollary I of Theorem 2.11 of [14], $(A_c, B_c)$ is completely controllable and $A_c = A_1$ and $B_c = B_1$.) From Lemma 2.1 $[\lambda I_p - A, -\overline{B}]$ and $[\lambda I_p - A_c, -B_c]$ are equivalent $\lambda$-matrices and their nontrivial invariant factors are those of $A_3 \in \mathbb{F}^{(p-s)\times(p-s)}$, where $s$ is the order of the square matrix $A_1$. As $d(\mu_1 \dots \mu_p) = m$, it follows that $p - s = m$.

Since $B_c = P\overline{B}Q$ we have that $t = \operatorname{rank} \overline{B} = \operatorname{rank} B_c = \operatorname{rank} B_1$, and, from Lemma 2.3, we know that $B_1$ has its first $t$ columns linearly independent. Let us denote them by $b_1,\dots,b_t$, and let $b_{t+1},\dots,b_r$ be $r-t$ columns such that

$$\overline{B}_1 = [b_1,\dots,b_r]$$

is a nonsingular matrix. It is clear that $(A_1, \overline{B}_1)$ is a completely controllable pair and rank $\overline{B}_1 = r$. By setting

$$\overline{B}_c = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{B} = P^{-1}B_c,$$

we have that $[A_c, \overline{B}_c]$ and $[A, \tilde{B}]$ are $P$-equivalent, and by Lemma 2.4 the nontrivial invariant factors of $[\lambda I_p - A_c, \ -\overline{B}_c]$ are those of $\lambda I_m - A_3$. Thus, $[\lambda I_p - A, \ -\tilde{B}]$ has $\mu_1, \ldots, \mu_p$ as invariant factors. So $B = [\tilde{B}, 0] \in \mathbb{F}^{p\times q}$ is the required matrix.

(ii): If $q \leqslant p - m$, then by (2.4) and part (i) we can obtain a matrix $B$ such that rank $B = q$. And if $p - m < q$, then the matrix $B_1$ in part (i) has rank less than or equal to $p - m$. Using the same procedure as in that part, we can obtain a matrix $\overline{B}_1$ whose rank is $p - m$. Then the proof follows as in part (i).

(iii): This part is an immediate consequence of (i) and (ii). ■

Now we need some results related to the Hardy-Littlewood-Polya majorization inequalities. We begin with

LEMMA 2.9 (Lemma 4.3 of [14]). *Let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a column degree dominant matrix. Let $d_i$ be the degree of its ith column, and let $m_1, \ldots, m_n$ be positive integers. If*

$$(m_1, \ldots, m_n) \prec (d_1, \ldots, d_n), \tag{2.6}$$

*then there exists a column degree dominant matrix $A'(\lambda)$ equivalent to $A(\lambda)$ with $m_i$ as the degree of its ith column.*

LEMMA 2.10. *Let $x_1, \ldots, x_q$ be nonnegative integers such that $\sum_{i=1}^{q} x_i = q + m$, where $m$ is a positive integer. If $m = tq + r$, $0 \leqslant r < q$, then*

$$(t+2, \ldots, t+2, t+1, \ldots, t+1) \prec (x_1, \ldots, x_q),$$

*where $t+2$ appears $r$ times.*

*Proof.* It is clear that $r(t+2) + (q-r)(t+1) = m + q = \sum_{i=1}^{q} x_i$. We can assume, without the loss of generality, that $x_1 \geqslant \cdots \geqslant x_q$. Firstly, we assume that $r > 0$. Now $x_1 \geqslant t+2$, because if $x_1 < t+2$ then $x_i < t+2$, $i = 1, \ldots, q$, and we come at the following contradiction:

$$\sum_{i=1}^{q} x_i \leqslant q(t+1) < qt + r + q = m + q.$$

If there is $k$, $1 < k \leqslant r$, such that $\sum_{i=1}^{k-1} x_i \geqslant (k-1)(t+2)$ but $\sum_{i=1}^{k} x_i < k(t +$

2), then $x_k \leqslant t+1$ and therefore $x_i \leqslant t+1$, $i=k+1,\dots,q$. So

$$\begin{aligned}\sum_{i=1}^{q} x_i &= \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{q} x_i \\ &< k(t+2)+(q-k)(t+1) = k+qt+q \leqslant qt+r+q = m+q,\end{aligned}$$

which contradicts the hypothesis. Hence $\sum_{i=1}^{k} x_i \geqslant k(t+2)$, $k=1,\dots,r$.

