Analytic Solution with \textit{a priori} Error Bounds for a Class of Mixed Coupled Partial Differential Equations

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Abstract—In this paper, the existence of a sequence of stable solutions $X_n$ of the parametric matrix equation $Z^2 + AZ - n^2B = 0$ so that $\{X_n/n\}$ converges to a stable matrix is studied. The result is used for solving mixed problems related to the coupled second-order partial differential system $u_{tt} + Au_{xx} + Bu_t = 0$.

Keywords—Algebraic matrix equation, Stable block diagonalization, Partial differential system, Approximate solution, \textit{a priori} error bound.

1. INTRODUCTION

Systems of coupled second-order partial differential equations appear in many different fields such as magnetohydrodynamic flows [1], diffusion problems [2–4], mechanics [5,6], nerve conduction problems [7], armament models [8], etc.

Discrete numerical methods for the study of such problems are widely studied in the literature [9,10]; however, the analytic solution may satisfy an important physical property and the numerical solution may not. This motivates the search of the analytic solution of the problem.

In this paper, we consider coupled second-order partial differential systems of the form

\begin{equation}
  u_{tt}(x,t) + Bu_{xx}(x,t) + Au_t(x,t) = 0, \quad -b < x < b, \quad t > 0, \quad (1.1)
\end{equation}

\begin{equation}
  u(-b,t) = u(b,t) = 0, \quad t > 0, \quad (1.2)
\end{equation}

\begin{equation}
  u(x,0) = f(x), \quad -b \leq x \leq b, \quad (1.3)
\end{equation}

where $f(x)$ and the unknown $u(x,t)$ are $m$-dimensional vector functions and $A, B$ are square complex matrices in $\mathbb{C}^{m \times m}$ such that

\begin{equation}
  \sigma(B) \cap \{-\infty, 0\} = \emptyset, \quad (1.4)
\end{equation}

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where \( \sigma(B) \) denotes the set of all eigenvalues of \( B \). Note that if matrices \( A \) and \( B \) are not simultaneously diagonalizable, the above problem cannot be transformed into a set of scalar independent equations.

This paper is organized as follows. In Section 2, we prove the existence of a sequence of solutions \( \{Z_n\} \) of the algebraic matrix equation

\[
Z^2 + AZ - n^2B = 0,
\]

satisfying the properties

\[
\text{Re } z < 0, \quad \text{for } z \in \sigma(Z_n), \quad \text{and } \text{Re } w < 0, \quad \text{for } w \in \sigma(W),
\]

\[
\lim_{n \to \infty} \frac{Z_n}{n} = W,
\]

where \( W \) is a square root of \( B \left( \frac{n}{2k} \right)^2 \). Apart from this fact, a constructive method for solving equations of the type

\[
Z^2 + A_1Z + A_0 = 0
\]

is proposed. In Section 2, we also introduce the concept of full rank cosolution of equation (1.8) that will be used below to construct explicit solutions of the matrix differential equation

\[
T'(t) + A_1T(t) + A_0T(t) = 0
\]

in terms of matrix exponentials, even in the case where the matrix equation (1.8) can be unsolvable. In Section 3, we propose a matrix separation of variables to the problem (1.1)-(1.3) constructing an exact series solution of the problem involving matrix exponentials of solutions \( Z_k \) of the equation

\[
Z^2 + AZ - \left( \frac{k\pi}{2b} \right)^2 B = 0,
\]

having the spectrum \( \sigma(Z_k) \) in the left half-plane, as well as matrix exponentials related to an appropriate set of full rank cosolutions of (1.10), for a finite number of positive integers \( k \). Given an admissible error \( \epsilon > 0 \) and fixed domain \( R(t_0, t_1) = [-b, b] \times [t_0, t_1] \) with \( t_1 > t_0 > 0 \), we construct an analytic approximate solution of the problem whose error with respect to the exact infinite series solution is smaller than \( \epsilon \), uniformly in \( R(t_0, t_1) \).

Throughout this paper, we say that a matrix \( Q \) in \( \mathbb{C}^{m \times m} \) is stable if its spectrum \( \sigma(Q) \) lies in the left half-plane \( \text{Re } z < 0 \). If \( P \in \mathbb{C}^{m \times n} \), we denote by \( \|P\| \) the 2-norm of \( P \), defined by

\[
\|P\| = \sup_{y \neq 0} \frac{\|Py\|_2}{\|y\|_2},
\]

where for a vector \( y \) in \( \mathbb{C}^n \), \( \|y\|_2 = (y^T y)^{1/2} \) is the usual Euclidean norm of \( y \). We denote by \( \|P\|_F \) its Frobenius norm defined by

\[
\|P\|_F = \left( \sum_{i=1}^m \sum_{j=1}^n |p_{ij}|^2 \right)^{1/2},
\]

and we recall that from [11, p. 57], one gets

\[
\|P\| \leq \|P\|_F \leq n^{1/2}\|P\|.
\]

If \( S \) is a matrix in \( \mathbb{C}^{m \times n} \), we denote by \( S^\dagger \) its Moore-Penrose pseudoinverse. An account of properties, examples, and applications of \( S^\dagger \) is found in [12]. In particular, \( S^\dagger \in \mathbb{C}^{n \times m} \) and \( SS^\dagger \) is the orthogonal projector of \( \mathbb{C}^m \) onto the range of \( S \). Finally, from [13, Theorem 2.3.2, p. 24], the algebraic system

\[
Sy = c; \quad c \in \mathbb{C}^m
\]

is compatible if and only if \( SS^\dagger c = c \), and in this case a solution of (1.11) is given by \( y = S^\dagger c \).
2. ON SOLUTIONS AND COSOLUTIONS OF ALGEBRAIC MATRIX EQUATIONS

