Discrete Mathematics 14 (1976) 297–304.

ALGEBRAIC AND COMBINATORIAL RESULTS FOR RANKING COMPETITORS IN A SEQUENCE OF RACES

W.J. WALKER

Department of Mathematics, University of Auckland, Auckland, New Zealand

Received 31 August 1973
Revised 20 March 1974
Final version received 25 April 1975

Suppose n competitors each compete in r races and a ranking function F assigns a score $F(j)$ to the competitor finishing in the jth position in each race. The sum of the scores over the r races gives each competitor a final ranking with equal rankings being possible. A series representation and an asymptotic estimate are obtained for a_n, the number of ways of ranking n competitors in order, given that equal rankings are permissible. Also algebraic results are obtained which give criteria for the construction of a ranking function F which ranks scores in a predetermined way.

1. Introduction

We propose to examine the algebraic and combinatorial properties of the problem of ranking n competitors each of whom participates in r races. A "ranking function" F for these competitors will be a positive function defined on the first n positive integers and satisfying the condition $F(j) > F(j+1)$, for $1 \leq j \leq n-1$. In each race the competitor finishing in the jth position is awarded a score $F(j)$. The sum of the scores over the r races gives each competitor a final score and the competitors are ranked by these final scores.

A "result" will be simply a finite set of positive integers $\{\alpha_k\}_{1 \leq k \leq r}$, where for each k, $1 \leq \alpha_k \leq n$. That is, a result represents the placings of a single competitor over the r races. Although we do not allow the possibility that two competitors be placed equal in a given race, if $r > 1$ then they will certainly be ranked equal if they achieve the same result and possibly in other cases depending on the ranking function F.

In Section 2 we obtain an asymptotic estimate for a_n, the number of ways of ranking n competitors in order, given that equal rankings are permissible.

In Section 3 we examine the problem, which is essentially algebraic, of determining if certain apparently reasonable rankings of results can in fact be achieved by choosing a suitable ranking function.

2. Combinatorial results

Let us denote by a_n the number of ways of ranking n competitors where equal rankings are permissible and for convenience take $a_0 = 1$. Then, for example, $a_1 = 1, a_2 = 3, a_3 = 13$ and $a_4 = 75$.

Although the following result does not seem to appear explicitly elsewhere it is probably not new and is certainly closely related to known results.

Proposition 2.1. *For each positive integer n,*

$$a_n = \sum_{m=1}^{\infty} \frac{m^n}{2^{m+1}} .$$

(It is of interest to note that the sum of the infinite series is an integer for all n.)

Proof. If a ranking is to contain i competitors ranked equally at the top there are $\binom{n}{i}$ ways of choosing such a set of i competitors and in fact $\binom{n}{i} a_{n-i}$ rankings which satisfy this criterion. Hence we obtain the identity

$$(1) \qquad a_n = \sum_{i=1}^{n} \binom{n}{i} a_{n-i} .$$

A well known technique to obtain an explicit solution from an identity of this nature is to find a generating function. See, for example, [1, p. 230] where the Catalan numbers are obtained in this way. We define $G(t) = \sum_{n=0}^{\infty} a_n t^n/n$ and then from eq. (1) we see that

$$(2) \qquad a_n \frac{t^n}{n!} = \sum_{i=1}^{n} \binom{n}{i} a_{n-i} \frac{t^n}{n!}$$

and by addition of terms in (2),

$$G(t) = 1 + \sum_{n=1}^{\infty} \sum_{i=1}^{n} \binom{n}{i} a_{n-i} \frac{t^n}{n!}$$

$$= 1 + \sum_{m=0}^{\infty} a_m \left\{ \sum_{j=1}^{\infty} \binom{m+j}{j} \frac{t^{m+j}}{(m+j)!} \right\}$$

$$= 1 + \sum_{m=0}^{\infty} a_m \left\{ \sum_{j=1}^{\infty} \binom{m+j}{m} \frac{t^{m+j}}{(m+j)!} \right\}$$