Finally, if there is $k$, $1 \leqslant k \leqslant q-r$, such that $\sum_{i=1}^{r+k-1} x_i \geqslant r(t+2)+(k-1)(t+1)$ but $\sum_{i=1}^{r+k} x_i < r(t+2)+k(t+1)$, then $x_{r+k} < t+1$ and $x_i \leqslant t$ for $i=r+k,\dots,q$. So

$$\begin{aligned}\sum_{i=1}^{q} x_i &= \sum_{i=1}^{r+k} x_i + \sum_{i=r+k+1}^{q} x_i \\ &< r(t+2)+k(t+1)+(q-r-k)t \\ &= tq+r+k+r = m+k+r \leqslant m+q,\end{aligned}$$

which is again a contradiction. Therefore $\sum_{i=1}^{r+k} x_i \geqslant r(t+2)+k(t+1)$, $k=1,\dots,q-r$.

A similar procedure enables us to prove that if $r=0$ then $\sum_{i=1}^{k} x_i \geqslant k(t+1)$ for $k=1,\dots,q-1$, and $\sum_{i=1}^{q} x_i = q(t+1) = m+q$. ■

LEMMA 2.11 (Theorem 5.1 of [14]). *Let $A \in \mathbb{F}^{p\times p}$, $B \in \mathbb{F}^{p\times q}$, $G \in \mathbb{F}^{n\times n}$, and $n=p+q$. Let us assume that* rank $B=r$. *Let $\mu_1 :> \cdots :> \mu_p$ and $\gamma_1 :> \cdots :> \gamma_n$ be the invariant factors of $[\lambda I_p - A, -B]$ and $\lambda I_n - G$, respectively. Let $k_i \geqslant \cdots \geqslant k_r > k_{r+1} = \cdots = k_q$ $(=0)$ be the controllability indices of $(A,B)$. Then there exist matrices $C \in \mathbb{F}^{q\times p}$ and $D \in \mathbb{F}^{q\times q}$ such that $G$ is similar to*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

*if and only if the following relations hold:*

$$\gamma_i :> \mu_i :> \gamma_{i+q}, \qquad i=1,\dots,p, \tag{2.6}$$

$$(k_1+1,\dots,k_q+1) \prec (d(\sigma_q),\dots,d(\sigma_1)), \tag{2.7}$$

*where $\sigma_j = \beta^j/\beta^{j-1}$, $j=1,\dots,q$; $\beta^j = \beta_1^j \cdots \beta_{p+j}^j$ and $\beta_i^j = \mathrm{lcm}(\mu_{i-j},\gamma_i)$, $i=1,\dots,p+j$, $j=0,1,\dots,q$.*

LEMMA 2.12 (Corollary I of Theorem 5.1 of [14]). *With the same notation as in Lemma* 2.11, *if* $B = 0$ *then there exist* $C \in \mathbb{F}^{q\times p}$ *and* $D \in \mathbb{F}^{q\times q}$ *such that* $G$ *is similar to*

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

*if and only if the relations* (2.6) *hold.*

## 3. MAIN RESULT

THEOREM 3.1. *Let* $A \in \mathbb{F}^{p\times p}$, *and let* $\alpha_1 :> \cdots :> \alpha_p$ *be its invariant factors. Let* $\mu_1 :> \cdots :> \mu_p$ *be* $p$ *monic polynomials, and let* $k_1 \geqslant \cdots \geqslant k_r$ *be* $r$ *positive integers. Then there exists a matrix* $B \in \mathbb{F}^{p\times q}$ *with* $\operatorname{rank} B = r$ *and such that* $\mu_1, \dots, \mu_p$ *are the invariant factors of* $[\lambda I_p - A, \ -B]$ *and* $k_1, \dots, k_r$ *are the nonzero controllability indices of* $(A, B)$ *if and only if the following relations hold*:

$$\alpha_i :> \mu_{i+r} :> \alpha_{i+r}, \qquad i = 1, \dots, p - r, \tag{3.1}$$

$$(k_1, \dots, k_r) \prec (d(\theta_r), \dots, d(\theta_1)), \tag{3.2}$$

*where* $\theta_j = \varepsilon^j / \varepsilon^{j-1}$, $j = 1, \dots, r$; $\varepsilon^j = \varepsilon_1^j \dots \varepsilon_{p-r+j}^j$, *and* $\varepsilon_i^j = \operatorname{lcm}(\mu_{i+r-j}, \alpha_i)$, $i = 1, \dots, p - r + j$, $j = 0, 1, \dots, r$.

