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Third-order nilpotency, nice reachability and asymptotic stability ☆

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Abstract

We consider an affine control system whose vector fields span a third-order nilpotent Lie algebra. We show that the reachable set at time T using measurable controls is equivalent to the reachable set at time T using piecewise-constant controls with no more than four switches. The bound on the number of switches is uniform over any final time T. As a corollary, we derive a new sufficient condition for stability of nonlinear switched systems under arbitrary switching. This provides a partial solution to an open problem posed in [D. Liberzon, Lie algebras and stability of switched nonlinear systems, in: V. Blondel, A. Megretski (Eds.), Unsolved Problems in Mathematical Systems and Control Theory, Princeton Univ. Press, 2004, pp. 203–207].

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1. Introduction

Consider the control system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + u\boldsymbol{g}(\boldsymbol{x}), \quad u \in \mathcal{U}, \tag{1}$$

where $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ are two analytic vector fields, and $\mathcal{U}$ is the set of measurable functions $u(\cdot) : \mathbb{R} \to [0, 1]$. Given an initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, and an admissible control u, we use $\boldsymbol{x}(t; u, \boldsymbol{x}_0)$ to denote the solution of (1) at time t.

For a subset of admissible controls $\mathcal{V} \subseteq \mathcal{U}$, let

$$R(t; \mathcal{V}, \boldsymbol{x}_0) := \{\boldsymbol{x}(t; v, \boldsymbol{x}_0): \ v \in \mathcal{V}\},$$

that is, the *reachable set* at time $t \geqslant 0$ using controls from $\mathcal{V}$. An important problem in the analysis of control systems can be stated as follows. Find conditions guaranteeing that

$$R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{V}, \boldsymbol{x}_0),$$

where $\mathcal{V} \subset \mathcal{U}$ is some subset of "nice" controls. In other words, any point that can be reached at time T using a control $u \in \mathcal{U}$ can also be reached, at the same time, using a "nice" control $v \in \mathcal{V}$. This property, sometimes referred to as *reachability with nice controls*, is important for both theoretical and practical reasons.

The main result of this paper is that a certain Lie-algebraic condition implies

$$R(t; \mathcal{U}, \boldsymbol{x}_0) = R(t; \mathcal{PC}_4, \boldsymbol{x}_0), \quad \forall t \geqslant 0, \ \forall \boldsymbol{x}_0 \in \mathbb{R}^n, \tag{2}$$

where $\mathcal{PC}_j \subset \mathcal{U}$ is the set of piecewise constant controls with no more than j discontinuities on their domain of definition. Note that (2) has an important practical application. It implies that any point-to-point control problem is reduced to the problem of determining a (small) set of parameters: the four switching times and the five control values between each two consecutive switchings.

An interesting feature of (2) is that the bound on the number of discontinuities is *uniform* over all $t \geqslant 0$ and all $\boldsymbol{x}_0 \in \mathbb{R}^n$. In this respect, the result is *global*. Many other reachability with nice controls results are local in the sense that either (1) they hold only for sufficiently small final time T, or (2) the complexity of the set $\mathcal{V}$ increases with T. A typical example of case (2) is the celebrated bang–bang theorem of linear control systems.

Theorem 1 *(Bang–bang theorem [2–4]). Consider the system* (1) *with* $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$ *and* $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{b}$. *Then, for any* $T > 0$ *there exists an integer* $j = j(T)$ *such that*

$$R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{BB}_j, \boldsymbol{x}_0), \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

Here $\mathcal{BB}_j \subset \mathcal{PC}_j$ *is the set of bang–bang controls with no more than* j *discontinuities.*

This is a local result in the sense that the number of required switches, $j(T)$, increases with T.[1]

[1] An exception is when all the eigenvalues of the matrix A are real [3, Chapter 15].

Our main result was motivated by an open problem on the stability analysis of switched systems with a nilpotent Lie algebra. We now briefly review the relevant issues.

1.1. Stability analysis of switched systems

Two vector fields $\boldsymbol{f}_0, \boldsymbol{f}_1$ give rise to the *switched system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_\sigma(\boldsymbol{x}), \tag{3}$$

where $\sigma : [0, \infty) \to \{0, 1\}$ is a piecewise constant function of time, called a *switching signal*. We say that (3) is *globally uniformly asymptotically stable* (GUAS) if there exists a class $\mathcal{KL}$ function[2] β such that for every initial condition $\boldsymbol{x}(0)$ *every* solution of (3) satisfies

$$\bigl|\boldsymbol{x}(t)\bigr| \leqslant \beta\bigl(\bigl|\boldsymbol{x}(0)\bigr|, t\bigr), \quad \forall t \geqslant 0. \tag{4}$$

The difficulty in analyzing the stability of (3) is that the switched system admits an infinite number of solutions for each initial condition. It is well known that the global asymptotic stability of the individual subsystems $\dot{\boldsymbol{x}} = \boldsymbol{f}_i(\boldsymbol{x})$ is necessary but *not* sufficient for GUAS of the switched system (3). An important problem is identifying conditions for the individual subsystems—apart from the obviously necessary requirement as to their global asymptotic stability—which guarantee GUAS of (3). This problem has received considerable attention in the literature; see [5, Chapter 2], [6] for some available results.

For the special case of *linear switched systems* (that is, when $\boldsymbol{f}_i = A_i\boldsymbol{x}$, $A_i \in \mathbb{R}^{n\times n}$) determining a necessary and sufficient condition for GUAS is equivalent to solving one of the oldest open problems in the theory of control: the problem of absolute stability (see, e.g., [7]). Pyatnitskiy and Rapoport [8] developed a variational approach, based on characterizing the "most unstable" solution of the switched system, to tackle the absolute stability problem (for a survey on the variational approach, see [9]). They applied the maximum principle to derive many powerful results on this worst-case solution [10]. For the case of second-order systems, it is possible to solve their optimal control problem using dynamic programming techniques [11] (see also [9]). This provides a necessary and sufficient condition for GUAS of switched linear systems in the plane [12] (see also [13]).