For the sake of clarity in the presentation, we recall the concept of cosolution introduced in [14] for a particular equation and developed in [15] for the general case. Let $A_0$ and $A_1$ be matrices in $\mathbb{C}^{m \times m}$ and let us consider the algebraic matrix equation

$$Z^2 + A_1Z + A_0 = 0. \quad (2.1)$$

**DEFINITION 2.1.** [15] We say that the pair $(X, T)$ is an $(m, r)$ cosolution of (2.1) if $X \in \mathbb{C}^{m \times r}, X \neq 0, T \in \mathbb{C}^{r \times r}$ and

$$XT^2 + A_1XT + A_0X = 0. \quad (2.2)$$

If $(X_i, T_i)$ is an $(m, r_i)$ cosolution of (2.1), $1 \leq i \leq k$, $r_1 + r_2 + \cdots + r_k = 2m$, we say that ${\{X_i, T_i; 1 \leq i \leq k\}}$ is a $k$-complete set of cosolutions of (2.1), if the block matrix $W = (W_{ij}) = (X_iT_j^{-1})$, $1 \leq i \leq 2, 1 \leq i \leq k$, is invertible in $\mathbb{C}^{2m \times 2m}$.

If $M_{ij} \in \mathbb{C}^{m \times r_j}, 1 \leq i \leq 2, 1 \leq j \leq k$ and $M = (M_{ij})$ is an invertible block matrix in $\mathbb{C}^{2m \times 2m}$, and $D_j \in \mathbb{C}^{r_j \times r_j}$ for $1 \leq j \leq k$ satisfy

$$M \text{ Diag } (D_1D_2\cdots D_k) = -A_0 - A_1 M, \quad (2.3)$$

then ${\{(M_{ij}, D_j), 1 \leq j \leq k\}}$ is a $k$-complete set of cosolutions of (2.1); see [15].

The following lemma provides a sufficient condition for the existence and a method for constructing solutions of (2.1).

**LEMMA 2.1.** Let ${\{(M_{ij}, D_j), 1 \leq j \leq k\}}$ be a $k$-complete set of cosolutions of equation (2.1) such that $M = (M_{ij})$ and $D = \text{Diag } (D_1D_2\cdots D_k)$ are matrices in $\mathbb{C}^{2m \times 2m}$ satisfying (2.3) where $M$ is invertible, $M_{ij} \in \mathbb{C}^{m \times r_j}, D_j \in \mathbb{C}^{r_j \times r_j}, r_1 + r_2 + \cdots + r_k = 2m$. If $r_1 = m$ and $M_{11}$ is invertible in $\mathbb{C}^{m \times m}$, then

$$Z = M_{11}D_1M_{11}^{-1} \quad (2.4)$$

is a solution of (2.1).

**PROOF.** The matrix $Z$ defined by (2.4) is a solution of (2.1) if and only if

$$\left(M_{11}D_1M_{11}^{-1}\right)^2 + A_1M_{11}D_1M_{11}^{-1} + A_0 = 0,$$

or

$$M_{11}D_1^2M_{11}^{-1} + A_1M_{11}D_1M_{11}^{-1} + A_0 = 0. \quad (2.5)$$

Postmultiplying equation (2.5) by $M_{11}$, one gets

$$M_{11}D_1^2 + A_1M_{11}D_1 + A_0M_1 = 0, \quad (2.6)$$

but (2.6) holds because $(M_{11}, D_1)$ is an $(n, m)$ cosolution of (2.1).

Now we recall the concept of minimal set of cosolutions recently given in [16].

**DEFINITION 2.2.** [16] Let $\Omega = \{(X_i, T_i); 1 \leq i \leq h\}$ be a set of rectangular cosolutions of (2.1). We define the rank of the set $\Omega$ as the rank of the matrix $[X_1, X_2, \ldots, X_h]$. We say that the set $\Omega$ is a minimal set of cosolutions of (2.1) if its rank is $m$ and any proper subset of cosolutions has a rank strictly smaller than $m$.

Note that minimal set of cosolutions always exist because any $k$-complete set of cosolutions of (2.1) provides a minimal set of cosolutions.
If \((X_i, T_i)\) is an \((m, r_i)\) cosolution of (2.1) such that \(\Omega = \{(X_i, T_i); 1 \leq i \leq h\}\) is a minimal set of cosolutions, and we consider the matrices

\[
X_0 = [X_1, X_2, \ldots, X_h], \quad T_0 = \text{Diag} [T_1, T_2, \ldots, T_h],
\]

then \((X_0, T_0)\) is an \((m, r)\) cosolution of (2.1) where \(r = r_1 + \cdots + r_k\), and the matrix \(X_0 = [X_1, X_2, \ldots, X_h] \in \mathbb{C}^{m \times r}\) has a full rank. This leads to the following concept that will be used in the following definition.

**Definition 2.3.** We say that an \((m, r)\) cosolution \((X_0, T_0)\) of equation (2.1) is a full rank cosolution if the rank of the matrix \(X_0\) is \(m\).

