doi:10.1006/jmaa.2001.7818, available online at http://www.idealibrary.com on IDEAL®

Hausdorff Dimension in Convex Bornological Spaces

J. Almeida and L. Barreira[1]

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisbon, Portugal
E-mail: jalmeida@math.ist.utl.pt, luis.barreira@math.ist.utl.pt

Submitted by John Horváth

Received January 17, 2001

For non-metrizable spaces the classical Hausdorff dimension is meaningless. We extend the notion of Hausdorff dimension to arbitrary locally convex linear topological spaces and thus to a large class of non-metrizable spaces. This involves a limiting procedure using the canonical bornological structure. In the case of normed spaces the new notion of Hausdorff dimension is equivalent to the classical notion.

Key Words: bornological linear space; Hausdorff dimension.

1. INTRODUCTION

In the late 1970s the attention of many physicists and applied mathematicians turned to the study of the complexity of strange attractors. The lack of appropriate techniques caused several important developments. Most notably, the notion of Hausdorff dimension, along with other quantities measuring complexity, grew up to the dimension theory of dynamical systems, which today plays a crucial role in the study of invariant sets and of their complexity. We refer the reader to [1, 3] for a list of references and for a detailed discussion.

This treatment of complexity uses several dimensional quantities, strongly based on the metric structure of the ambient space. On the other hand there exists the clear interest in spaces of transformations that may not be metrizable. This is the case, for example, when one considers spaces of distributions, which often occur in the study of partial differential equations. In these situations one can use the notion of topological dimension in order

[1] Partially supported through FCT's Research Units Funding Program.

0022-247X/02 $35.00

to characterize the complexity of complicated subsets, but this will provide a very rough classification. Furthermore, these spaces may not have the Baire property and thus topological dimension may in general lack good embedding properties.

These observations motivate our interest in a notion of Hausdorff dimension in non-metrizable spaces. In this paper we introduce such a notion in arbitrary locally convex linear topological spaces. In the particular case of normed spaces the new notion is equivalent to the classical notion of Hausdorff dimension.

The new notion of Hausdorff dimension is introduced through a limiting procedure which requires the canonical von Neumann bornological structure. In fact, the natural framework for the introduction of the new notion is precisely that of convex bornological linear spaces. Namely, given a disk D we consider the subspace X_D spanned by D and the associated Minkowski semi-norm p_D (see Section 2.1 for details). The Hausdorff dimension of a set Z shall be defined by

$$\dim_H Z = \lim_{D} \lim_{D'} \dim(Z \cap X_D,\ X_{D'}),$$

where $\dim(Z \cap X_D, X_{D'})$ denotes the classical Hausdorff dimension of $Z \cap X_D$ in the semi-normed space $\langle X_{D'}, p_{D'} \rangle$ and the limits run through all bounded disks. The new notion applies, for example, to the space of germs of holomorphic functions on a compact set and to the space of tempered distributions.

2. HAUSDORFF DIMENSION IN BORNOLOGICAL CONVEX SPACES

2.1. *Convex Bornological Spaces*

In this section we recall several basic notions from the theory of bornological linear spaces. A *bornology* on a set X is a family $\mathcal{B}$ of subsets of X such that $\mathcal{B}$ is a cover of X, finite unions of elements of $\mathcal{B}$ are in $\mathcal{B}$, and any subset of an element of $\mathcal{B}$ is also in $\mathcal{B}$. The elements of $\mathcal{B}$ are called *bounded sets.*

A *bornological linear space* is a linear space over the field $\mathbb{K}$ (the real or complex field) together with a bornology on the underlying set of vectors such that the sum of vectors and the product of elements of $\mathbb{K}$ by vectors are bounded operations; i.e., the sets $A + B$ and $C \cdot B$ are bounded sets whenever A and B are bounded subsets of X and C is a bounded subset of $\mathbb{K}$.