Another promising approach for addressing the GUAS problem is based on studying the commutation relations between the two vector fields using Lie-algebraic techniques. The *Lie bracket* of two vector fields is another vector field defined by

$$[\boldsymbol{f}_0, \boldsymbol{f}_1](\boldsymbol{x}) := \frac{\partial \boldsymbol{f}_1(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}_0(\boldsymbol{x}) - \frac{\partial \boldsymbol{f}_0(\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{f}_1(\boldsymbol{x}). \tag{5}$$

The Lie-algebra spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$ is

$$\{\boldsymbol{f}, \boldsymbol{g}\}_{LA} := \operatorname{span}\bigl\{\boldsymbol{f}, \boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}], \bigl[\boldsymbol{f}, [\boldsymbol{f}, \boldsymbol{g}]\bigr], \bigl[\boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}]\bigr], \ldots\bigr\}$$

[2] Recall that a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geqslant 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geqslant 0$.

where the dots indicate all iterated Lie brackets. We say that $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ is *kth-order nilpotent* if all iterated Lie brackets containing $k+1$ terms vanish (and there exists a Lie bracket containing k terms that does not vanish).

Gurvits [14] showed that first- and second-order nilpotency is a sufficient condition for GUAS of the switched system. Liberzon, Hespanha and Morse [15] showed that a sufficient condition for GUAS is that the Lie-algebra is *solvable* (see also [16]). This includes the special case of nilpotent Lie-algebras (of any order).

Nonlinear switched systems are much less thoroughly understood. In particular, the methods used to tackle nilpotent linear switched systems do not seem to apply. These issues are explained in [1] where the question whether nilpotency implies GUAS for *nonlinear* switched systems is posed as an open problem. Mancilla-Aguilar [17] showed that the answer is affirmative for the case of first-order nilpotency (that is, when the vector fields commute).

It is well known that there is a strong connection between the maximum principle (MP) of optimal control and $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ (see, e.g., [3,18]). This suggests that the variational and Lie-algebraic approaches are actually related. Indeed, Margaliot and Liberzon [19] showed that if $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ is second-order nilpotent, then the worst-case solution of the switched system contains no more than two switches. The proof is based on a Lie-algebraic analysis of the switching function defined in the MP. In particular, this implies that second-order nilpotency implies GUAS for nonlinear switched systems.

An analysis of the arguments in [19] shows that their approach cannot be used to address the case of third-order nilpotency, that is, when

$$\big[\boldsymbol{f}_i, \big[\boldsymbol{f}_j, [\boldsymbol{f}_k, \boldsymbol{f}_l]\big]\big](\boldsymbol{x}) = \boldsymbol{0}, \quad \forall i, j, k, l \in \{0, 1\}. \tag{6}$$

Our main result is that if $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ is third-order nilpotent then (2) holds. To prove this, we apply a new approach based on (1) a product expansion for the Chen series derived by Sussmann [20]; and (2) a *second-order* MP (see, e.g., [21–23]).

Note that setting $u \equiv 0$ [$u \equiv 1$] in (1) yields $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ [$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})$]. Hence, for $\boldsymbol{f}_0 = \boldsymbol{f}$ and $\boldsymbol{f}_1 = \boldsymbol{f} + \boldsymbol{g}$, every trajectory of the switched system (3) is also a solution of (1). Using this we derive, as a corollary of our main result, a new sufficient condition for the stability of nonlinear switched systems.

The remainder of the paper is organized as follows. Our main result is stated in Section 2. Section 3 reviews several known results that are used later on. These allow reducing the proof to the analysis of time-optimal controls for a specific control system. Section 4 is devoted to the analysis of the regularity properties of these time-optimal controls. The proof of our main result is completed in Section 5. Section 6 demonstrates the ideas using an example. Section 7 contains some concluding remarks.

2. Main result

Fix an arbitrary point $\boldsymbol{x}_0 \in \mathbb{R}^n$. We use $\boldsymbol{x}(\cdot; u, \boldsymbol{x}_0)$ to denote the solution of the system (1) with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ corresponding to a control $u \in \mathcal{U}$. Since the right-hand side of (1) is bounded on every bounded ball in $\mathbb{R}^n$, there exists a largest time $T_{\max} \in (0, \infty]$ (that depends on $|\boldsymbol{x}_0|$) such that $\boldsymbol{x}(\cdot; u, \boldsymbol{x}_0)$ is well defined for all $u \in \mathcal{U}$ and all $t \in [0, T_{\max})$. We are now ready to state our main result.

Theorem 2. *If* $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ *is third-order nilpotent then*

$$R(t; \mathcal{U}, \boldsymbol{x}_0) = R(t; \mathcal{PC}_4, \boldsymbol{x}_0), \quad \forall t \in [0, T_{\max}).$$

The next result will allow us to apply Theorem 2 to the stability analysis of (1). We say that the control system (1) is GUAS if there exists a class $\mathcal{KL}$ function β such that for any $u \in \mathcal{U}$

$$\big|\boldsymbol{x}(t; u, \boldsymbol{x}_0)\big| \leqslant \beta\big(|\boldsymbol{x}_0|, t\big), \quad \forall t \geqslant 0. \tag{7}$$

In other words, the bound (7) holds for *every* solution of the control system.

For $c \in [0, 1]$, we use $\boldsymbol{x}(t; c, \boldsymbol{x}_0)$ to denote the solution of (1) corresponding to the constant control $u(t) \equiv c$.