$$= 1 + \sum_{m=0}^{\infty} a_m \frac{t^m}{m!} \left\{ \sum_{j=1}^{\infty} \frac{t^j}{j!} \right\}$$

$$= 1 + (e^t - 1)\, G(t)\,.$$

Thus $G(t) = (2 - e^t)^{-1}$ and we note that for $|t| < \log_e 2$ we may rewrite $G(t)$ in the form

$$G(t) = \sum_{m=0}^{\infty} \frac{1}{2^{m+1}} e^{tm}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left\{ \sum_{m=1}^{\infty} \frac{m^n}{2^{m+1}} \right\}$$

and conclude that

$$a_n = \sum_{m=1}^{\infty} \frac{m^n}{2^{m+1}}\,.$$

Proposition 2.2.

$$\frac{a_n}{n!} = \tfrac{1}{2} (\log_e 2)^{-n-1} + o((2\pi)^{-n})\,.$$

Proof. If we let z be complex then $G(z) = 1/(2 - e^z)$ has simple poles at the points $z_k = \log_e 2 + 2k\pi i$. The nearest singular point to $z = 0$ is $z_0 = \log_e 2$ with residue $-\frac{1}{2}$. The result then follows by considering $G(z) - 1/2(\log_e 2 - z)$. For further details see a similar proof [3, pp. 28–30].

3. Algebraic results

We begin by stating the following proposition the proof of which is immediate.

Proposition 3.1. *Let C be a set of results which have been ranked by applying a ranking function (with of course the possibility of equal rankings).*

(i) *If any two elements of C say* $\{\alpha_k\}_{1\leq k\leq r}$ *and* $\{\beta_k\}_{1\leq k\leq r}$ *(relabelled if necessary) satisfy the condition* $\alpha_k \leq \beta_k$, *for* $1 \leq k \leq r$, *with at least one strict inequality, then the result* $\{\alpha_k\}_{1\leq k\leq r}$, *is ranked above the result* $\{\beta_k\}_{1\leq k\leq r}$.

(ii) *(A substitution condition) Suppose two results* $\{\alpha_k\}_{1\leq k\leq r}$ *and* $\{\beta_k\}_{1\leq k\leq r}$ *have a subset of common elements* $\{\gamma_j\}_{1\leq j\leq p}$. *Let* $\{\alpha'_k\}_{1\leq k\leq r}$ *and* $\{\beta'_k\}_{1\leq k\leq r}$ *be results produced by replacing the elements* $\{\gamma_j\}_{1\leq j\leq p}$ *by elements* $\{\gamma'_j\}_{1\leq j\leq p}$ *in each of the results* $\{\alpha_k\}_{1\leq k\leq r}$ *and* $\{\beta_k\}_{1\leq k\leq r}$ *respectively. Then the ranking of* $\{\alpha'_k\}_{1\leq k\leq r}$ *relative to* $\{\beta'_k\}_{1\leq k\leq r}$ *must be the same as the ranking of* $\{\alpha_k\}_{1\leq k\leq r}$ *relative to* $\{\beta_k\}_{1\leq k\leq r}$.

We now pose a question which is in a sense a converse of the above proposition; namely if from a set of results we construct a ranking which satisfies the conditions (i) and (ii), then is there a ranking function $F(j)$ which gives this ranking? The answer is negative and in fact we shall see that this is so even if we require that the ranking satisfies the further condition:

(iii) *Whenever* $\{\alpha_k\}_{1\leq k\leq r}$ *and* $\{\beta_k\}_{1\leq k\leq r}$ *are results with* $\Sigma_{k=1}^{r} \alpha_k < \Sigma_{k=1}^{r} \beta_k$ *then the result* $\{\alpha_k\}_{1\leq k\leq r}$ *is ranked above the result* $\{\beta_k\}_{1\leq k\leq r}$.