Note that if \((X_0, T_0)\) is a full rank cosolution of (2.1), then a solution of the problem

\[
X''(t) + A_1 X'(t) + A_0 X(t) = 0, \quad X(0) = C \in \mathbb{C}^m
\]

is given by

\[
X(t) = X_0 \exp(t T_0) v; \quad v = X_0^t C.
\]

Let us consider the parametrized algebraic matrix equation

\[
Z^2 + \lambda A Z - B = 0,
\]

where \(\lambda\) is a complex parameter, \(B\) is an invertible matrix in \(\mathbb{C}^{m \times m}\) satisfying the condition (1.4), and the unknown \(Z\) and \(A\) are matrices in \(\mathbb{C}^{m \times m}\). If \(D\) is the Jordan canonical form of \(B\), and \(P\) is an invertible matrix such that

\[
B = P D P^{-1}, \quad D = \text{Diag} \{D_1, \ldots, D_r\},
\]

where

\[
D_i = [\alpha_i]_{1 \times 1}, \quad \text{or} \quad D_i = \begin{bmatrix}
\alpha_i & 1 \\
\alpha_i & 1 \\
\vdots & \ddots & \ddots \\
\alpha_i & 1
\end{bmatrix} \in \mathbb{C}^{r_i \times r_i}, \quad 1 \leq i \leq r,
\]

then the matrix \(Z_0\) defined by

\[
Z_0 = PD^{1/2} P^{-1}, \quad D^{1/2} = \text{Diag} \left[ D_1^{1/2}, \ldots, D_r^{1/2} \right],
\]

where

\[
D_i^{1/2} = \sqrt{\alpha_i}, \quad \text{or} \quad D_i^{1/2} = \begin{bmatrix}
\sqrt{\alpha_i} & R_i \\
\sqrt{\alpha_i} & R_i \\
\vdots & \ddots & \ddots \\
\sqrt{\alpha_i} & R_i
\end{bmatrix}, \quad \text{Re} \sqrt{\alpha_i} < 0, \quad 1 \leq i \leq r,
\]

is a square root of \(B\) as well as \(-Z_0\). From (1.4), the pair \(\{Z_0, -Z_0\}\) is a complete pair of solutions of the equation

\[
Z^2 - B = 0,
\]

because \(Z_0\) is invertible [17], and satisfies

\[
\begin{bmatrix}
I & I \\
Z_0 & -Z_0
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
B & 0
\end{bmatrix} = \begin{bmatrix}
0 & I \\
Z_0 & -Z_0
\end{bmatrix}.
\]
From (2.11), we can write
\[ \tilde{M} \text{ Diag } (D(Z_0), -D(Z_0)) = \tilde{C} \tilde{M}, \] (2.14)
where
\[ \tilde{M} = \begin{bmatrix} P & P \\ Z_0 P & -Z_0 P \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}, \] (2.15)
and \( D(Z_0) \) is the Jordan form of the matrix \( Z_0 \). From (2.13)–(2.15), the companion matrix \( \tilde{C} \) has \( s \leq r \) eigenvalues \( \rho_1, \rho_2, \ldots, \rho_s \) with multiplicities \( m_1, m_2, \ldots, m_s \) and \( \sum_{i=1}^{s} m_i = m \), lying in the left half-plane \( \text{Re } z < 0 \), corresponding to the solution \( Z_0 \) of equation (2.12), and the eigenvalues \( -\rho_1, -\rho_2, \ldots, -\rho_s \) with multiplicities \( m_1, m_2, \ldots, m_s \), lying in the right half-plane \( \text{Re } z > 0 \), corresponding to the solution \(-Z_0\).

Note that the companion matrix \( \tilde{C}(\lambda) \) associated to (2.9) verifies
\[ \tilde{C}(\lambda) = \begin{bmatrix} 0 & I \\ B & -\lambda A \end{bmatrix}; \quad \tilde{C}(0) = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}. \] (2.16)

From the continuity of the eigenvalues of a matrix as functions of the elements of the matrix [18, p. 45], (2.16), and the previous comments, there exists a positive number \( \delta \) and \( m \) eigenvalues of \( \tilde{C}(\lambda) \), not necessarily distinct, contained in \( \text{Re } z < 0 \) and other \( m \) eigenvalues of \( \tilde{C}(\lambda) \) lying in \( \text{Re } z > 0 \), for every \( \lambda \) with \( |\lambda| < \delta \).

Thus, there are relative prime monic polynomials \( p_1(z, \lambda) \) and \( p_2(z, \lambda) \) of degree \( m \) in \( z \) such that
\[ \det (\tilde{C}(\lambda) - zI) = p_1(z, \lambda)p_2(z, \lambda), \quad 0 \leq |\lambda| \leq \delta, \]
where the roots \( \alpha_i(\lambda) \) of \( p_1(z, \lambda) \) belong to the left half-plane \( \text{Re } z < 0 \) and those of \( p_2(z, \lambda) \) denoted by \( \beta_i(\lambda) \) with \( \text{Re } \beta_i(\lambda) > 0 \) for \( 1 \leq i \leq m \).

Hence, there are relative prime monic polynomials \( p_1(z, \lambda) \) and \( p_2(z, \lambda) \) of degree \( m \) in \( z \) such that
\[ \det (\tilde{C}(\lambda) - zI) = p_1(z, \lambda)p_2(z, \lambda), \quad 0 \leq |\lambda| \leq \delta, \]
where the roots \( \alpha_i(\lambda) \) of \( p_1(z, \lambda) \) belong to the left half-plane \( \text{Re } z < 0 \) and those of \( p_2(z, \lambda) \) denoted by \( \beta_i(\lambda) \) with \( \text{Re } \beta_i(\lambda) > 0 \) for \( 1 \leq i \leq m \).