Proposition 1. *Suppose that* (1) *there exists function* $\bar{\beta} \in \mathcal{KL}$ *such that for any* $c \in [0, 1]$

$$\big|\boldsymbol{x}(t; c, \boldsymbol{x}_0)\big| \leqslant \bar{\beta}\big(|\boldsymbol{x}_0|, t\big), \quad \forall t \geqslant 0 \tag{8}$$

and (2) *for any* $t_f < T_{\max}$ *there exists a finite integer* d (*that does not depend on* t_f) *such that*

$$R(t_f; \mathcal{U}, \boldsymbol{x}_0) = R(t_f; \mathcal{PC}_d, \boldsymbol{x}_0). \tag{9}$$

Then the control system (1) *is GUAS.*

Note that (8) implies in particular that for any *fixed* $c \in [0, 1]$ the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + c\boldsymbol{g}(\boldsymbol{x})$ is GAS.

Proof. Fix an arbitrary $t_f < T_{\max}$. Consider the problem of finding a control $u \in \mathcal{U}$ that maximizes $J(u) := |\boldsymbol{x}(t_f)|^2$. Such a control exists, and we use $u^*, \boldsymbol{x}^*$ to denote an optimal control and the corresponding trajectory, respectively. It follows from (9) that we may assume, without loss of generality, that $u^* \in \mathcal{PC}_d$. The interval $[0, t_f]$ is thus divided into a maximum of $d + 1$ subintervals: $[t_0, t_1), [t_1, t_2), \ldots, [t_d, t_{d+1}]$, with $t_0 = 0$ and $t_{d+1} = t_f$, on each of which $\boldsymbol{x}^*$ satisfies $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + c_j \boldsymbol{g}(\boldsymbol{x})$ for some $c_j \in [0, 1]$. Using (8) yields

$$\big|\boldsymbol{x}^*(t_f)\big| \leqslant \bar{\beta}\big(\cdots \bar{\beta}\big(\bar{\beta}\big(\bar{\beta}\big(|\boldsymbol{x}_0|, t_1\big), t_2 - t_1\big), t_3 - t_2 - t_1\big), \ldots, t_{d+1} - t_d - \cdots - t_1\big).$$

Lemma 2.2 in [17] implies that there exists $\beta \in \mathcal{KL}$ such that

$$\bar{\beta}\big(\cdots \bar{\beta}\big(\bar{\beta}\big(\bar{\beta}(r, t_1), t_2 - t_1\big), t_3 - t_2 - t_1\big), \ldots, t_{d+1} - t_d - \cdots - t_1\big) \leqslant \beta(r, t_{d+1})$$

for all $r \geqslant 0$ and all $t_1 \geqslant 0, t_2 \geqslant t_1, \ldots, t_{d+1} \geqslant t_1 + \cdots + t_d$. Hence,

$$\big|\boldsymbol{x}^*(t_f)\big| \leqslant \beta\big(|\boldsymbol{x}_0|, t_f\big). \tag{10}$$

In view of the bound $\beta(|\boldsymbol{x}_0|, t_f) \leqslant \beta(|\boldsymbol{x}_0|, 0)$ and the fact that $\boldsymbol{x}_0$ and $t_f < T_{\max}$ were *arbitrary*, we conclude that $\boldsymbol{x}^*(t)$ is bounded for all t and so exists globally in time. In other words, $T_{\max} = \infty$, and t_f could be an arbitrary positive number. By the definition of $\boldsymbol{x}^*$, we conclude that *all* solutions of (1) satisfy the bound (10). □

Loosely speaking, Proposition 1 states that to obtain instability in a control system that "switches" between GAS subsystems, we must never stop switching. Combining Theorem 2 and Proposition 1 yields the following.

Corollary 1. *Suppose that* (1) *there exists a function* $\bar{\beta} \in \mathcal{KL}$ *such that for any* $c \in [0, 1]$

$$\left|\boldsymbol{x}(t; c, \boldsymbol{x}_0)\right| \leqslant \bar{\beta}\left(|\boldsymbol{x}_0|, t\right), \quad \forall t \geqslant 0$$

and (2) $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ *is third-order nilpotent. Then the control system* (1) *is GUAS, and, in particular, the switched system* (3) *with* $\boldsymbol{f}_0 = \boldsymbol{f}$ *and* $\boldsymbol{f}_1 = \boldsymbol{f} + \boldsymbol{g}$ *is GUAS.*

The remainder of this paper is devoted to the proof of Theorem 2.

3. Preliminaries

In this section, we review two results that will be used later on.

3.1. The Hall–Sussmann system

The first result shows that the study of a general third-order nilpotent system can be reduced to the study of a *specific* third-order nilpotent system.

We use the notation $\exp(t\boldsymbol{f})(\boldsymbol{x}_0)$ for the solution at time t of the differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}(0) = \boldsymbol{x}_0$.

Proposition 2. *If* $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ *is third-order nilpotent, then the solution of* (1) *satisfies*

$$\begin{aligned}\boldsymbol{x}(t; u, \boldsymbol{x}_0) = {} & \exp\left(C_1(t; u)\boldsymbol{f}\right) \circ \exp\left(C_2(t; u)\boldsymbol{g}\right) \circ \exp\left(C_3(t; u)[\boldsymbol{f}, \boldsymbol{g}]\right) \\ & \circ \exp\left(C_4(t; u)\left[\boldsymbol{f}, [\boldsymbol{f}, \boldsymbol{g}]\right]\right) \circ \exp\left(C_5(t; u)\left[\boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}]\right]\right)(\boldsymbol{x}_0),\end{aligned} \tag{11}$$

where the C_i*s are the solution of*

$$\dot{\boldsymbol{C}}(t) = \boldsymbol{p} + u\boldsymbol{q}\left(\boldsymbol{C}(t)\right), \qquad \boldsymbol{C}(0) = \boldsymbol{0}, \tag{12}$$

with $\boldsymbol{p} = (1, 0, 0, 0, 0)^T$ *and* $\boldsymbol{q} = (0, 1, C_1, C_1^2/2, C_1C_2)^T$.

