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Some Classes of Multivariate Life Distributions in Discrete Time

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New classes of multivariate survival distribution functions based on monotonic behaviour of a multivariate failure rate are developed in the discrete set up. Relationship among the classes along with multivariate geometric distributions that act as boundaries of the various classes are identified. © 1997 Academic Press

1. INTRODUCTION

The importance of failure rate and mean residual life in the context of modelling lifetime data is well established and various classes of life distributions have been identified (for example, [1–6]) on the basis of the monotone behaviour of these reliability concepts. In the multivariate case, the pattern of ageing can be viewed from different angles and accordingly there is more than one way in which monotonicity of failure rates can be analysed. When lifetime is treated as continuous, often various forms of multivariate exponential distributions appear as the boundary separating increasing failure rate and decreasing failure rate models. Recently there has been considerable interest in developing reliability models in the discrete time domain. For justification of such models and the analysis involving them, we refer to [2, 7–13].

In the present note, we attempt a theoretical investigation of the classes of life distributions using the nature of a multivariate discrete failure rate and derive the chain of implications among them. In the process forms of multivariate geometric distributions that are characterized by the no-ageing property are identified. The results are useful in developing tests for constant failure rate against increasing (decreasing) failure rate alternatives.

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Let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be a discrete random vector in the support of $I_n^+ = \{\mathbf{x} = (x_1, x_2, ..., x_n) \mid x_i = 0, 1, 2, \ldots; i = 1, ..., n\}$ representing the lifetimes of an n-component system with the joint survival function

$$R(\mathbf{x}) = P(\mathbf{X} \geqslant \mathbf{x}), \tag{1.1}$$

where the ordering $\mathbf{X} \geqslant \mathbf{x}$ is understood component-wise. We consider the multivariate failure rate [8]

$$h(\mathbf{x}) = (h_1(\mathbf{x}), h_2(\mathbf{x}), ..., h_n(\mathbf{x})),$$

where

$$\begin{aligned} h_i(\mathbf{x}) &= P[X_i = x_i \mid \mathbf{X} \geqslant \mathbf{x}] \\ &= 1 - [R(x_1, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_n)/R(\mathbf{x})] \end{aligned} \tag{1.2}$$

and derive various classes of increasing failure rate (IFR) and decreasing failure rate (DFR) models. The failure rate $h(\mathbf{x})$ determines the distribution of $\mathbf{X}$ uniquely through the formula

$$\begin{aligned} R(\mathbf{x}) = & \prod_{r=0}^{x_1 - 1} [1 - h_1(x_1 - r - 1, x_2, ..., x_n)] \\ & \times \prod_{r=0}^{x_2 - 1} [1 - h_2(0, x_2 - r - 1, ..., x_n)] \cdots \\ & \times \prod_{r=0}^{x_n - 1} [1 - h_n(0, 0, ..., 0, x_n - r - 1)] \end{aligned} \tag{1.3}$$

as seen from Eq. (1.2).

2. IFR$_1$ (DFR$_1$) CLASS

Definition 2.1. A random vector $\mathbf{X}$ in the support of I_n^+ is said to have IFR$_1$ (DFR$_1$) distribution if for all $\mathbf{x}$, $\mathbf{t}$ in I_n^+,

$$h(\mathbf{x} + \mathbf{t}) \geqslant (\leqslant)\, h(\mathbf{x}), \tag{2.1}$$

where $\mathbf{t} = (t_1, t_2, ..., t_n)$.

Theorem 2.1 *The only class of distributions in I_n^+ which is both* IFR$_1$ *and* DFR$_1$ *is the multivariate geometric distribution with independent marginals.*

Proof. Let $\mathbf{X}$ be such that

$$R(\mathbf{x}) = p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}, \qquad 0 < p_i < 1, \quad i = 1, 2, ..., n. \tag{2.2}$$