Thus, the hypothesis H1 used in [19, p. 1077] is satisfied and from [19], there exists a global blockdiagonalization of \( \tilde{C}(\lambda) \), of the form
\[ M(\lambda) \text{ Diag } (D_1(\lambda), D_2(\lambda)) = \tilde{C}(\lambda)M(\lambda), \] (2.17)
where \( M(\lambda) = (M_{ij}(\lambda)) \) with \( M_{ij}(\lambda) \in \mathbb{C}^{m \times m} \) for \( 1 \leq i, j \leq 2 \), \( M(\lambda) \) is invertible and continuous, \( D_i(\lambda) \in \mathbb{C}^{m \times m} \) for \( i = 1, 2 \),
\[ \sigma(D_1(\lambda)) = \{ \alpha_i(\lambda); 1 \leq i \leq m \}, \quad \sigma(D_2(\lambda)) = \{ \beta_i(\lambda); 1 \leq i \leq m \}, \] (2.18)
and \( M(\lambda) \) is the solution of the matrix differential equation
\[ M'(\lambda) = [Q'(\lambda)Q(\lambda) - Q(\lambda)Q'(\lambda)] M(\lambda); \quad M(0) = \tilde{M}, \] (2.19)
where \( Q(\lambda) \) is a projection matrix commuting with \( \tilde{C}(\lambda) \).

From the continuity of \( M(\lambda) \) at \( \lambda = 0 \), it follows that
\[ \lim_{n \to \infty} M \left( \frac{1}{n} \right) = M(0) = \tilde{M}; \quad \lim_{n \to \infty} D_1 \left( \frac{1}{n} \right) = D(Z_0), \] (2.20)
and by (2.15)
\[ \lim_{n \to \infty} M_{11} \left( \frac{1}{n} \right) = M_{11}(0) = \tilde{M}_{11} = P \text{ invertible}. \] (2.21)
From the invertibility of \( M_{11}(0) = P \) and the perturbation lemma [18, p. 32], there exists a positive integer \( k_0 \) such that for \( n \geq k_0 \), the matrix \( M_{11}(1/n) \) is invertible. From Lemma 2.1, it follows that
\[ Z(n) = M_{11} \left( \frac{1}{n} \right) D_1 \left( \frac{1}{n} \right) \left[ M_{11} \left( \frac{1}{n} \right) \right]^{-1} \] (2.22)
is a solution of the matrix equation
\[ Z^2 + \frac{1}{n}AZ - B = 0, \quad n \geq k_0. \quad (2.23) \]

Hence,
\[ X(n) = nZ(n) = M_{11} \left( \frac{1}{n} \right) D_1 \left( \frac{1}{n} \right) \left[ M_{11} \left( \frac{1}{n} \right) \right]^{-1}, \quad n \geq k_0 \quad (2.24) \]
is a solution of the equation
\[ X^2 + AX - n^2B = 0, \quad n \geq k_0, \quad (2.25) \]
and satisfies the conditions
\[ \text{Re } z < 0, \text{ for every eigenvalue } z \text{ of } X(n), \quad n \geq k_0, \quad (2.26) \]
\[ \lim_{n \to \infty} \frac{X(n)}{n} = Z_0. \]

Summarizing, the following result has been established.

**THEOREM 2.1.** Let \( A, B \) be matrices in \( C^{m \times m} \) where \( B \) satisfies the condition (1.4). Then there exists a sequence of stable matrix solutions \( \{X(n)\}_{n \geq k_0} \) of equation (2.25) satisfying the condition (2.26), where \( Z_0 \) is a stable square root of \( B \).

**REMARK.** Theorem 2.1 ensures the existence of a sequence of eventually stable roots \( X(n) \) of the matrix equation (2.25) so that \( \{X(n)/n\} \) converges to a stable matrix. However, such a theorem does not provide a constructive method for obtaining such matrices \( X(n) \), because the result is based on the theoretical global block diagonalization given by Gingold in [19]. In practice, the conclusion of Theorem 2.1 can be guaranteed in another way. Let us suppose that we are able to construct a sequence \( X(n) \) of solutions of (2.25) so that \( \{X(n)/n\} \) converges to a stable matrix \( S \). Then, by [20, Problem 86], the sequence \( X(n) \) is eventually stable. Recall that Lemma 2.1 provides a method for constructing solutions of equations of the type (2.25). The following example illustrates this fact.

**EXAMPLE 1.** Let \( A \) and \( B \) be the matrices in \( C^{2 \times 2} \) defined by
\[ A = \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
where \( \sigma(B) = \{ -i, i \} \). Then, equation (2.5) takes the form
\[ X^2 + \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \quad X - n^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0, \quad (2.27) \]
and its companion matrix \( C(n) \) is
\[ C(n) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & n^2 & i & -1 \\ -n^2 & 0 & -1 & -i \end{pmatrix}, \]
where
\[ \sigma(C(n)) = \left\{ \frac{n}{\sqrt{2}} (1 + i), \frac{n}{\sqrt{2}} (-1 - i), \frac{n}{\sqrt{2}} (1 + i), \frac{n}{\sqrt{2}} (1 - i) \right\}. \]
Taking \( D(n) \) with the notation of Lemma 2.1, the diagonal matrix
\[ D(n) = \frac{n}{\sqrt{2}} \text{ Diag} \left( -1 + i, -1 - i, 1 + i, 1 - i \right), \]
and
\[ D_1(n) = \frac{n}{\sqrt{2}} \begin{bmatrix} -1 + i & 0 \\ \vdots & \vdots & \vdots \\ 0 & -1 - i \end{bmatrix}; \]

straightforward computations yield
\[ M_{11}(n) = \frac{1 - i}{\sqrt{2}} + n \begin{bmatrix} \frac{1 + i}{\sqrt{2}} + n \\ \frac{1 + i}{\sqrt{2} - in} \end{bmatrix}, \]

and
\[ \det M_{11}(n) = 2n \left( \frac{i}{\sqrt{2}} - 1 + in \right) \neq 0, \quad n > 0. \]