Proof. Note that if $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ is third-order nilpotent, then the set of vector fields $\{\boldsymbol{f}, \boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}], [\boldsymbol{f}, [\boldsymbol{f}, \boldsymbol{g}]], [\boldsymbol{g}, [\boldsymbol{f}, \boldsymbol{g}]]\}$ is a P. Hall basis of $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$. Hence, Proposition 2 is a special case of the main theorem in [20]. □

We refer to (12) as the *Hall–Sussmann system*.[3] It is easy to see that its solutions $\boldsymbol{C}(t; u) \in \mathbb{R}^5$ exist for any $u \in \mathcal{U}$ and all $t \in \mathbb{R}$. A direct calculation yields

$$\left[\boldsymbol{q}, [\boldsymbol{p}, \boldsymbol{q}]\right] = (0, 0, 0, 0, 1)^T, \qquad \left[\boldsymbol{p}, [\boldsymbol{p}, \boldsymbol{q}]\right] = (0, 0, 0, 1, 0)^T,$$

[3] For more details on the Hall–Sussmann system and its applications, see [24,25].

and all higher-order brackets vanish. Hence, $\{\boldsymbol{p}, \boldsymbol{q}\}_{LA}$ is also third-order nilpotent. Note also that the vector $\boldsymbol{q}$ is polynomial in the C_is (see [26] for some related considerations).

3.2. Reachability and time-optimality

It will be useful to define another control system by introducing the variables $y_1(t) := C_1(t) - t$, and $y_i(t) := C_i(t)$ for $i = 2, \dots, 5$. Using (12) yields $y_1(t) \equiv 0$ and

$$\begin{aligned} \dot{y}_2 &= u, & y_2(0) &= 0, \\ \dot{y}_3 &= ut, & y_3(0) &= 0, \\ \dot{y}_4 &= ut^2/2, & y_4(0) &= 0, \\ \dot{y}_5 &= uty_2, & y_5(0) &= 0. \end{aligned} \tag{13}$$

Note that this is a time-varying driftless control system.

The next result is a special case of a construction introduced by Sussmann in [27]. It will allow us to study the reachable set of (12) using variational principles.

Proposition 3. *Fix an arbitrary $T > 0$ and $u \in \mathcal{U}$. There exists a time $T' \in [0, T]$ and a control*

$$v(t) := \begin{cases} w^*(t), & t \in [0, T'), \\ 0, & t \in [T', T] \end{cases} \tag{14}$$

such that: (1) w^* *is a time-optimal control for the system* (13); *and* (2) $\boldsymbol{C}(T; v) = \boldsymbol{C}(T; u)$.

In other words, the control v, which is a concatenation of a time-optimal control and the zero control, steers the Hall–Sussmann system to the same point as the control u does at the same final time T.

Proof. Note that $C_1(T; u) = C_1(T; v) = T$, so we only need to prove the result for $C_2, \dots, C_5$. Clearly, it is enough to prove that $\boldsymbol{y}(T; u) = \boldsymbol{y}(T; v)$ where $\boldsymbol{y} := (y_2, \dots, y_5)^T$. Denote $\boldsymbol{y}^u := \boldsymbol{y}(T; u)$, and consider the problem of finding a time-optimal control $w^* \in \mathcal{U}$ that steers (13) to $\boldsymbol{y}^u$ in *minimal time*. It follows from [28, §7, Theorem 3] that such a control does exist, so there exists $T' \in [0, T]$ such that $\boldsymbol{y}(T'; w^*) = \boldsymbol{y}^u$. Now (13) and (14) yield $\boldsymbol{y}(T; v) = \boldsymbol{y}(T'; v) = \boldsymbol{y}(T'; w^*) = \boldsymbol{y}^u$. □

In the next section, we analyze the properties of time optimal controls of (13).

4. Time optimal controls

To apply the MP, we introduce the adjoint vector $\boldsymbol{\lambda}(t) = (\lambda_2(t), \dots, \lambda_5(t))^T$ and the Hamiltonian

$$H(t, \boldsymbol{\lambda}, u, \boldsymbol{y}) := u\big(\lambda_2 + \lambda_3 t + \lambda_4 t^2/2 + \lambda_5 t y_2\big). \tag{15}$$

Applying the MP yields the following result.

Proposition 4. *Suppose that* $u^* : [0, T'] \to [0, 1]$ *is a time-optimal control for* (13)*, and let* $\mathbf{y}^*$ *denote the corresponding trajectory. Then there exist absolutely continuous functions* $\lambda_i(t)$*,* $i = 2, \dots, 5$*, such that* $\boldsymbol{\lambda}(t) \neq \mathbf{0}$ *for all* t*,*

$$\begin{aligned} \dot{\lambda}_2 &= -\lambda_5 t u, \\ \dot{\lambda}_3 = \dot{\lambda}_4 &= \dot{\lambda}_5 = 0, \end{aligned} \tag{16}$$

and

$$u^*(t) = \begin{cases} 0, & \varphi(t) < 0, \\ 1, & \varphi(t) > 0, \end{cases} \tag{17}$$

where

$$\varphi := \lambda_2 + \lambda_3 t + \lambda_4 t^2/2 + \lambda_5 t y_2^*. \tag{18}$$

Note that (16), combined with the absolute continuity of $\boldsymbol{\lambda}$, implies that λ_3, λ_4, and λ_5 are constants. It follows from (18) that φ is absolutely continuous, and differentiating it yields

$$\dot{\varphi}(t) = \lambda_3 + \lambda_4 t + \lambda_5 y_2^*(t) \tag{19}$$

for almost all t. It follows from (19) that $\dot{\varphi}$ is also absolutely continuous and differentiating again yields

$$\ddot{\varphi}(t) = \lambda_4 + \lambda_5 u^*(t). \tag{20}$$

We use $Z(\varphi, [0, T'])$ to denote the set of zeros of φ on $[0, T']$. It is well known that this set can be, in general, very complex (see, e.g., [29,30] and the references therein). We now analyze the possible cases.