*Proof.* Let us assume that $d(\mu_1 \cdots \mu_p) = m$.

(1) *Necessity.* From Lemma 2.8, (3.1) is obviously necessary, and it is easy to see (Lemma 4.4 of [14]) that we can prove (3.2) is a necessary condition by showing that the following relations are verified:

$$\sum_{i=1}^{j} d(\theta_i) \leqslant \sum_{i=1}^{j} k_{r-i+1}, \qquad j = 1, \dots, r-1, \tag{3.3}$$

$$\sum_{i=1}^{r} d(\theta_i) = \sum_{i=1}^{r} k_i. \tag{3.4}$$

Firstly we are going to see that (3.4) is necessary. In fact,

$$\sum_{i=1}^{r} d(\theta_i) = d(\varepsilon^r) - d(\varepsilon^0) = d(\alpha_1 \cdots \alpha_p) - d(\mu_{r+1} \cdots \mu_p).$$

But as rank $B = r$, we have that $\mu_1 = \cdots = \mu_r = 1$, and therefore $\sum_{i=1}^{r} d(\theta_i) = p - m$. On the other hand, by Lemma 2.3, $[A, B]$ is $P, Q$-equivalent to a matrix $[A_c, B_c]$ with the form

$$A_c = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad B_c = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where $A_1 \in \mathbb{F}^{s \times s}$ and $s = k_1 + \cdots + k_r$. Since $m = p - s$ (see, for example Theorem 2.11 of [14] or the proof of Lemma 2.8 above), we have $k_1 + \cdots + k_r = p - m$ and (3.4) follows.

Moreover, if $r = p - m$ then $k_i = 1$, $i = 1, \ldots, r$, and therefore (3.2) is obviously verified. Hence we are going to assume that $r < p - m$. By Lemma 2.5, $[\lambda I_p - A_c, -B_c]$, and therefore $[\lambda I_p - A, -B]$, is equivalent to a $\lambda$-matrix with the form

$$\begin{bmatrix} I_{p-m-r} & 0 & 0 & 0 & 0 \\ 0 & \lambda I_m - A_3 & 0 & 0 & 0 \\ 0 & C & T(\lambda) & I_r & 0 \end{bmatrix}, \tag{3.5}$$

where $T(\lambda)$ is a column degree dominant matrix such that the degree of its $i$th column is $k_i$, and $\lambda I_p - A_c$ is equivalent to the submatrix of (3.5) formed by its first $p$ rows and columns. As $A$ and $A_c$ are similar, we conclude that $\alpha_1 :> \cdots :> \alpha_p$ are the invariant factors of

$$\begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - A_3 & 0 \\ 0 & C & T(\lambda) \end{bmatrix}$$

Let $P$ be the permutation matrix such that $T_1(\lambda) = PT(\lambda)P^T$ has as columns and rows the columns and rows of $T(\lambda)$ but placed in reverse order. Hence, $T_1(\lambda)$ is a column degree dominant matrix such that the degree of its

$i$th column is $k_{r-i+1}$, $i=1,\dots,r$. So $\alpha_1,\dots,\alpha_p$ are also the invariant factors of

$$A(\lambda)=\begin{bmatrix} I_{p-r} & 0 \\ 0 & P \end{bmatrix}\begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - A_3 & 0 \\ 0 & C & T(\lambda) \end{bmatrix}\begin{bmatrix} I_{p-r} & 0 \\ 0 & P^T \end{bmatrix}$$

$$=\begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - A_3 & 0 \\ 0 & PC & T_1(\lambda) \end{bmatrix}.$$

We denote by $A_j(\lambda)$ the submatrix of $A(\lambda)$ formed by its first $p+j-r$ rows and columns, and by $\tau_1^j :> \cdots :> \tau_{p+j-r}^j$ the invariant factors of $A_j(\lambda)$, $j=1,\dots,r$. Consider

$$\begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - A_3 & 0 \end{bmatrix}, \tag{3.6}$$

the submatrix of $A_j(\lambda)$ formed with its first $p-r$ rows and $p-r+j$ columns. Since the nontrivial invariant factors of $[\lambda I_p - A_c, -B_c]$ are those of $\lambda I_m - A_3$, we can conclude that $\mu_{p-m+1},\dots,\mu_p$ are the invariant factors of $\lambda I_m - A_3$, $\mu_{r+1},\dots,\mu_p$ are those of the matrix in (3.6), and $\mu_i = 1$ for $i=r+1,\dots,p-m$. So, by Lemma 2.6,

$$\tau_i^j :> \mu_{i+r} :> \tau_{i+j}^j, \qquad i=1,\dots,p-r, \quad j=0,1,\dots,r.$$