A *disk* in a linear space X is a convex subset of X closed under products by numbers λ with $|\lambda| \leq 1$. The *disked hull* of a set $A \subset X$ is the intersection of all disks in X containing A. A *convex bornological space* is a bornological linear space for which the disked hull of every bounded set is bounded.

It is well known that in any topological linear space there is a canonical notion of bounded set—the von Neumann bornology. In this case the underlying linear space constitutes a bornological linear space when endowed with this bornology. Moreover, if the topological linear space is a locally convex space, then the associated bornological linear space is a convex bornological space. Furthermore, linear continuous maps between topological linear spaces are bounded maps between the corresponding bornological linear spaces, in the sense that bounded sets are mapped onto bounded sets.

Given a disk D in a linear space X, we denote by X_D the subspace of X spanned by D. Moreover, we denote by p_D the *Minkowski semi-norm* defined on X_D by

$$p_D(x) = \inf\{r > 0 : x \in rD\}.$$

Henceforth, whenever there is no danger of confusion we shall write X_D to indicate the semi-normed space $\langle X_D, p_D \rangle$.

If D and D' are disks of X with $D \subset D'$, the embedding of X_D into $X_{D'}$ is a continuous mapping. Furthermore, given two disks D_1 and D_2, there exists a disk D containing D_1 and D_2—the disked hull of $D_1 \cup D_2$. It can be shown that any convex bornological space X is the canonical inductive limit—in the category of convex bornological spaces and linear functions that map bounded sets onto bounded sets—of the family of semi-normed spaces $\langle X_D, p_D \rangle$, where the index D ranges over all bounded disks of X.

2.2. *Classical Hausdorff Dimension*

We now briefly recall the classical notion of Hausdorff dimension. Let X be a metric space and consider a subset Z of X. Given $\alpha \geq 0$, we set

$$H(Z, \alpha) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha,$$

where the infimum is taken over all countable covers $\mathcal{U}$ of Z by sets of diameter at most δ, with the usual convention that $\inf \emptyset = +\infty$. The *Hausdorff dimension* of Z is the unique number $\dim Z$ such that $H(Z, \alpha) = +\infty$ for every $\alpha < \dim Z$; moreover,

$$\dim Z = \sup\{\alpha : H(Z, \alpha) = +\infty\} = \inf\{\alpha : H(Z, \alpha) = 0\}.$$

Observe that if for some δ there exists no countable cover of Z by sets of diameter at most δ, then $\dim Z = +\infty$. One can readily extend the notion of Hausdorff dimension to semi-metric spaces, and this will be used in the following. Henceforth, given a set Z, we shall also denote by $\dim(Z, X)$

the Hausdorff dimension of $Z \cap X$ in the space X, especially when there is a need to stress the space.

Clearly, if $\phi: E \to F$ is a Lipschitz map between semi-metric spaces and $Z \subset E$, then $\dim \phi(Z) \le \dim Z$. This is the case, for example, when ϕ is a linear continuous map between semi-normed spaces. This also happens when ϕ is the natural embedding into a semi-metric space F of a semi-metric subspace E. In these cases, we have

$$\dim(Z, E) = \dim(\phi(Z), \phi(E)) = \dim(\phi(Z), F).$$

2.3. *Hausdorff Dimension in Convex Bornological Spaces*

Let X be a convex bornological space and let Z be a subset of X. Let D be a bounded disk of X. By associating to each bounded disk D' the value of $\dim(Z \cap X_D, X_{D'})$, we define a net, since the union of two bounded disks is contained in some bounded disk. Moreover, this net is non-increasing when restricted to disks containing D, since for $D'' \supset D'$ the embedding of $X_{D'}$ into $X_{D''}$ is continuous. We set

$$d_D(Z) = \lim_{D'} \dim(Z \cap X_D, X_{D'}).$$

On the other hand, given bounded disks D_1 and D_2 such that $D_1 \subset D_2$, the inclusion $Z \cap X_{D_1} \cap X_{D'} \subset Z \cap X_{D_2} \cap X_{D'}$ holds for every bounded disk D'. It follows that