From Lemma 2.1, a solution of (2.27) is given by
\[ X(n) = M_{11}(n)D_1(n)[M_{11}(n)]^{-1}, \]

Note that
\[ \lim_{n \to \infty} \frac{X(n)}{n} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = S, \]

and \( S \) is stable because
\[ \sigma(S) = \left\{ \frac{1}{\sqrt{2}}(-1 + i), \frac{1}{\sqrt{2}}(-1 - i) \right\}. \]

### 3. COUPLED MIXED PARTIAL DIFFERENTIAL PROBLEMS

In this section, we construct a series solution of the problem (1.1)–(1.3) under hypothesis (1.4) for the matrix \( B \) and where \( A \) is an arbitrary matrix in \( \mathbb{C}^{m \times m} \). Let us seek solutions of the homogeneous boundary value problem
\[ u_{tt}(x,t) + Bu_{xx}(x,t) + Au_t(x,t) = 0, \quad -b < x < b, \quad t > 0, \quad (3.1) \]
\[ u(-b,t) = u(b,t) = 0, \quad t > 0 \quad (3.2) \]

of the form
\[ u(x,t) = T(t)X(t), \quad T(t) \in \mathbb{C}^{m \times m}, \quad X(x) \in \mathbb{C}^m. \quad (3.3) \]

Note that if \( T(t) \) satisfies the matrix differential equation
\[ T''(t) + AT'(t) - A^2 BT(t) = 0; \quad \lambda \in \mathbb{C}, \quad t > 0, \quad (3.4) \]

and \( X(x) \) is a vector solution of the boundary value problem
\[ X''(x) + \lambda^2 X(x) = 0, \quad X(b) = X(-b) = 0, \quad -b \leq x \leq b, \quad (3.5) \]

then \( u(x,t) \) defined by (3.3) is a solution of (3.1), (3.2) because
\[ u_{tt}(x,t) + Bu_{xx}(x,t) + Au_t(x,t) = T''(t)X(x) + BT(t)X''(x) + AT'(t)X(x) \]
\[ = T''(t)X(x) - BT(t)\lambda^2 X(x) + AT'(t)X(x) \]
\[ = (T''(t) + AT'(t) - \lambda^2 BT(t)) X(x) = 0, \]
and 

\[ u(-b, t) = T(t)X(-b) = 0, \quad u(b, t) = T(t)X(b) = 0. \]

It is easy to show that the eigenvalues of the Sturm-Liouville problem (3.6) are \( \lambda_k = \frac{k\pi}{2b}, \ k \geq 1 \) and the corresponding eigenfunctions

\[ X_{2j}(x) = \sin \left( \frac{j\pi x}{b} \right) d_{2j}; \quad X_{2j-1}(x) = \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) d_{2j-1}; \quad d_j \in \mathbb{C}^m, \ j \geq 1. \] (3.6)

Taking \( \lambda = \lambda_k \) into (3.4), one gets

\[ T''(t) + AT'(t) - \left( \frac{k\pi}{2b} \right)^2 BT(t) = 0, \quad k \geq 1. \] (3.7)

We are interested in obtaining a sequence of nonzero solutions \( T_k(t) \) of (3.7). Let us assume that \( B \) satisfies the condition (1.4). Then, from Theorem 2.1, there exists a positive integer \( k_0 \) such that the algebraic matrix equation

\[ Z^2 + AZ - \left( \frac{k\pi}{2b} \right)^2 B = 0 \] (3.8)

admits a stable solution \( Z_k \) for \( k \geq k_0 \), and

\[ \lim_{k \to \infty} \frac{Z_k}{k} = W, \] (3.9)

where \( W \) is stable square root of \( B \left( \frac{\pi}{2b} \right)^2 \). Hence, the matrix function \( T_k(t) = \exp(tZ(k)) \) is a solution of (3.7) for \( k \geq k_0 \).

Now let us take, for each \( k = 1, 2, \ldots, k_0 - 1 \), an \((m, r_k)\) full rank cosolution \((S_k, Z_k)\) of the algebraic matrix equation (1.10) for \( 1 \leq k \leq k_0 - 1 \). Then

\[ T_k(t) = S_k \exp(tZ_k)d_k, \quad d_k \in \mathbb{C}^{r_k}, \quad 1 \leq k \leq k_0 - 1 \]

is a \( \mathbb{C}^m \) solution of equation (3.7). Summarizing, we have the following set of solutions of the problem (1.1)-(1.3):