4.1. Bang arcs

If $\varphi(t) = 0$ holds only on discrete points, then (17) implies that $u^*(t) \in \{0, 1\}$ for almost all t, that is, u^* is a bang–bang control.

Proposition 5. *Suppose that* Z *is a discrete set of points. Then any time-optimal control* $u^* : [0, T'] \to [0, 1]$ *is bang–bang and either*: (1) u^* *is periodic*; *or* (2) u^* *contains no more than three switches on* $[0, T']$.

Proof. It is sufficient to prove that any bang–bang control u^* with more than three switches is periodic. Suppose that u^* has exactly four switches at times $\tau_1 < \tau_2 < \tau_3 < \tau_4$. For notational convenience, we set $\tau_0 = 0$ and $\tau_5 = T'$. We assume, without loss of generality, that $u^*(t) = 0$ for $t \in (\tau_0, \tau_1)$, so $u^*(t) = 1$ for $t \in (\tau_1, \tau_2)$, and so on.

We use $\varphi_i(t)$ to denote the restriction of the absolutely continuous function φ on the interval (τ_i, τ_{i+1}). It follows from (20) that $\varphi_1(t) = (\lambda_4 + \lambda_5)t^2 + c_1 t + c_2$ for some constants c_1 and c_2. Combining this with the fact that φ_1 must vanish on τ_1, τ_2 yields $\varphi_1(t) = (\lambda_4 + \lambda_5)(t - \tau_1) \times (t - \tau_2)$. Since $u(t) = 1$ on (τ_1, τ_2) it follows from (17) that $\lambda_4 + \lambda_5 < 0$. Analyzing the next interval, we find that $\varphi_2(t) = \lambda_4(t-\tau_2)(t-\tau_3)$, and that $\lambda_4 > 0$. Similarly, $\varphi_3(t) = (\lambda_4+\lambda_5) \times (t-\tau_3)(t-\tau_4)$.

The absolute continuity of $\dot{\varphi}$ implies that $\dot{\varphi}_1(\tau_2) = \dot{\varphi}_2(\tau_2)$ and $\dot{\varphi}_2(\tau_3) = \dot{\varphi}_3(\tau_3)$. This yields

$$(\lambda_4+\lambda_5)(\tau_2-\tau_1) = \lambda_4(\tau_2-\tau_3) \quad \text{and} \quad \lambda_4(\tau_3-\tau_2) = (\lambda_4+\lambda_5)(\tau_3-\tau_4). \tag{21}$$

Hence, $\tau_2-\tau_1 = \tau_4-\tau_3$ so φ is periodic. It now follows from (17) that u^* is periodic. □

The analysis based on the classical, first-order, MP provides considerable information on any time optimal control u^*. However, the fact that u^* might be periodic implies that the number of switches can increase with the final time T'. In order to rule out this possibility, we need to apply a more accurate analysis.

4.1.1. Second-order analysis

In this section, we apply a second-order analysis to prove that any bang–bang control with more than three switches is not optimal.

Proposition 6. *Suppose that Z is a discrete set of points. Then any time-optimal control $u^*: [0, T'] \to [0, 1]$ is bang–bang and contains no more than three switches on* $[0, T']$.

Proof. Assume that u^* is a time-optimal control with exactly four switches on $[0, T']$. We use $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < T'$ to denote the switching times, and assume, without loss of generality, that $u^*(t) = 0$ on $t \in [0, \tau_1)$. Let $\boldsymbol{y}^*$ denote the corresponding trajectory. Using (13) yields

$$\begin{aligned} y_2^*(T') &= \tau_4-\tau_3+\tau_2-\tau_1, \\ y_3^*(T') &= \left(\tau_4^2-\tau_3^2+\tau_2^2-\tau_1^2\right)/2, \\ y_4^*(T') &= \left(\tau_4^3-\tau_3^3+\tau_2^3-\tau_1^3\right)/6, \\ y_5^*(T') &= \left((\tau_2-\tau_1)^2(\tau_1+2\tau_2)+\tau_3^2(3\tau_1-3\tau_2+\tau_3)+3(\tau_2-\tau_1-\tau_3)\tau_4^2+2\tau_4^3\right)/6. \end{aligned} \tag{22}$$

For $\boldsymbol{\alpha} \in \mathbb{R}^4$ and $s > 0$, define a new control $\tilde{u}(t; \boldsymbol{\alpha}, s)$ by perturbing the switching times of u^* to $\tilde{\tau}_i := \tau_i + s\alpha_i$, $i = 1, \ldots, 4$. In other words, $\tilde{u}(t) = 0$ for $t \in (0, \tilde{\tau}_1)$, $\tilde{u}(t) = 1$ for $t \in (\tilde{\tau}_1, \tilde{\tau}_2)$, and so on.

It is clear that for any $\boldsymbol{\alpha} \in \mathbb{R}^4$, there exists a sufficiently small $s_0 > 0$ such that $\tilde{u}(s, \boldsymbol{\alpha})$: $[0, T'] \to [0, 1]$ is an admissible control for all $s \in [0, s_0]$. Let $\tilde{\boldsymbol{y}}(T'; s, \boldsymbol{\alpha})$ denote the value of the corresponding trajectory at time T'. It is easy to obtain an explicit expression for $\tilde{\boldsymbol{y}}(T')$ by substituting $\tilde{\tau}_i$ for τ_i in (22).