$$\dim(Z \cap X_{D_1}, X_{D'}) \le \dim(Z \cap X_{D_2}, X_{D'})$$

for all D', and thus the net $(d_D(Z))_D$ is non-decreasing. We define the *Hausdorff dimension* of Z (on the space X) by

$$\dim_H Z = \lim_D d_D(Z) = \lim_D \lim_{D'} \dim(Z \cap X_D, X_{D'}).$$

We shall also write $\dim_H(Z, X)$ when there is a need to stress the space X.

The notion above can be lifted to locally convex spaces via the von Neumann bornology: For any locally convex space E, we define the Hausdorff dimension of a subset Z of E as the value of $\dim_H Z$ in the convex bornological space corresponding to E.

If Z is contained in some X_D—in particular, if Z is bounded—then the definition simplifies to

$$\dim_H Z = \lim_{D'} \dim(Z, X_{D'}).$$

Given a convex bornological space X, a *bornological linear subspace* of X is any linear subspace Y of X endowed with the *induced bornology*, i.e., the family of intersections of Y with all bounded subsets of X; a bornological linear subspace of a convex bornological space is still a convex bornological space. We now show that the Hausdorff dimension is preserved by embeddings of bornological linear subspaces.

THEOREM 1. *Let X be a convex bornological space and let Y be a bornological linear subspace of X. If Z is a subset of Y, then* $\dim_H(Z, Y) = \dim_H(Z, X)$.

Proof. For every bounded disk D of X, write $D^- = D \cap Y$. Since $X_D \cap Y = Y_{D^-}$ and $Z \subset Y$, it follows that

$$\dim(Z \cap X_D, X_{D_1}) = \dim(Z \cap Y_{D^-}, X_{D_1})$$

for all bounded disks D and D_1 of X. It is easy to see that $p_{D_1}(x) = p_{D_1^-}(x)$ for $x \in Y_{D_1^-}$, which means that $Y_{D_1^-}$ is a semi-normed subspace of X_{D_1}. Therefore,

$$\dim(Z \cap Y_{D^-}, Y_{D_1^-}) = \dim(Z \cap Y_{D^-}, X_{D_1}) = \dim(Z \cap X_D, X_{D_1}).$$

Hence,

$$\dim_H(Z, X) = \lim_D \lim_{D_1} \dim(Z \cap X_D, X_{D_1}) = \lim_D \lim_{D_1} \dim(Z \cap Y_{D^-}, Y_{D_1^-}).$$

For fixed D, we observe that

$$\lim_{D_1} \dim(Z \cap Y_{D^-}, Y_{D_1^-}) = \lim_{\Delta_1} \dim(Z \cap Y_{D^-}, Y_{\Delta_1}),$$

where Δ_1 ranges over all bounded disks of Y, because the former net depends on D_1 only through $D_1 \cap Y$ and we know that both limits exist. Similarly we obtain

$$\lim_D \lim_{D_1} \dim(Z \cap X_D, X_{D_1}) = \lim_\Delta \lim_{\Delta_1} \dim(Z \cap Y_\Delta, Y_{\Delta_1}) = \dim_H(Z, Y),$$

where Δ and Δ_1 range over all bounded disks of Y. This implies that $\dim_H(Z, X) = \dim_H(Z, Y)$. ∎

The following corollary is a consequence of the definition of Hausdorff dimension in locally convex spaces, together with the fact that the von Neumann bornology associated with a locally convex subspace Y of a locally convex space X coincides with the bornology induced on the linear subspace Y of X by the von Neumann bornology associated with X.

COROLLARY 2. *Let X be a locally convex space and let Y be a linear subspace of X with the induced topology. If Z is a subset of Y, then* $\dim_H(Z, Y) = \dim_H(Z, X)$.