\[ u_{2k}(x, t) = S_{2k} \exp(tZ_{2k}) \sin \left( \frac{k\pi x}{b} \right) c_{2k}, \quad 1 \leq 2k < k_0, \]
\[ u_{2k+1}(x, t) = S_{2k+1} \exp(tZ_{2k+1}) \cos \left( \left( k + \frac{1}{2} \right) \frac{\pi x}{b} \right) c_{2k+1}, \quad 2k + 1 < k_0, \] (3.10)
\[ u_{2k}(x, t) = \exp(tZ_{2k}) \sin \left( \frac{k\pi x}{b} \right) d_{2k}, \quad 2k \geq k_0, \]
\[ u_{2k+1}(x, t) = \exp(tZ_{2k+1}) \cos \left( \left( k + \frac{1}{2} \right) \frac{\pi x}{b} \right) d_{2k+1}, \quad 2k + 1 \geq k_0, \]

where \( c_j \in \mathbb{C}^{r_j} \) for \( 1 \leq j \leq k_0 - 1 \) and \( d_j \in \mathbb{C}^m \) for \( j \geq k_0 \). If we denote by \( v_j(x) \) the functions

\[ v_j(x) = \begin{cases} \sin \left( \frac{j\pi x}{2b} \right) & \text{if } j = 2k, \\ \cos \left( \frac{j\pi x}{2b} \right) & \text{if } j = 2k + 1, \end{cases} \] (3.11)

then we can write the functions appearing in (3.10) in the form

\[ u_j(x, t) = \begin{cases} S_j \exp(tZ_j)c_jv_j(x); & c_j \in \mathbb{C}^{r_j}, \quad 1 \leq j \leq k_0 - 1, \\ \exp(tZ_j)d_jv_j(x); & d_j \in \mathbb{C}^m, \quad j \geq k_0. \end{cases} \] (3.12)
By superposition, we construct a candidate series solution of the problem (1.1)–(1.3), of the form

\[ u(x, t) = \sum_{j=1}^{k_0-1} S_j \exp(tZ_j) c_j v_j(x) + \sum_{j \geq k_0} \exp(tZ_j) d_j v_j(x), \]  

(3.13)

where vectors \( c_j \in \mathbb{C}^{r_j}, \ 1 \leq j \leq k_0 - 1 \) and \( d_j \in \mathbb{C}^m, \ j \geq k_0 \), have to be chosen so that

\[ u(x, 0) = f(x), \quad -b \leq x \leq b, \]  

(3.14)

\[ \sum_{j=1}^{k_0-1} S_j c_j v_j(x) + \sum_{j \geq k_0} d_j v_j(x) = f(x). \]  

(3.15)

If \( f(x) \) satisfies any of the sufficient conditions for the convergence to this function of its Fourier series, and we denote by

\[ a_j = \frac{1}{b} \int_{-b}^{b} f(x) \cos \left( \frac{j\pi x}{b} \right) dx, \quad j \geq 0, \]  

\[ b_j = \frac{1}{b} \int_{-b}^{b} f(x) \sin \left( \frac{j\pi x}{b} \right) dx, \quad j \geq 1, \]  

then the coefficients \( c_j \) and \( d_j \) of the series (3.15) must verify

\[ S_{2k} c_{2k} = b_{2k}, \quad 1 \leq 2k \leq k_0 - 1, \]  

\[ S_{2k+1} c_{2k+1} = a_{2k+1}, \quad 1 \leq 2k + 1 \leq k_0 - 1, \]  

(3.17)

\[ d_{2k} = b_{2k}, \quad 2k \geq k_0, \]  

\[ d_{2k+1} = a_{2k+1}, \quad 2k + 1 \geq k_0, \]

and \( f(x) \) must verify

\[ a_0(f) = \frac{1}{b} \int_{-b}^{b} f(x) dx = 0. \]  

(3.18)

Since \( S_j \) is a full rank \( \mathbb{C}^{m \times r_j} \) matrix for \( 1 \leq j \leq k_0 - 1 \), the solution of (3.17) is given by

\[ c_{2k} = S_{2k}^t b_{2k}, \quad 1 \leq 2k \leq k_0 - 1, \]  

\[ c_{2k+1} = S_{2k+1}^t a_{2k+1}, \quad 1 \leq 2k + 1 \leq k_0 - 1, \]  

(3.19)

\[ d_{2k} = b_{2k}, \quad d_{2k+1} = a_{2k+1}, \quad 2k \geq k_0. \]

If \( n_0 = \lfloor k_0/2 \rfloor \) is the entire part of \( k_0/2 \), then the candidate series solution (3.13) takes the form

\[ u(x, t) = \sum_{j=1}^{n_0} \left\{ S_{2j} \exp(tZ_{2j}) S_{2j}^t b_{2j} \sin \left( \frac{j\pi x}{b} \right) + S_{2j-1} \exp(tZ_{2j-1}) \right\} \]

\[ \times S_{2j-2}^t a_{2j-2} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \]  

\[ + \sum_{j \geq n_0+1} \left\{ \exp(tZ_{2j}) b_{2j} \sin \left( \frac{j\pi x}{b} \right) \right\} \]  

\[ + \exp(tZ_{2j-1}) a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \]  

(3.20)