Suppose $x = (x_1, x_2, ..., x_n)$ and $V_i(x)$, $1 \leq i \leq p$, are linear functionals on Euclidean n-space E^n. Then the following lemma is due to Carver [2] and is also proved in the paper of Fan [4, p. 115].

Lemma 3.2. *The system of inequalities*

$$V_i(x) > 0, \qquad 1 \leq i \leq p,$$

has a solution if and only if the zero-functional is not in the convex hull of $V_1, V_2, ..., V_p$.

Let C be a set of results and $R(C)$ a ranking of the set which satisfies conditions (i), (ii) and (iii). Further suppose no two results are given an equal ranking and whenever $\{\alpha_k\}_{1\leq k\leq r}$ and $\{\beta_k\}_{1\leq k\leq r}$ satisfy $\Sigma_{k=1}^{r} \alpha_k = \Sigma_{k=1}^{r} \beta_k$ with $\{\alpha_k\}_{1\leq k\leq r}$ ranked above $\{\beta_k\}_{1\leq k\leq r}$, then we define a linear functional f on E^n by

$$f(x) = \sum_{k=1}^{r} x_{\alpha_k} - \sum_{k=1}^{r} x_{\beta_k} .$$

Let the linear functionals defined in this way be $\{f_i(x)\}_{1 \leq i \leq q}$.

Proposition 3.3. *Let $R(C)$ be a ranking of C which satisfies conditions* (i), (ii) *and* (iii) *and gives no two results an equal ranking and let $\{f_i(x)\}_{1 \leq i \leq q}$ be the linear functionals defined above. Then there is a ranking function which gives C the ranking $R(C)$ if and only if the zero-functional is not in the convex hull of $f_1, f_2, ..., f_q$.*

Proof. For $1 \leq j \leq n-1$, let $g_j(x)$ denote the linear functional on E^n defined by $g_j(x) = x_j - x_{j+1}$. Further suppose that for every pair $\{\alpha_k\}_{1 \leq k \leq r}$ and $\{\beta_k\}_{1 \leq k \leq r}$ of successive results in the ranking $R(C)$ we define a linear functional

$$h(x) = \sum_{k=1}^{r} x_{\alpha_k} - \sum_{k=1}^{r} x_{\beta_k} .$$

Let the linear functions defined in this way be $\{h_i(x)\}_{1 \leq i \leq p}$ and then of course $\{f_i(x)\}_{1 \leq i \leq q} \subseteq \{h_i(x)\}_{1 \leq i \leq p}$. Now by Lemma 3.2 there exists a ranking function which gives C the ranking $R(C)$ if and only if the zero-functional is not in the convex hull of

$$\{g_j\}_{1 \leq j \leq n-1} \cup \{h_i\}_{1 \leq i \leq p} .$$

If we define the "moment" of a linear function

$$v(x) = \sum_{k=1}^{n} c_k x_k \quad \text{to be} \quad \mu(v) = \sum_{k=1}^{n} k\, c_k ,$$

we note that each $f_i(x)$ has zero moment while each $g_j(x)$ and the remainder of the $h_i(x)$ have a negative moment.

Since the zero-functional has zero moment, it is clear that if it belongs to the convex hull of $\{g_j\}_{1 \leq j \leq n-1} \cup \{h_i\}_{1 \leq i \leq p}$, then in fact it belongs to the convex hull of $\{f_i(x)\}_{1 \leq i \leq q}$ and the proof is complete.

Example 3.4. An example of a ranking which cannot be induced by a ranking function is ($r = 2$ and $n = 6$)

$$\{1,5\}, \ \{2,4\}, \ \{3,4\}, \ \{1,6\}, \ \{2,6\}, \ \{3,5\} .$$

It is interesting to note that these results can arise from the two races:

1	2	3	4	5	6
D	B	F	C	A	E
A	E	C	B	F	D

Here

$$f_1(x) = x_1 + x_5 - x_2 - x_4 ,$$

$$f_2(x) = x_3 + x_4 - x_1 - x_6 ,$$

$$f_3(x) = x_2 + x_6 - x_3 - x_5 ,$$

and the zero-functional belongs to the convex hull of f_1, f_2, f_3, since $f_1(x) + f_2(x) + f_3(x) = 0$.