The Hausdorff dimension in convex bornological spaces can also be introduced through a limiting procedure involving Hausdorff measures in each of the semi-normed spaces X_D. Consider a subset $Z \subset X$. Given $\alpha \geq 0$, we define

$$H_D(Z, \alpha) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam}_D U)^\alpha,$$

where diam_D denotes the diameter with respect to the semi-norm of X_D and where the infimum is taken over all covers $\mathcal{U}$ of $Z \cap X_D$ by sets of diameter diam_D at most δ. One can easily show that the Hausdorff dimension of Z satisfies

$$\begin{aligned}\dim_H Z &= \sup\Big\{\alpha : \lim_D \lim_{D'} H_{D'}(Z \cap X_D, \alpha) = +\infty\Big\}\\ &= \inf\Big\{\alpha : \lim_D \lim_{D'} H_{D'}(Z \cap X_D, \alpha) = 0\Big\}. \qquad (1)\end{aligned}$$

This characterization of Hausdorff dimension implies that

$$\dim_H \bigcup_{n=1}^{\infty} Y_n = \sup\{\dim_H Y_n : n \in \mathbb{N}\}, \qquad (2)$$

as for the classical Hausdorff dimension. A simple consequence of this identity is that countable subsets of convex bornological spaces have zero Hausdorff dimension (see also Proposition 3 below). In order to establish (2), observe first that the inequality

$$\dim_H \bigcup_{n=1}^{\infty} Y_n \geq \sup\{\dim_H Y_n : n \in \mathbb{N}\}$$

is immediate. Whenever the supremum is infinite we have the desired identity. Otherwise, choosing $\alpha > 0$ such that $\dim_H Y_n < \alpha$ for every n, we obtain

$$\lim_D \lim_{D'} H_{D'}(Y_n \cap X_D, \alpha) = 0.$$

The σ-subadditivity of the Hausdorff measures (which is valid in semimetric spaces) yields

$$H_{D'}\left(\bigcup_{n=1}^{\infty} Y_n \cap X_D, \alpha\right) \leq \sum_{n=1}^{\infty} H_{D'}(Y_n \cap X_D, \alpha) = 0.$$

It follows from (1) that

$$\lim_D \lim_{D'} H_{D'}\left(\bigcup_{n=1}^{\infty} Y_n \cap X_D, \alpha + \varepsilon\right) = 0$$

for every $\varepsilon > 0$. Therefore $\dim_H \bigcup_{n=1}^{\infty} Y_n \leq \alpha + \varepsilon$ and thus

$$\dim_H \bigcup_{n=1}^{\infty} Y_n \leq \sup\{\dim_H Y_n : n \in \mathbb{N}\}.$$

Now let X be a semi-normed space. Denote by B_n the ball of radius $n \in \mathbb{N}$ centered at the origin. For every n, the linear space X_{B_n} coincides with X. Moreover, the balls B_n with $n \in \mathbb{N}$ constitute a fundamental system of bounded sets, in the sense that every bounded subset of X is contained in some B_n. Therefore, for any subset Z of X,

$$\dim_H Z = \lim_{D'} \dim(Z, X_{D'}) = \lim_{n\to\infty} \dim(Z, X_{B_n}).$$

The identity between the semi-normed spaces X_{B_n} and X is bi-Lipschitz, since $p_{B_n}(x) = \|x\|/n$ for every $x \in X$. Hence,

$$\dim_H Z = \lim_{n\to\infty} \dim(Z, X) = \dim(Z, X).$$

This shows that for semi-normed spaces the classical notion of Hausdorff dimension and the notion of Hausdorff dimension introduced in Section 2.3 are equivalent. The following statement further justifies that it is appropriate to call the number $\dim_H Z$ introduced above the Hausdorff dimension of Z, thus maintaining the designation used in the classical theory.