And as $\mu_i = 1$ for $i \leqslant r$, we can put

$$\mu_{r+i-j} :> \tau_i^j, \qquad i=1,\dots,p-r+j, \quad j=0,1,\dots,r.$$

Since $A_j(\lambda)$ is a principal submatrix of $A(\lambda)$, by Lemma 2.6 we have that

$$\alpha_i :> \tau_i^j, \qquad i=1,\dots,p-r+j, \quad j=0,1,\dots,r.$$

Therefore $\tau_i^j$ is a common multiple of $\alpha_i$ and $\mu_{r+i-j}$. Since $\varepsilon_i^j = \operatorname{lcm}(\mu_{r+i-j},\alpha_i)$, we can conclude that

$$\varepsilon_i^j :> \tau_i^j, \qquad i=1,\dots,p-r+j, \quad j=0,1,\dots,r. \tag{3.7}$$

On the other hand, it is clear that

$$d\left(|A_j(\lambda)|\right) = m + \sum_{i=1}^{j} k_{r-i+1}, \qquad j = 1,\dots,r, \tag{3.8}$$

and

$$d\left(|A_j(\lambda)|\right) = \sum_{i=1}^{p-r+j} d\left(\tau_i^j\right), \qquad j = 1,\dots,r. \tag{3.9}$$

From (3.7)

$$\sum_{i=1}^{p-r+j} d\left(\varepsilon_i^j\right) \leqslant \sum_{i=1}^{p-r+j} d\left(\tau_i^j\right), \qquad j = 1,\dots,r.$$

So, from (3.8), we can write

$$d(\varepsilon^j) \leqslant m + \sum_{i=1}^{j} k_{r-i+1}, \qquad j = 1,\dots,r. \tag{3.10}$$

As $m = d(\mu_{r+1}\cdots\mu_p) = d(\varepsilon^0)$ and $d(\varepsilon^j) - d(\varepsilon^0) = \sum_{i=1}^{j} d(\theta_i)$, (3.3) follows from (3.10).

(2) *Sufficiency.* Now we assume that (3.1) and (3.2) are verified. We put $\nu_i = \mu_{r+i}$, $i = 1,\dots,p-r$. Hence (3.1) can be written as

$$\alpha_i :> \nu_i :> \alpha_{i+r}, \qquad i = 1,\dots,p-r. \tag{3.1$'$}$$

Assume that $\mu_1 = \cdots = \mu_t = 1$ and $d(\mu_{t+1}) \geqslant 1$, and put $N = \operatorname{diag}(M_1,\dots,M_{p-t})$, where $M_i$ is the companion matrix of $\mu_{t+i}$, $i = 1,\dots,p-t$. Thus $N \in \mathbb{F}^{m\times m}$ and $\mu_{p-m+1},\dots,\mu_p$ are its invariant factors.

(i) Assume $p - m = r$. This means that $k_i = 1$ for $i = 1,\dots,r$. By (3.1$'$) and Lemma 2.12, there exist matrices $C, D$ such that

$$\begin{bmatrix} N & 0 \\ -C & D \end{bmatrix}$$

has $\alpha_1,\dots,\alpha_p$ as invariant factors. Therefore, this matrix is similar to $A$, and $\lambda I_r - D$ is a column degree dominant matrix such that the degree of its $i$th column is $k_i$, $i=1,\dots,r$. From Lemma 2.5

$$\begin{bmatrix} \lambda I_m - N & 0 & 0 & 0 \\ C & \lambda I_r - D & I_r & 0 \end{bmatrix} \tag{3.11}$$

is equivalent to a matrix $[\lambda I_p - A_c, -B_c]$ where $(A_c, B_c)$ has $k_1,\dots,k_r$ as nonzero controllability indices and rank $B_c = r$.