By construction, the condition (3.14) is satisfied, and note that in order to prove that (3.20) is well defined and defines a continuous function, twice continuously differentiable with respect to \( x \) and \( t \), it is sufficient to prove these properties for the series

\[ V(x, t) = \sum_{j \geq n_0} \left\{ \exp(tZ_{2j}) b_{2j} \sin \left( \frac{j\pi x}{b} \right) + \exp(tZ_{2j-1}) a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \right\}. \]
The series obtained by termwise partial differentiation of $V(x, t)$ with respect to the variables $x$ and $t$ take the form

$$V_z(x, t) = \sum_{j \geq n_0} \left\{ \left( \frac{j \pi}{b} \right) \exp \left( tZ_{2j} b \right) \cos \left( \frac{j \pi x}{b} \right) \right. $$

$$- \left( j - \frac{1}{2} \right) \frac{\pi}{b} \exp(tZ_{2j-1})a_{2j-1} \sin \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right),$$

$$V_{xx}(x, t) = \sum_{j \geq n_0} \left\{ - \left( \frac{j \pi}{b} \right)^2 \exp \left( tZ_{2j} b \right) \sin \left( \frac{j \pi x}{b} \right) \right. $$

$$- \left( j - \frac{1}{2} \right) \frac{\pi^2}{b} \exp(tZ_{2j-1})a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \},$$

$$V_t(x, t) = \sum_{j \geq n_0} \left\{ Z_{2j} \exp \left( tZ_{2j} b \right) \right. $$

$$+ Z_{2j-1} \exp \left( tZ_{2j-1} b \right) \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \},$$

$$V_{tt}(x, t) = \sum_{j \geq n_0} \left\{ Z_{2j}^2 \exp \left( tZ_{2j} b \right) \right. $$

$$+ Z_{2j-1}^2 \exp \left( tZ_{2j-1} b \right) \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \}.$$ (3.21)

From the derivation theorem of functional series [21, p. 37], to prove the above termwise differentiations, it is sufficient to show that such series are uniformly convergent in a rectangle $[-b, b] \times [t_0, t_1] = R(t_0, t_1)$ with $t_1 > t_0 > 0$.

From the Riemann-Lebesgue lemma, there exists a positive constant $M$ such that

$$\|a_n\| \leq M, \quad \|b_n\| \leq M, \quad n \geq 1. \quad (3.22)$$

If $Q_k^H Z_k Q_k = D_k + N_k$ is the Schur decomposition of $Z_k$, for $k \geq n_0 + 1$, from [11, pp. 336,556], it follows that

$$\|N_k\| \leq \|N_k\|_F \leq \|Z_k\|_F, \quad (3.23)$$

and if

$$\alpha(Z_k) = \max \{\text{Re} (z); \ z \in \sigma(Z_k)\}, \quad (3.24)$$

it follows that

$$\| \exp(tZ_k)\| \leq \exp(\alpha(Z_k)) \sum_{j=0}^{m-1} \frac{\|N_k t\|_F}{j!}. \quad (3.25)$$

From (3.9), the sequence $\{Z_k/k\}_{k>n_0}$ converges to a stable square root $W$ of $B \left( \frac{\pi}{2b} \right)^2$ and by [20, Problem 86], we also have that $\{\alpha(Z_k/k)\}_{k \geq n_0}$ converges to $\alpha(W) < 0$. Hence, there exists a positive integer $n_1 \geq n_0$ such that

$$\|Z_k\| \leq \|Z_k\|_F \leq 2k\|W\|_F, \quad \alpha(Z_k) \leq 2k\alpha(W) < 0, \quad k \geq k_1 \geq n_0. \quad (3.26)$$
From (3.25) and (3.26) for \( k \geq k_1 \) and \((x,t) \in R(t_0,t_1)\), it follows that

\[
\| Z_k \exp (t Z_k) \| \leq 2k \| W \|_F \exp (2kt_0 \alpha(W)) \sum_{j=0}^{m-1} \frac{(2kt_1 \| W \|_F)j}{j!}.
\]

\[
\| Z_k^2 \exp (t Z_k) \| \leq (2k \| W \|_F)^2 \exp (2kt_0 \alpha(W)) \sum_{j=0}^{m-1} \frac{(2k \| W \|_F)j}{j!}.
\]

\[
\left\| \left( \frac{k \pi}{b} \right)^2 \exp (t Z_k) \right\| \leq \left( \frac{k \pi}{b} \right)^2 \exp (2kt_0 \alpha(W)) \sum_{j=0}^{m-1} \frac{(2k \| W \|_F)j}{j!}.
\]

From (3.27) and the derivation theorem for functional series [21, p. 37], it follows that the series (3.25) defines a continuous function, twice partially differentiable with respect to the variables \( x \) and \( t \).

If \( f(x) \) is a \( \mathbb{C}^m \) valued function defined in \([-b, b]\) and satisfies the conditions

\[
\begin{align*}
f(x) & \text{ is continuous and admits right-hand and left-hand} \\
& \text{derivatives at every point } x \text{ in } [-b,b], \\
f(-b) &= f(b) = 0, \\
\int_{-b}^{b} f(x) \, dx &= 0,
\end{align*}
\]

then its Fourier series has the coefficient \( a_0 = 0 \), and converges at every point \( x \) of \([-b, b]\) to \( f(x) \) [22, p. 75]. From the previous comments, the following result has been established.

**Theorem 3.1.** Let \( A, B \) be matrices in \( \mathbb{C}^{m \times m} \) where \( B \) satisfies the condition (1.4). Let \( f(x) \) be a \( \mathbb{C}^m \) valued function satisfying the conditions (3.28)–(3.30) and let us suppose that \( \{ Z_k \}_{k \geq k_0} \) are matrix solutions of equation (2.25) such that \( Z_k \) is stable for \( k \geq k_0 \) and \( \{ Z_k/k \} \) converges to a stable matrix \( W \). Let \( \{ S_k, Z_k \} \) be a full rank \((m, r_k)\) cosolution of (2.25) for \( k = 1, 2, \ldots, k_0 - 1 \) and let \( n_0 \) be the entire part of \( k_0/2 \). If \( a_j, b_j \) are the Fourier coefficients of \( f(x) \) given by (3.16), then \( u(x, t) \) defined by (3.20) defines an exact solution of the problem (1.1)–(1.3).