PROPOSITION 3. *Let $\phi\colon E \to X$ be an embedding of a finite-dimensional space E into a Hausdorff locally convex space X. For every subset Z of E, the classical Hausdorff dimension of Z coincides with* $\dim_H \phi(Z)$.

Proof. Since $\phi(E)$ is finite-dimensional, the topology induced on $\phi(E)$ by the locally convex space X is the unique Hausdorff topology for which $\phi(E)$ is a linear topological space. Such topology is normable, whence it follows that ϕ yields a bi-Lipschitz map from E onto $\phi(E)$. Therefore, for any subset Z of E,

$$\dim(Z, E) = \dim_H(\phi(Z), \phi(E)).$$

We conclude by Corollary 2 that $\dim(Z, E) = \dim_H(\phi(Z), X)$. ∎

A priori the notion of Hausdorff dimension on locally convex spaces could very well be trivial. For example, it could be always zero or always infinity. The following statement shows that this is never the case.

PROPOSITION 4. *In every infinite-dimensional Hausdorff locally convex space X, for each $\alpha \geq 0$ there exists a subset of X with Hausdorff dimension equal to α.*

Proof. Choose $n \in \mathbb{N}$ such that $n \geq \alpha$, and consider a set $A \subset \mathbb{R}^n$ with (classical) Hausdorff dimension equal to α. Take n linearly independent vectors $e_1, \ldots, e_n$ of X, and define an embedding $\phi\colon \mathbb{R}^n \to X$ by $\phi(a_1, \ldots, a_n) = \sum_{i=1}^n a_i e_i$. By Proposition 3 we obtain $\dim_H \phi(A) = \alpha$. ∎

2.4. *Hausdorff Dimension in Non-Metrizable Spaces*

In convex bornological spaces with a fundamental sequence of bounded sets, the limits of nets in the definition of Hausdorff dimension can be replaced by limits of sequences. In fact, any increasing fundamental sequence $(D_n)_n$ of bounded disks in a space X constitutes a family cofinal to the family of all bounded disks D of X, and therefore to each D-indexed net is associated a D_n-indexed subnet. Writing $X_n = X_{D_n}$, we obtain

$$\begin{aligned}\dim_H Z &= \lim_{D} \lim_{D'} \dim(Z \cap X_D, X_{D'}) = \lim_{D_n} \lim_{D_k} \dim(Z \cap X_{D_n}, X_{D_k}) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap X_n, X_k)\end{aligned}$$

for every subset Z of X.

A simple argument shows that it is impossible for a non-normable metrizable locally convex space X to have a fundamental sequence of bounded sets: There would exist a decreasing fundamental sequence $(V_n)_n$ of neighborhoods of 0 and an increasing fundamental sequence $(D_n)_n$ of bounded sets. For each n one might choose $x_n \in V_n \backslash D_n$, for otherwise there would be a bounded neighborhood of 0, whence X would be normable. The sequence thus obtained would converge to zero and would be unbounded, which is a contradiction.

The so-called *Silva spaces* constitute a particular class of locally convex spaces for which a fundamental sequence of bounded sets does exist. Given a sequence $(X_n)_n$ of linear subspaces of a space X such that $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$, we shall denote by $i_n\colon X_n \to X_{n+1}$ and $j_n\colon X_n \to X$ the associated linear embeddings. We say that a locally convex space X is a *Silva space* if there exists a sequence $(X_n)_n$ of normed subspaces of X such that

1. $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} X_n$;
2. i_n is compact; i.e., i_n maps bounded subsets of X_n onto relatively compact subsets of X_{n+1}, for each $n \in \mathbb{N}$;
3. j_n is continuous for each $n \in \mathbb{N}$, and the topology of X is the finest locally convex topology on X that makes all maps j_n continuous.

Since compact operators between normed spaces are continuous, every Silva space is the inductive limit of any of its associated sequences $(X_n)_n$.

We remark that no infinite-dimensional Silva space is a Baire space (see [2, Corollary 7.2.10]; note that Silva spaces are special cases of the (LF)-spaces in [2]), and thus topological dimension may in general lack good embedding properties.