It is easy to compute the invariant factors of the matrix in (3.11). We find that they are $\mu_1,\dots,\mu_p$. Thus, these are the invariant factors of $[\lambda I_p - A_c, -B_c]$. Also, by Lemma 2.5, $\lambda I_p - A_c$ is equivalent to

$$\begin{bmatrix} \lambda I_m - N & 0 \\ C & \lambda I_r - D \end{bmatrix}.$$

So $A$ and $A_c$ are similar matrices. Thus, there exists $P \in \mathrm{Gl}_p(\mathbb{F})$ such that $A = PA_cP^{-1}$. By setting $B = PB_c$ we have that $[A, B]$ and $[A_c, B_c]$ are $P$-equivalent, and, consequently $[\lambda I_p - A, -B]$ has $\mu_1,\dots,\mu_p$ as invariant factors and $(A, B)$ has $k_i = 1$, $i=1,\dots,r$ as nonzero controllabilty indices.

(ii) Assume $p - m > r$. In this case it is clear that the invariant factors of $\mathrm{diag}(I_{p-m-r}, \lambda I_m - N)$ are $\mu_{r+1},\dots,\mu_p$. Hence, according to (3.1′) and Lemma 2.7, there exist matrices $C_1 \in \mathbb{F}^{r\times m}$ and $T_1(\lambda) \in \mathbb{F}^{r\times r}$ such that $\alpha_1,\dots,\alpha_p$ are the invariant factors of

$$\begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - N & 0 \\ 0 & C_1 & T_1(\lambda) \end{bmatrix}. \tag{3.12}$$

Moreover, $T_1(\lambda)$ is a column degree dominant triangular matrix such that the degree of its $i$th column is $d(\theta_i)$, $i=1,\dots,r$. By (3.2) and Lemma 2.9, $T_1(\lambda)$ is equivalent to a degree dominant matrix $T(\lambda)$ such that the degree of its $i$th column is $k_i$, $i=1,\dots,r$. So there exist unimodular matrices $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$ such that $T(\lambda) = P(\lambda)T_1(\lambda)Q(\lambda)$. Hence, the matrix in (3.12) and

$$\begin{bmatrix} I_{p-r} & 0 \\ 0 & P(\lambda) \end{bmatrix} \begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - N & 0 \\ 0 & C_1 & T_1(\lambda) \end{bmatrix} \begin{bmatrix} I_{p-r} & 0 \\ 0 & Q(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - N & 0 \\ 0 & P(\lambda)C_1 & T(\lambda) \end{bmatrix}$$

are equivalent. As $\lambda I_m - N$ is regular, there exist $S(\lambda) \in \mathbb{F}[\lambda]^{r\times m}$ and $C \in \mathbb{F}^{r\times m}$ such that

$$P(\lambda)C_1 = S(\lambda)(\lambda I_m - N) + C.$$

Then, $\alpha_1, \dots, \alpha_p$ are the invariant factors of

$$\begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & -S(\lambda) & I_r \end{bmatrix} \begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - N & 0 \\ 0 & P(\lambda)C_1 & T(\lambda) \end{bmatrix}$$
$$= \begin{bmatrix} I_{p-m-r} & 0 & 0 \\ 0 & \lambda I_m - N & 0 \\ 0 & C & T(\lambda) \end{bmatrix}.$$

According to Lemma 2.5 and its Remark,

$$\begin{bmatrix} I_{p-m-r} & 0 & 0 & 0 & 0 \\ 0 & \lambda I_m - N & 0 & 0 & 0 \\ 0 & C & T(\lambda) & I_r & 0 \end{bmatrix}$$

is equivalent to a matrix $[\lambda I_p - A_c, -B_c]$ where $(A_c, B_c)$ has $k_1, \dots, k_r$ as nonzero controllability indices and rank $B_c = r$. Now, the proof of the theorem follows as in case (i). ∎

Corollary I. *Let* $A \in \mathbb{F}^{p\times p}$ *and* $\alpha_1 :> \cdots :> \alpha_p$ *be its invariant factors. There exists* $B \in \mathbb{F}^{p\times q}$ *such that* $(A, B)$ *is completely controllable with* $k_1 \geqslant \cdots \geqslant k_r\ (>0)$ *as nonzero controllability indices and* rank $B = r$ *if and only if the following conditions hold*:

$$\alpha_i = 1, \qquad i = 1, \dots, p-r, \tag{3.13}$$

$$(k_1, \dots, k_r) \prec \left(d(\alpha_p), \dots, d(\alpha_{p-r+1})\right). \tag{3.14}$$