Let $\boldsymbol{z}(T'; s, \boldsymbol{\alpha}) := \tilde{\boldsymbol{y}}(T'; s, \boldsymbol{\alpha}) - \boldsymbol{y}^*(T')$. Note that the definition of $\tilde{\tau}_i$ implies that $\boldsymbol{z}(T'; 0, \boldsymbol{\alpha}) = \boldsymbol{0}$. We now expand $\boldsymbol{z}$ as a Taylor series about $s = 0$. A calculation yields

$$\boldsymbol{z}(T'; s, \boldsymbol{\alpha}) = sA\boldsymbol{\alpha} + o(s) \tag{23}$$

where

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 \\ -\tau_1 & \tau_2 & -\tau_3 & \tau_4 \\ -\tau_1^2/2 & \tau_2^2/2 & -\tau_3^2/2 & \tau_4^2/2 \\ a & \tau_2(\tau_2-\tau_1)-b & \tau_3(\tau_1-\tau_2)+b & (\tau_2-\tau_1+\tau_4-\tau_3)\tau_4 \end{pmatrix},$$

with $a := (\tau_1^2 - \tau_2^2 + \tau_3^2 - \tau_4^2)/2$, $b := (\tau_3^2 - \tau_4^2)/2$, and $o(\epsilon)$ denotes terms that satisfy $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$.

Let $Q := \{A\boldsymbol{\alpha}\colon \boldsymbol{\alpha} \in \mathbb{R}^4\}$. Then, every $\boldsymbol{r} \in Q$ is a tangent direction for the difference $\tilde{\boldsymbol{y}}(T') - \boldsymbol{y}^*(T')$. Recall that the proof of the MP is based on the fact that if u^* is optimal then there exists a direction $\boldsymbol{v}$ such that $\boldsymbol{v}^T\boldsymbol{r} \leqslant 0$ for any $\boldsymbol{r} \in Q$. This $\boldsymbol{v}$ is actually the value of the adjoint at the final time T'. Thus, it is possible to choose a function $\boldsymbol{\lambda}(\cdot)$ satisfying all the conditions in Proposition 4 such that

$$\boldsymbol{\lambda}^T(T')A\boldsymbol{\alpha} \leqslant 0, \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^4. \tag{24}$$

If $\boldsymbol{\lambda}^T(T')A\boldsymbol{\alpha}^0 < 0$ for some $\boldsymbol{\alpha}^0 \in \mathbb{R}^4$, then $\boldsymbol{\lambda}^T(T')A(-\boldsymbol{\alpha}^0) > 0$ and this contradicts (24). Hence, $\boldsymbol{\lambda}^T(T')A\boldsymbol{\alpha} = 0$, $\forall \boldsymbol{\alpha} \in \mathbb{R}^4$, so

$$\boldsymbol{\lambda}^T(T')A = \mathbf{0}. \tag{25}$$

A calculation yields $\det(A) = (\tau_1 - \tau_2)(\tau_2 - \tau_3)^2(\tau_3 - \tau_4)(\tau_1 - \tau_2 - \tau_3 + \tau_4)/4$. If $\det(A) \neq 0$ then (25) yields $\boldsymbol{\lambda}(T') = \mathbf{0}$ which is a contradiction of the MP (Proposition 4). Hence,

$$\tau_4 = \tau_3 + \tau_2 - \tau_1 \tag{26}$$

(which, not surprisingly, is the result we already derived using the MP). Substituting (26) in the expression for A, we find that A has a single eigenvalue that is zero. The corresponding eigenvector is

$$\boldsymbol{v}^1 = (\tau_2 - \tau_3, 2\tau_1 - \tau_2 - \tau_3, 2\tau_1 - \tau_2 - \tau_3, \tau_2 - \tau_3)^T.$$

It follows from (25) that $\boldsymbol{\lambda}(T')$ is an eigenvector of A^T corresponding to the zero eigenvalue. This yields

$$\boldsymbol{\lambda}(T') = c\begin{pmatrix} (\tau_2 - \tau_1)(\tau_1 - 2\tau_3)(\tau_1 - \tau_2 - \tau_3) \\ (\tau_1 - \tau_2)(2\tau_1 + \tau_2 - \tau_3) \\ 2(\tau_2 - \tau_1) \\ 2(\tau_1 - \tau_3) \end{pmatrix},$$

for some constant c. Hence, $\lambda_4 = 2c(\tau_2 - \tau_1)$, but we already know that $\lambda_4 > 0$ so $c > 0$. Thus, λ is uniquely defined up to multiplication by a positive scalar.

We now analyze the effect of the specific perturbation $\boldsymbol{\alpha} = \boldsymbol{v}^1$, that is, we consider $\tilde{u} = \tilde{u}(s, \boldsymbol{v}^1)$. For this perturbation the first term in (23) vanishes and a calculation using (22) yields

$$\boldsymbol{z}(T'; s, \boldsymbol{v}^1) = s^2(\tau_2 - \tau_1)(\tau_3 - \tau_2)\big(0, 0, 2\tau_1 - \tau_2 - \tau_3, 2(\tau_1 - \tau_2)\big)^T + o(s^2).$$

Hence,

$$\begin{aligned} \boldsymbol{\lambda}^T(T')\boldsymbol{z}(T'; s, \boldsymbol{v}^1) &= s^2(\tau_2 - \tau_1)(\tau_3 - \tau_2)\big(\lambda_4(2\tau_1 - \tau_2 - \tau_3) + 2\lambda_5(\tau_1 - \tau_2)\big) + o(s^2) \\ &= s^2(\tau_2 - \tau_1)(\tau_3 - \tau_2)^2\lambda_4 + o(s^2) \end{aligned}$$

where the second equation follows from (21). We already know that $\lambda_4 > 0$ so $\boldsymbol{\lambda}^T(T')z(T'; s, \boldsymbol{v}^1) > 0$ for all sufficiently small $s > 0$.