The designation "Silva space" was proposed by Köthe after Sebastião e Silva, who introduced these spaces in particular in connection with his study of distributions. Every finite-dimensional normed space is a Silva space: in this case one can simply consider the constant sequence $X_n = X$ for each $n \in \mathbb{N}$. Silva spaces are a natural generalization of finite-dimensional normed spaces, although no infinite-dimensional Silva space is metrizable.

We now collect several well-known properties of Silva spaces.

If $(X_n)_n$ is a sequence of normed spaces associated with the Silva space X, then the following properties hold:

1. A subset $Z \subset X$ is open (respectively, closed) if and only if $Z \cap X_n$ is an open (respectively, closed) subset of X_n for every $n \in \mathbb{N}$.
2. A subset $Z \subset X$ is bounded (respectively, compact) if and only if it is a bounded (respectively, compact) subset of X_n for some $n \in \mathbb{N}$.
3. A sequence converges in X if and only if it converges in X_n for some $n \in \mathbb{N}$.

Moreover, Silva spaces are complete Hausdorff spaces, and the closed graph theorem holds for linear maps between Silva spaces.

THEOREM 5. *Let X be a Silva space and let $(X_n)_n$ be a sequence of normed spaces associated with X. For every subset Z of X,*

$$\dim_H Z = \lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap X_n, X_k).$$

Proof. Let Z be a subset of X. We shall prove the following facts, from which the statement readily follows: (i) The identity in the theorem holds for a particular sequence of normed spaces associated with X; (ii) the identity holds for a sequence $(E_n)_n$ if and only if it holds for $(\overline{E}_n)_n$, where $\overline{E}_n$ is the completion of E_n; and (iii) if the identity holds for a particular sequence of Banach spaces associated with X, then it also holds for any other such sequence.

Proof of (i). Let $(X_n)_n$ be a sequence as in the theorem. Denote by B_n the unit ball in X_n. Set $D_1 = B_1$ and let D_{n+1} be the disked hull of $(n+1)(D_n \cup B_{n+1})$, for every n. The sequence $(D_n)_n$ is an increasing fundamental sequence of bounded disks, since the ball of X_n with radius m and center at the origin is contained in D_{n+m}. Therefore, the identity holds when X_n is replaced by X_{D_n}. We observe that p_{D_n} is a norm, since the space X is Hausdorff.

Proof of (ii). Let $(E_n)_n$ be a sequence of normed spaces associated with X. Since X is complete, the completion $\overline{E}_n$ of E_n is identifiable to a subspace of X. Hence $\overline{E}_n$ is embedded into $\overline{E}_{n+1}$, and it is easy to prove that this embedding is compact. Thus the sequence $(\overline{E}_n)_n$ yields a Silva space Y. The linear spaces X and Y are clearly identical, and the closed graph theorem for Silva spaces ensures that X and Y are identical as locally convex spaces.

By Corollary 2 and by taking into account that $Z \cap E_n \subset Z \cap \overline{E}_n$ and that $Z \cap E_n \subset E_k$ for $k > n$, we obtain

$$\dim(Z \cap E_n, E_k) = \dim(Z \cap E_n, \overline{E}_k) \leq \dim(Z \cap \overline{E}_n, \overline{E}_k),$$

whence

$$\lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap E_n, E_k) \leq \lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap \overline{E}_n, \overline{E}_k).$$