The exact series solution provided by Theorem 3.1 has the computational drawback of the infiniteness. In the following, we are interested in addressing the following problem. Given \( t_0 \) and \( t_1 \) with \( 0 < t_0 < t_1 \) and \( \epsilon > 0 \), how can an approximate solution of problem (1.1)–(1.3) be constructed, with an error smaller than \( \epsilon \) uniformly for \((x,t) \in R(t_0,t_1) = [-b,b] \times [t_0,t_1]? \)

Let \( n \) be a positive integer with \( n > n_0 \), where \( n_0 \) is defined by Theorem 3.1, and let us consider the function

\[
E(x,t,n) = \sum_{j \geq n} \left\{ \exp (t Z_j) b_{2j} \sin \left( \frac{j \pi x}{b} \right) + \exp (t Z_j) a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \right\}.
\]

Note that from (3.22) and (3.27), for \((x,t) \in R(t_0,t_1)\) and \( j \geq n_0 \), it follows that

\[
\left\| \exp (t Z_j) b_{2j} \sin \left( \frac{j \pi x}{b} \right) + \exp (t Z_j) a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \right\|
\leq 2M \exp (j \gamma) \sum_{h=0}^{m-1} \frac{j^h \rho^h}{h!} = 2M \exp (j \gamma) \sum_{h=0}^{m-1} \exp (h \ln j) \frac{\rho^h}{h!}.
\]
where
\[
\gamma = 2t_0 \alpha (W) < 0, \quad \rho = \sum_{h=0}^{m-1} \frac{(2t_1 \|W\|_P)^h}{h!}.
\] (3.33)

Let us introduce the scalar functions \( g_h : [1, \infty] \to \mathbb{R} \) defined by
\[
g_h(s) = \gamma \frac{s}{2} + h \ln s, \quad 0 \leq h \leq m - 1, \quad s \geq 1,
\]
and note that \( g_0(s) < 0 \), and for \( 1 \leq h \leq m - 1 \), \( g_h(s) < 0 \) if and only if
\[
\frac{\ln s}{s} < - \frac{2\gamma}{h}.
\] (3.34)

Since \( \lim_{s \to \infty} (\ln s / s) = 0 \) for each \( h \) with \( 1 \leq h \leq m - 1 \), there exists a real number \( s_h \) such that (3.34) holds for \( s \geq s_h \). Let \( n_1 \) be an integer such that
\[
n_1 \geq \max \{ n_0, s_1, s_2, \ldots, s_{m-1} \},
\] (3.35)
then from (3.32)–(3.35) for \( j > n_1 \), it follows that
\[
\| \exp(tZ_j)b_{2j} \sin \left( \frac{j\pi x}{b} \right) + \exp(tZ_j)a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \| \leq 2M \sum_{h=0}^{m-1} \exp \left( g_h(j) + \frac{j \gamma}{2} \right) \frac{\rho^h}{h!} \leq 2M \exp \left( \frac{j \gamma}{2} \right) \sum_{h=0}^{m-1} \frac{\rho^h}{h!} = 2M \exp \left( \frac{j \gamma}{2} \right),
\]
where \( \rho \) is given by (3.33). From (3.31), for \( n \geq n_1 \) it follows that
\[
\| E(x,t,n) \| \leq 2M \rho \sum_{j=n}^{\infty} \exp \left( \frac{\gamma j}{2} \right) = \frac{2M \rho}{1 - \exp(\gamma/2)} \exp \left( \frac{n \gamma}{2} \right).
\] (3.36)

Given an admissible error \( \epsilon > 0 \), taking a positive integer \( n_2 \) such that
\[
n_2 \geq n_1, \quad n_2 > \frac{2}{\gamma} \ln \left[ \frac{1 - \exp(\gamma/2) \epsilon}{2M \rho} \right],
\] (3.37)
then from (3.37), one gets
\[
\| E(x,t,n_2) \| \leq \epsilon, \quad \text{uniformly for } (x,t) \in R(t_0,t_1).
\]

Thus, the following result has been established.

**Corollary 3.1.** With the hypotheses of Theorem 3.1 and the previous notation, if \( \epsilon > 0 \) and \( n_2 \) satisfies (3.37), then the function
\[
u(x,t, n_2) = \sum_{j=1}^{n_0} \left\{ S_{2j} \exp(tZ_{2j})S_{2j}^t b_{2j} \sin \left( \frac{j\pi x}{b} \right) + S_{2j-1} \exp(tZ_{2j-1})S_{2j-1}^t a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \right\} + \sum_{j=n_0+1}^{n_2} \left\{ \exp(tZ_{2j})b_{2j} \sin \left( \frac{j\pi x}{b} \right) + \exp(tZ_{2j-1})a_{2j-1} \cos \left( \left( j - \frac{1}{2} \right) \frac{\pi x}{b} \right) \right\}
\]
is an approximate solution of problem (1.1)–(1.3) whose error with respect to the exact series solution \( u(x,t) \) provided by Theorem 3.1 is uniformly bounded by \( \epsilon \) in \( R(t_0,t_1) = [-b,b] \times [t_0,t_1] \), where \( t_1 > t_0 > 0 \).
REFERENCES