When confronted with the practical problem of ranking competitors one possible procedure is to choose a ranking function which distinguishes at least some of the possible results according to the preference of the designer although the above example demonstrates that his expectations must necessarily be limited. We do however give a further proposition which shows that if an appropriate set of results C is chosen, then any ranking of C which satisfies conditions (i), (ii) and (iii) can be induced by a ranking function.

Proposition 3.5. *Let C be a set of results. For each pair of results $\{\alpha_k\}_{1<k<r}$ and $\{\beta_k\}_{1<k<r}$ with $\Sigma_{k=1}^r \alpha_k = \Sigma_{k=1}^r \beta_k$ define a linear functional on* E^n *by*

$$f(x) = \sum_{k=1}^{r} x_{\alpha_k} - \sum_{k=1}^{r} x_{\beta_k}$$

(here the sign of $f(x)$ can be arbitrarily fixed). If the linear functionals defined in this way are linearly independent, then for any ranking $R(C)$ of C which satisfies conditions (i), (ii) *and* (iii) *(including the possibility of equal rankings) there exists a ranking function which induces this ranking.*

Proof. If $R(C)$ is a ranking in which no two results are given equal ranking, then we may apply Proposition 3.3. (Since the linear functionals to which this proposition refers will in fact be linearly independent, it follows that the zero-functional does not belong to their convex hull.)

Now we suppose $R(C)$ is a ranking which assigns equal ranks to certain pairs of results. In particular suppose there is just one such pair of results $\{\alpha_k\}_{1 \leqslant k \leqslant r}$ and $\{\beta_k\}_{1 \leqslant k \leqslant r}$. Then again by Proposition 3.3 there exist ranking functions F and G such that

$$\sum_{k=1}^{r} F(\alpha_k) - \sum_{k=1}^{r} F(\beta_k) = \delta_1 ,$$

$$\sum_{k=1}^{r} G(\alpha_k) - \sum_{k=1}^{r} G(\beta_k) = -\delta_2 ,$$

where δ_1 and δ_2 are positive and F and G rank the remaining results as required. If we define

$$H(j) = F(j)/\delta_1 + G(j)/\delta_2, \qquad 1 \leqslant j \leqslant n ,$$

then H ranks $\{\alpha_k\}_{1 \leqslant k \leqslant r}$ and $\{\beta_k\}_{1 \leqslant k \leqslant r}$ to be equal and the ranking of the remaining results is unchanged.

To complete the proof we observe that if $R(C)$ assigns equal rankings to a number of pairs of results we may apply the above procedure to each pair in succession and thus construct a ranking function which gives the prescribed ranking $R(C)$.

We note that condition (iii) is certainly a restriction which one might wish to avoid in practice and conclude by pointing out a further combinatorial problem.

Two ranking functions can be said to be "equivalent" for fixed n and r, if for any set of race results they give the same final ranking of the n competitors.

Example 3.6. If F is a ranking function and m and c are positive constants an equivalent ranking function G may be defined by setting

$$G(j) = m\,F(j) + c \quad \text{for } 1 \leqslant j \leqslant n.$$

The problem posed is to determine asymptotic estimates in terms of n and r for the number of equivalence classes.

References

[1] Bateman Manuscript Project, Higher Transcendental Functions, Vol. 3 (McGraw-Hill, New York, 1955).

[2] W.B. Carver, Systems of linear inequalities, Annals of Math. 23 (1921/22) 212–220.
[3] M.A. Evgrafov, Asymptotic Estimates and Entire Functions (Gordon and Breach, New York, 1963).
[4] K. Fan, Systems of linear inequalities, in: Linear Inequalities and Related Systems (Princeton Univ. Press, Princeton, N.J., 1956).