*Proof.* $(A, B)$ is completely controllable if and only if the invariant factors of $[\lambda I_p - A, -B]$ are all equal to 1 (Corollary I of Theorem 2.11 of [14]); i.e., $\mu_i = 1$, $i = 1, \dots, p$. So, in this case (3.1) and (3.2) are equivalent to (3.13) and (3.14), respectively. ∎

COROLLARY II. *Let $\chi$ be a monic polynomial of degree $p$. Then there exist matrices $A \in \mathbb{F}^{p\times p}$ and $B \in \mathbb{F}^{p\times q}$ such that $\chi$ is the characteristic polynomial of $A$,* rank $B = r$, *$[\lambda I_p - A, -B]$ has $\mu_1 :> \cdots :> \mu_p$ as invariant factors, and $(A, B)$ has $k_1 \geqslant \cdots \geqslant k_r$ $(>0)$ as nonzero controllability indices, if and only if the following conditions hold:*

$$\prod_{i=1}^{p-r} \mu_{i+r} :> \chi, \tag{3.15}$$

$$\sum_{i=1}^{r} k_i = p - m, \tag{3.16}$$

*where $m = d(\mu_1 \cdots \mu_p)$.*

*Proof.* If $\alpha_1 :> \cdots : \alpha_p$ are the invariant factors of $A$, then (3.1) is verified, and so (3.15) follows. Moreover, (3.16) follows immediately from (3.2).

Conversely, if (3.16) is verified, then $p - m \geqslant r$ because $k_i \geqslant 1$; and since $p \geqslant m = d(\mu_1 \cdots \mu_p)$, we have that $\mu_{p-m} = 1$. So $\mu_r = 1$. If (3.15) holds, then there exists $\nu \in \mathbb{F}[\lambda]$ such that

$$\chi = \nu \prod_{i=1}^{p-r} \mu_{i+r}.$$

We define $\alpha_i = \mu_i$ for $i = 1, \ldots, p-1$ and $\alpha_p = \mu_p \nu$.

Suppose that $\mu_1 = \cdots = \mu_t = 1$, $d(\mu_{t+1}) \geqslant 1$, and set $A = \operatorname{diag}(A_1, \ldots, A_{p-t})$, where $A_i$ is the companion matrix of $\alpha_{t+i}$, $i = 1, \ldots, p-t$. It is clear that $\chi$ is the characteristic polynomial of $A$. Moreover

$$\alpha_i :> \mu_{i+r} :> \alpha_{i+r}, \qquad i = 1, \ldots, p-r, \tag{3.17}$$

and by defining $\varepsilon_i^j$ as in Theorem 3.1, it is easy to see that for $i = 1, \ldots, p - r + j$ and $j = 0, 1, \ldots, r-1$, $\alpha_i :> \mu_{i+r-j}$ and $\varepsilon_i^j = \mu_{i+r-j}$. So $\varepsilon^j = \mu_{r-j+1} \cdots \mu_p$ for $j = 0, 1, \ldots, r-1$. Since $\mu_i = 1$ for $i \leqslant r$, we have that $d(\varepsilon^j) = m$ for $j = 0, 1, \ldots, r-1$. Hence

$$(d(\theta_r), \ldots, d(\theta_1)) = (p - m, 0, \ldots, 0).$$

And as $k_1 + \cdots + k_r = p - m$, then

$$(k_1, \ldots, k_r) \prec (d(\theta_r), \ldots, d(\theta_1)) \tag{3.18}$$

is always true. Now, the existence of $B$ verifying the prescribed requirements follows from (3.17), (3.18), and Theorem 3.1. ■

We are now ready to give a new proof of the Sà-Thompson interlacing inequalities.