For each $n \in \mathbb{N}$, let B_n be the unit ball in $\overline{E}_n$. Since B_n is a bounded subset of X, it is also a bounded subset of E_{m_n} for some $m_n \in \mathbb{N}$. Since every vector of $\overline{E}_n$ is collinear with some element of B_n and E_{m_n} is a linear space, it follows that $\overline{E}_n \in E_{m_n}$. Moreover, one can choose m_n so that the sequence $(m_n)_n$ is strictly increasing. For $k > m_n$ the identity $\dim(Z \cap E_{m_n}, E_k) = \dim(Z \cap E_{m_n}, \overline{E}_k)$ holds. Hence $\dim(Z \cap \overline{E}_n, \overline{E}_k) \leq \dim(Z \cap E_{m_n}, E_k)$, and therefore

$$\begin{aligned}\lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap \overline{E}_n, \overline{E}_k) &\leq \lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap E_{m_n}, E_k) \\ &\leq \lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap E_n, E_k).\end{aligned}$$

Proof of (iii). Let $(X_n)_n$ and $(Y_m)_m$ be two sequences of Banach spaces associated with X. For some increasing sequence $(m_n)_n$ of natural numbers we have $X_n \subset Y_{m_n}$ (see Proof of (ii)). We now show that the embedding of X_n into Y_{m_n} is continuous. Let $(x_j)_j$ be a sequence of elements of X_n such that $x_j \to 0$ as $j \to \infty$ with respect to the norm of X_n, and suppose that $(x_j)_j$ converges in Y_{m_n} to some element x of Y_{m_n}. By the closed graph theorem, it suffices to prove that $x = 0$. Since the embeddings of X_n and Y_{m_n} into X are continuous, we must have $x_j \to 0$ and $x_j \to x$ in X as $j \to \infty$, and hence $x = 0$.

By a similar argument one can obtain an increasing sequence $(\ell_n)_n$ of natural numbers such that each Y_{m_n} is continuously embedded into X_{ℓ_n}.

For every n, k, the inclusions $Y_{m_k} \hookrightarrow X_{\ell_k}$ and $X_n \subset Y_{m_n}$ entail

$$\dim(Z \cap X_n, X_{\ell_k}) \leq \dim(Z \cap X_n, Y_{m_k}) \leq \dim(Z \cap Y_{m_n}, Y_{m_k}).$$

It follows that

$$\lim_{k\to\infty} \dim(Z \cap X_n, X_k) \leq \lim_{k\to\infty} \dim(Z \cap Y_{m_n}, Y_k),$$

since both limits exist and $(X_{\ell_k})_k$ and $(Y_{m_k})_k$ are subsequences of $(X_k)_k$ and $(Y_k)_k$, respectively. Similarly one obtains

$$\lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap X_n, X_k) \leq \lim_{n\to\infty} \lim_{k\to\infty} \dim(Z \cap Y_n, Y_k).$$

By interchanging the roles of $(X_n)_n$ and $(Y_m)_m$ we prove the desired identity. ∎

An example of infinite-dimensional Silva space is given by the space of tempered distributions $\mathcal{S}'$. This is the dual of the space $\mathcal{S}$ of functions $\psi \in C^\infty(\mathbb{R})$ such that ψ and each of its derivatives tend to zero at infinity faster than any polynomial. In this case, each space X_n comprises the distributions of the form $D^n(p_n F)$, where p_n is the polynomial $p_n(x) = (1+x^2)^n$ and F is any bounded continuous function. The norm in X_n is defined as the infimum of the L^∞-norms corresponding to all functions F associated with a given distribution. The compactness of the embedding of X_n into X_{n+1} is a consequence of the theorem of Ascoli. Other examples of Silva spaces are provided by some spaces of ultradistributions.

Another example of Silva space is provided by the space $H(C)$ of germs of holomorphic functions on a non-empty compact set $C \subset \mathbb{C}$. The elements of $H(C)$ are equivalence classes of holomorphic functions defined on some open set containing C, two functions being equivalent when they coincide on some open neighborhood of C.

For each n, let B_n be the $\frac{1}{n}$-open neighborhood of C. The space X_n of all bounded holomorphic functions defined on B_n is a Banach space when endowed with the supremum norm. Every element of X_n can be considered as an element of X_{n+1}, since the restriction to the set B_{n+1} yields a one-to-one operator. By Montel's theorem on normal families, the restriction operator is compact. The space $H(C)$ is the inductive limit of the sequence $(X_n)_n$.