To complete the proof, we need the following result, which is an immediate corollary of Theorems 1 and 2 in [22] (see also [21,31]). We use the notation $z^{(k)}(T'; 0, \boldsymbol{\alpha}) := \frac{d^k z(T'; s, \boldsymbol{\alpha})}{ds^k}|_{s=0}$.

Corollary 2. *Suppose that $u^* : [0, T'] \to [0, 1]$ is a time-optimal control for* (13)*, and $\mathbf{y}^*$ is the corresponding trajectory. Suppose that there exists $\boldsymbol{\alpha}_0 \in \mathbb{R}^4$ such that: $z^{(1)}(T'; 0, \boldsymbol{\alpha}_0) = \mathbf{0}$ and $z^{(j)}(T'; 0, \boldsymbol{\alpha}_0) \neq \mathbf{0}$ for some $j > 0$. Let $\boldsymbol{q}_0 := z^{(k)}(T'; 0, \boldsymbol{\alpha}_0)$ where $k > 0$ is the smallest integer for which the derivative does not vanish. Then, there exists an absolutely continuous $\boldsymbol{\lambda}(t) \neq \mathbf{0}$ on $[0, T']$ which satisfies the conditions in Proposition* 4 *such that*: (1) $\boldsymbol{\lambda}^T(T')\boldsymbol{q}_0 \leqslant 0$; *and* (2) $\boldsymbol{\lambda}^T(T')z^{(1)}(T'; 0, \boldsymbol{\alpha}) \leqslant 0,\ \forall \boldsymbol{\alpha} \in \mathbb{R}^4$.

We already showed that for the specific perturbation $\boldsymbol{\alpha}_0 = \boldsymbol{v}^1$, Corollary 2 does not hold. Summarizing, we conclude that in the bang–bang case, any bang–bang control with exactly four switches is *not* time optimal.

Suppose now that u^* is a bang–bang control with j switches for some $j > 4$. It is easy to verify that perturbing the first four switching times, and reasoning exactly as above, leads to similar results, namely, that u^* cannot be optimal. We conclude that in the bang–bang case $u^* \in \mathcal{BB}(T', 3)$. This completes the proof of Proposition 6. □

We note that it is possible to apply a similar analysis directly on the nilpotent control system (1) using a powerful second-order MP developed by Agrachev and Gamkrelidze [23] (see also [31]). This is the approach we used in the abridged version of this paper [32]. However, Propositions 2 and 3 allow us to reduce the general case to the study of the specific system (13) and this makes it possible to derive the simpler proof presented above.

4.2. Singular arcs

Consider now the case where $\varphi(t) = 0$ on an interval $I \subseteq [0, T']$. In this case $\dot{\varphi} = \ddot{\varphi} = 0$ on I, so (20) yields $\lambda_4 + u\lambda_5 = 0$. If $\lambda_5 = 0$ then (18)–(20) yield $\boldsymbol{\lambda}(t) = \mathbf{0}$ which is a contradiction of the MP. Hence, $\lambda_5 \neq 0$ and $u = -\lambda_4/\lambda_5$. Recalling that λ_4 and λ_5 are constants, we conclude that on a singular arc $u(t)$ is constant.

4.3. Junctions

In this section, we show that every optimal trajectory is composed of a finite concatenation of bang–bang and singular arcs.

Proposition 7. *If $u^* : [0, T'] \to [0, 1]$ is a time-optimal control for the system* (13) *then*

$$u^* \in \mathcal{PC}(T', 3). \tag{27}$$

Proof. We consider several cases.

Case 1. Suppose that u^* contains no bang arcs. The (absolute) continuity of φ implies that in this case φ is identically zero on $[0, T']$. It follows from the discussion in Section 4.2 that $u^* \equiv c$ on $[0, T']$ and, in particular, (27) holds.

Case 2. Suppose that u^* contains a bang arc. Without loss of generality, we may assume that there exist $0 \leqslant t_1 < t_2 \leqslant T'$ such that $\varphi(t) < 0$ for $t \in J := (t_1, t_2)$.

Case 2.1. J is strictly contained in $(0, T')$. It follows from (17) that $u^*(t) \equiv 0$ on J and $\varphi(t_1) = \varphi(t_2) = 0$. Now (20) implies that φ is a second-order polynomial on J, so $\varphi(t) = a(t - t_1)(t - t_2)$ with $a \neq 0$. Differentiating yields $\dot{\varphi}(t_1) = a(t_1 - t_2)$ and $\dot{\varphi}(t_2) = a(t_2 - t_1)$. Both derivatives are different from zero, and since φ is absolutely continuous, we conclude that t_1 (t_2) is the upper (lower) bound of another bang arc. Thus, u^* is composed of a concatenation of bang-arcs, and Proposition 6 implies that in this case $u^* \in \mathcal{BB}(T', 3)$ and, in particular, (27) holds.

Case 2.2. Suppose that no bang arc is strictly contained in $[0, T']$. Thus, if (t_1, t_2) is a bang arc then either $t_1 = 0$ or $t_2 = T'$. The most general case possible is that we have two bang arcs: one on $(0, t_1)$ and the second on (t_2, T'), with $0 < t_1 < t_2 < T'$, and the interval (t_1, t_2) does not contain any bang arc. It follows from the discussion above that $u^*(t) \equiv c$ for $t \in (t_1, t_2)$. Hence, we conclude that in this case $u^* \in \mathcal{PC}(T', 2)$.