Then $h(\mathbf{x}) = (1-p_1, 1-p_2, ..., 1-p_n)$, implying $h(\mathbf{x}+\mathbf{t}) = h(\mathbf{x})$ for all $\mathbf{x}$, $\mathbf{t}$ in I_n^+. Conversely if $h(\mathbf{x}+\mathbf{t}) = h(\mathbf{x})$ holds, from (1.2) for $i=1$ we have

$$\frac{R(x_1+t_1+1, x_2+t_2, ..., x_n+t_n)}{R(x_1+t_1, x_2+t_2, ..., x_n+t_n)} = \frac{R(x_1+1, x_2, ..., x_n)}{R(\mathbf{x})}. \tag{2.3}$$

Taking $\mathbf{x} = (0, 0, ..., 0)$ and writing e_i for the n-dimensional unit vector with the ith coordinate as unity

$$R(t_1+1, t_2, ..., t_n) = p_1^{t_1+1} R(0, t_2, ..., t_n), \tag{2.4}$$

where $p_1 = R(e_1)$ and $0 < p_1 < 1$.

By considering (1.2) for $i=2$, we can have an expression similar to (2.3) in which if we set $\mathbf{x} = (0, 0, ..., 0)$ and $t_1 = 0$, $R(0, t_2, ..., t_n) = p_2^{t_2} R(0, 0, t_3, ..., t_n)$ with $p_2 = R(e_2)$. Continuing the iteration, we arrive at (2.2) and this proves the assertion.

The implication of Theorem 2.1 is that Definition 2.1 is highly restrictive in the sense that the components of the system are independent of one another. In order to include more realistic situations of ageing we consider alternative definitions.

3. IFR_2 (DFR_2) CLASS

Definition 3.1. The distribution of $\mathbf{X}$ belongs to the IFR_2 (DFR_2) class if for all $\mathbf{x}$ in I_n^+ and s in I_1^+,

$$h_i(x_1, ..., x_i+s, x_{i+1}, ..., x_n) \geqslant (\leqslant)\, h_i(\mathbf{x}), \qquad i = 1, 2, ..., n. \tag{3.1}$$

Theorem 3.1. *A multivariate distribution with support I_n^+ is both* IFR_2 *and* DFR_2 *if and only if it is multivariate geometric with survival function*

$$R(\mathbf{x}) = \left(\prod_{i=1}^{n} p_i^{x_i}\right)\left(\prod_{i<j} p_{ij}^{x_i x_j}\right) \cdots (p_{12\cdots n}^{x_1 x_2 \cdots x_n}), \tag{3.2}$$

where

$$0 < p_i < 1, \qquad 0 < p_{ij}, p_{ijk}, ..., p_{12\cdots n} < 1$$

and

$$1 - \sum_i p_i - \sum_{i<j}\sum p_i p_j p_{ij} + \cdots + (-1)^n p_{12\cdots n} \geqslant 0. \tag{3.3}$$

Proof. When the distribution is as stated in (3.2)

$$\begin{aligned} 1-h_i(\mathbf{x}) &= p_i\left(\prod_{i<j} p_{ij}^{x_j}\right)\left(\prod_{i<j<k} p_{ijk}^{x_j x_k}\right)\cdots\left(p_{12\cdots n}^{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}\right) \\ 1-h_i(\mathbf{x}) &= p_i\left(\prod_{i<j} p_{ij}^{x_j}\right)\left(\prod_{i<j<k} p_{ijk}^{x_j x_k}\right)\cdots\left(p_{12\cdots n}^{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}\right) \end{aligned} \tag{3.4}$$

which is independent of x_i for all $i=1, 2, \ldots, n$ and hence for all $\mathbf{x}$ and s

$$h_i(x_1, \ldots, x_{i-1}, x_i+s, x_{i+1}, \ldots, x_n) = h_i(\mathbf{x}), \tag{3.5}$$

Thus the condition is necessary. To establish the sufficiency part, we use the method of induction. It is proved in [10] that in the bivariate case the failure rate having functional form $h_i(x_1, x_2) = a_i(x_j)$, $i, j=1, 2$, $i \neq j$, characterizes the distribution (3.2) for $n=2$. Thus the theorem holds for $n=2$. Assume that the condition holds for every subset of m variables in $\mathbf{X}$. The same condition holds for $(m+1)$ variables $(X_1, \ldots, X_{m+1})$ if and only if $1-h_i(x_1, \ldots, x_{m+1}) = B_i(x_i^*)$, where $x_i^* = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+1})$.