We now consider the particular case of $C = \{0\}$. Set $X = H(\{0\})$. The space X can be identified with the set of Mac Laurin series with positive radius of convergence, and each element $f \in X$ is represented by the sequence of numbers $a_n(f)$ that are the coefficients of the Mac Laurin series $f(z) = \sum_{n=0}^\infty a_n(f) z^n$. Therefore, the Silva space X can be identified with the subspace of sequences

$$Y = \left\{ (a_n)_n \in \mathbb{R}^{\mathbb{N}} : 1/ \varlimsup_{n\to\infty} \sqrt[n]{|a_n|} > 0 \right\}.$$

It is straightforward to see that Y comprises the sequences $(a_n)_n$ with at most exponential growth. The topology of X induces a non-metrizable topology on Y. Furthermore, Y becomes a Silva space. The following example considers the notion of Hausdorff dimension on the space X (and hence, also on Y).

Example 6. Consider the Silva space $X = H(\{0\})$. For each $\varepsilon > 0$, let f_ε be the element of X represented by

$$f_\varepsilon(z) = \sum_{n=0}^\infty \varepsilon^{-n} z^n.$$

Its radius of convergence is equal to ε, and hence $f_\varepsilon \in X_n$ whenever $\frac{1}{n} < \varepsilon$. Let $K \subset \mathbb{R}$ be a bounded set of positive numbers such that $0 \in \overline{K}$, and consider the set $Z = \{f_\varepsilon : \varepsilon \in K\}$. (We note that if $0 \notin \overline{K}$ then $Z \subset X_n$ for some n, and thus in this case one could apply the classical notion of Hausdorff dimension.) We shall show that

$$\dim_H Z = \dim_H K. \tag{3}$$

For every n, let $Z_n = \{f_\varepsilon \in X : \varepsilon > \frac{1}{n}\}$ and $K_n = \{\varepsilon \in K : \varepsilon > \frac{1}{n}\}$. Set $c_1 = 1/(\sup K + \frac{1}{n+k})^2$ and $c_2 = (\frac{n(n+k)}{k})^2$. Using the identity $f_\varepsilon(z) - f_\delta(z) = \frac{\delta-\varepsilon}{(\varepsilon-z)(\delta-z)}$ and the maximum principle, we easily obtain

$$c_1|\delta - \varepsilon| \leq \|f_\delta - f_\varepsilon\|_{n+k} \leq c_2|\delta - \varepsilon|.$$

Therefore, for every k and n the mapping from K_n into X_{n+k} given by $\varepsilon \mapsto f_\varepsilon$ is bi-Lipschitz, and $\dim_H K_n = \dim(Z_n, X_{n+k})$. It follows that

$$\dim_H Z = \lim_{n\to\infty} d_n(Z) = \lim_{n\to\infty} \dim_H K_n = \dim_H K.$$

This establishes the identity in (3). ∎

We remark that the set of sequences in Y corresponding to the set $Z \subset X$ in Example 6 is not in $\ell^p(\mathbb{N})$ for any $p \in [1, \infty]$. Therefore, the classical notion of Hausdorff dimension in $\ell^p(\mathbb{N})$ cannot be applied, and we need the new notion introduced in Section 2.3.

REFERENCES

1. L. Barreira, Ya. Pesin, and J. Schmeling, Dimension and product structure of hyperbolic measures, *Ann. Math.* **149** (1999), 755–783.
2. M. Kunzinger, "Barrelledness, Baire-Like- and (LF)-Spaces," Pitman Research Notes in Mathematics, Vol. 298, Longman, Harlow/New York, 1993.
3. Ya. Pesin, "Dimension Theory in Dynamical Systems: Contemporary Views and Applications," Chicago Lectures in Mathematics, Chicago Univ. Press, Chicago, 1997.