This completes the proof of Proposition 7. □

5. Proof of Theorem 2

We are now ready to prove our main result. Fix $t_f < T_{\max}$ and $u \in \mathcal{U}$. It follows from Proposition 3 that we can find a control v in the form (14) such that $\boldsymbol{C}(t_f; v) = \boldsymbol{C}(t_f; u)$, and by Proposition 7 that $v \in \mathcal{PC}_4$. Applying Proposition 2 yields $\boldsymbol{x}(t_f; u, \boldsymbol{x}_0) = \boldsymbol{x}(t_f; v, \boldsymbol{x}_0)$. This completes the proof of Theorem 2. □

6. An example

As noted above, Theorem 2 has an important practical implication as any point-to-point control problem is reduced to the problem of determining a small set of parameters. This is demonstrated by the following example.

Example 1. Consider the control system (1) with $n = 2$, $\boldsymbol{f}(\boldsymbol{x}) = (-1, -x_1^2)^T$, $\boldsymbol{g}(\boldsymbol{x}) = (2, 0)^T$, and $\boldsymbol{x}(0) = \boldsymbol{p} := (0, 2)^T$. It is easy to verify that $\{\boldsymbol{f}, \boldsymbol{g}\}_{LA}$ is third-order nilpotent. Consider the following optimal control problem: find a control $u^* \in \mathcal{U}$ that maximizes $J(u) := |\boldsymbol{x}(1; u, \boldsymbol{p})|^2$.

Applying the MP to this optimal control problem yields that the adjoint satisfies

$$\begin{aligned} \dot{\lambda}_1(t) &= 2x_1(t)\lambda_2(t), \\ \dot{\lambda}_2(t) &= 0 \end{aligned} \tag{28}$$

with $\boldsymbol{\lambda}(1) = \boldsymbol{x}(1)$. Hence, λ_2 is constant. Furthermore, since $|\dot{x}_1| \leqslant 1$ and $\dot{x}_2 = -x_1^2$, we see that $\boldsymbol{x}(0) = \boldsymbol{p}$ implies $x_2(1) > 0$, so $\lambda_2 > 0$.

The switching function is $\varphi(t) = 2\lambda_1(t)$ so $\dot{\varphi}(t) = 4x_1(t)\lambda_2$ and $\ddot{\varphi} = -4\lambda_2 + 8\lambda_2 u$. Since $\lambda_2 > 0$, we see that on a singular arc $u(t) \equiv 1/2$.

In the bang–bang case, $u \in \{0, 1\}$ for almost all t and has no more than three switches. However, since $\lambda_2 > 0$ it is easy to see that on a bang–bang arc starting with $x_1(0) = 0$, $\operatorname{sgn}(\varphi(t)) = \operatorname{sgn}(\dot{\varphi}(t))$, so there cannot be any more switches. Thus, $u(t) \equiv v$, $\forall t \in [0, 1]$, where

v is either zero or one. For $v = 0$, the resulting trajectory satisfies $\boldsymbol{x}(1) = (-1, 5/3)^T$, and for $v = 1$, $\boldsymbol{x}(1) = (1, 5/3)^T$. Thus, in the bang–bang case,

$$\left|\boldsymbol{x}(1)\right|^2 = 34/9. \tag{29}$$

If a singular arc exists, then it follows from the analysis above that the general form of the optimal control is

$$u(t) = \begin{cases} v_1, & t \in [0, \tau_1), \\ 1/2, & t \in [\tau_1, \tau_2), \\ v_2, & t \in [\tau_2, 1], \end{cases} \tag{30}$$

where $v_1, v_2 \in \{0, 1\}$ and $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant 1$. If $\tau_1 > 0$ then $\dot{\varphi}(\tau_1) = 4x_1(\tau_1)\lambda_2 \neq 0$ so u cannot be singular on $[\tau_1, \tau_2]$. Hence, the singular case is only possible if $\tau_1 = 0$. For $v_2 = 0$, the corresponding trajectory satisfies $\boldsymbol{x}(1) = (\tau_2 - 1, 2 - (1 - \tau_2)^3/3)^T$, and for $v_2 = 1$, $\boldsymbol{x}(1) = (1 - \tau_2, 2 - (1 - \tau_2)^3/3)^T$. In both cases, the maximal value of $|\boldsymbol{x}(1)|^2$ is obtained for $\tau_2 = r$, where r is the smallest real root of the equation $z^4 - 4z^3 + 6z^2 + 2z - 2 = 0$ ($r \approx 0.4886$). Then

$$\left|\boldsymbol{x}(1)\right|^2 = (r-1)^2 + \left(2 - (1-r)^3/3\right)^2 \approx 4.08519.$$

Comparing this to (29), we conclude that there are exactly two optimal controls

$$u^*(t) = \begin{cases} 1/2, & t \in [0, r], \\ v, & t \in [r, 1]. \end{cases} \tag{31}$$

with $v = 0$ or $v = 1$.

It is interesting to note that in the second-order nilpotent case there always exists an optimal control that is bang–bang [19]. Example 1 demonstrates that this is no longer true in the third-order nilpotent case. In this case, there exist points in $R(T; \mathcal{U})$ that can be reached using piecewise constant controls, but cannot be reached using bang–bang controls. Another example that demonstrates this phenomena can be found in [33].

7. Conclusions

We considered a nonlinear control system that is affine in the control. We showed that if the Lie-algebra spanned by the vector fields is third-order nilpotent, then any point that can be reached at time T using a measurable control can also be reached, at the same time T, using a piecewise constant control with no more than four switches. The bound on the number of switches is uniform over any final time T.

As a corollary, we derived a new sufficient condition for global uniform asymptotic stability of the control system and, therefore, of the corresponding switched system. This is a promising step toward a solution of the open problem described in [1].

Interesting topics for further research include the following. First, the study of reachability with nice controls for higher orders of nilpotency. Second, the combination of our results with the approach of *feedback-nilpotentization*. The key idea is that some systems that are not nilpotent can be made nilpotent by means of a feedback transformation [34]. Third, there are methods for approximating nonnilpotent control systems using nilpotent ones [34,35]. Further study of the implications of our results in this context may be of interest.

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