THEOREM 3.2 (Interlacing inequalities for invariant factors: Marques de Sà [9–10], R. C. Thompson [11]). *Let $A \in \mathbb{F}^{p\times p}$, $G \in \mathbb{F}^{n\times n}$, and $n = p + q$. Let $\alpha_1 :> \cdots :> \alpha_p$, and $\gamma_1 :> \cdots :> \gamma_n$ be the invariant factors of $A$ and $G$, respectively. Then there exist matrices $B \in \mathbb{F}^{p\times q}$, $C \in \mathbb{F}^{q\times p}$, and $D \in \mathbb{F}^{q\times q}$ such that $G$ is similar to*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

*if and only if the following condition holds:*

$$\gamma_i :> \alpha_i :> \gamma_{i+2q}, \qquad i = 1,\ldots,p, \tag{3.19}$$

*where we agree that $\gamma_i = 0$ for $i > n$.*

*Proof.* The necessity of (3.19) is an immediate consequence of Lemma 2.6. We are going to see that (3.19) is a sufficient condition. We begin by defining $p$ polynomials in the following way:

$$\mu_i = \operatorname{lcm}(\alpha_{i-q}, \gamma_i) \qquad i = 1,\ldots,p.$$

These polynomials verify the following conditions (as one can see easily):

$$\mu_i :> \mu_{i+1}, \qquad i = 1,\ldots,p-1, \tag{3.20}$$

$$\mu_i :> \alpha_i :> \mu_{i+q}, \quad i = 1,\ldots,p \qquad (\mu_i := 0 \quad \text{for } i > p), \tag{3.21}$$

$$\gamma_i :> \mu_i :> \gamma_{i+q}, \quad i = 1,\ldots,p. \tag{3.22}$$

Set $d(\mu_1 \cdots \mu_p) = m$ and $r = \min(p - m, q)$. According to (3.21) and (iii) of Lemma 2.8, we have that

$$\alpha_i :> \mu_{i+r} :> \alpha_{i+r}, \qquad i = 1,\ldots,p-r. \tag{3.23}$$

If $r = p - m$, then we put $k_i = 1$, $i = 1,\dots,r$ and it is clear that

$$(k_1,\dots,k_r) \prec (d(\theta_r),\dots,d(\theta_1)), \tag{3.24}$$

where $\theta_j$ is defined as in Theorem 3.1.

If $r = q$, then we divide $(p-m) - r$ by $r$:

$$(p-m) - r = tr + s, \qquad 0 \leqslant s < r. \tag{3.25}$$

Putting $k_i = t+2$ for $i = 1,\dots,s$ and $k_i = t+1$ for $i = s+1,\dots,r$, from Lemma 2.10, we have that

$$(k_1,\dots,k_r) \prec (d(\theta_r),\dots,d(\theta_1)). \tag{3.24$'$}$$

From (3.23) and either (3.24) or (3.24′), and by applying Theorem 3.1, there exists $B \in \mathbb{F}^{p\times q}$ such that $[\lambda I_p - A, -B]$ has $\mu_1 :> \cdots :> \mu_p$ as invariant factors and $(A, B)$ has $k_1,\dots,k_r$ as nonzero controllability indices.

If $r = p - m$, we put $k_{r+1} = \cdots = k_q = 0$, and since $p - m = 0q + (p - m)$, Lemma 2.10 enables us to affirm that in this case

$$(k_1+1,\dots,k_q+1) \prec (d(\sigma_q),\dots,d(\sigma_1)), \tag{3.26}$$

where $\sigma_j$ is defined as in Lemma 2.11.

And if $r = q$, then, from (3.25), $p - m = (t+1)q + s$. Again by Lemma 2.10 we have

$$(k_1+1,\dots,k_q+1) \prec (d(\sigma_q),\dots,d(\sigma_1)). \tag{3.26$'$}$$

Now, the proof of the sufficiency follows from (3.22), from either (3.26) or (3.26′), and by applying Lemma 2.11. ■

## 4. FINAL REMARK

In this paper we have proceeded according to the following idea: In [12] Wimmer solved the problem of the existence of a matrix with some prescribed rows (or columns) and prescribed characteristic polynomial, and he used the solution to give a new proof of an earlier result of de Oliveira [6] related to the existence of a matrix with prescribed characteristic polynomial and a prescribed principal submatrix. Improving Wimmer's result, in [14] we solved the same problem but when the invariant factors are prescribed instead of the characteristic polynomial. It seems natural to think that the solution of this problem might serve to give a new proof of the interlacing inequalities for the invariant factors of a matrix over a field and those of a prescribed principal submatrix. We think that the proof we provide in this paper gives some insight into the relation between the invariant factors prescription problems and linear control theory.

Finally, we remark that by means of our method the convexity problems which Marques de Sà and Thompson had to solve are subsumed in the Hardy-Littlewood-Polya majorization inequalities which appear in the main theorem of [14] and in this paper. So it has not been necessary to make any explicit reference to convexity in our proofs.

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