Thus from (1.2) and by induction hypothesis

$$\begin{aligned} &R(x_1, x_2, \ldots, x_{m+1}) \\ &\quad = [B_i(x_i^*)]^{x_i} R(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{m+1}) \\ &\quad = [B_i(x_i^*)]^{x_i}\left(\prod_{j=1}^{m+1} p_j^{x_j}\right)\left(\prod_{j<k} p_{jk}^{x_j x_k}\right)\cdots\left(p_{12\cdots m+1}^{x_1 x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+1}}\right), \\ &\qquad\quad j=1, 2, \ldots, m+1; j \neq i. \end{aligned} \tag{3.6}$$

For $i=1, 2, \ldots, n$ we can write equivalent expressions for $R(x_1, x_2, \ldots, x_{m+1})$. Dividing each of them by

$$\left(\prod_{i=1}^{m+1} p_i^{x_i}\right)\left(\prod_{\substack{j=1\\ i<j}}^{m+1} p_{p_{ij}}^{x_i x_j}\right)\cdots\left(p_{12\cdots m}^{x_1 x_2\cdots x_m}\right)$$

and taking $(x_1, x_2, \ldots, x_{m+1})$th root we find

$$\begin{aligned} &\left[B_1(x_1^*)\left[p_1^{x_1}\left(\prod_{j\neq 1} p_{1j}^{x_1 x_j}\right)\cdots\left(p_{13\cdots m+1}^{x_1 x_3\cdots x_{m+1}}\right)\right]^{-1}\right]^{(x_1 x_2\cdots x_{m+1})^{-1}} \\ &\quad = \left[B_2(x_2^*)\left[p_2^{x_2}\left(\prod_{j\neq 2} p_{2j}^{x_2 x_j}\right)\cdots\left(p_{23\cdots m+1}^{x_2 x_3\cdots x_{m+1}}\right)\right]^{-1}\right]^{(x_1 x_2\cdots x_{m+1})^{-1}} \\ &\quad = \cdots \\ &\quad = \left[B_{m+1}(x_{m+1}^*)\left[p_{m+1}^{x_{m+1}}\left(\prod_{j\neq m+1} p_{m+1,j}^{x_{m+1} x_j}\right)\cdots\left(p_{1\cdots m}^{x_1\cdots x_m}\right)\right]^{-1}\right]^{(x_1\cdots x_{m+1})^{-1}}. \end{aligned}$$

This, however, means that

$$\left[B_i(x_i^*)\left[p_i^{x_i}\left(\prod_{j\neq i} p_{ij}^{x_i x_j}\right)\cdots(p_{1\cdots i\cdots m}^{x_1\cdots x_1\cdots x_m})\right]^{-1}\right]^{(x_1 x_2\cdots x_{m+1})^{-1}=\text{const}},$$

independent of $x_1, ..., x_{m+1}$, say, $p_{12\cdots m+1}$. Then

$$B_i(x_i^*)=p_i^{x_i}\left(\prod_{j\neq i} p_{ij}^{x_j}\right)\cdots(p_{12\cdots m+1}^{x_1\cdots x_{i-1}x_{i+1}\cdots x_{m+1}}). \tag{3.7}$$

Substituting (3.7) into (3.6) we recover the survival function (3.2) for $n=m+1$. The conditions on the parameters are obtained from the relationships

$$R(x_1, ..., x_{i-1})\geqslant R(x_1, ..., x_i), \qquad i=1, 2, ..., n; \quad f(0, 0, ..., 0)\geqslant 0.$$

4. IFR_3 (DFR_3) CLASS

An alternative way of relaxing the Definition 2.1 is realized by assuming that the failure rate of system with n components of different ages is observed after each component has worked through the same time. Thus we have

DEFINITION 4.1. The distribution of $\mathbf{X}$ belongs to the $IFR_3(DFR_3)$ class if for all $m\leqslant n$, $\mathbf{x}$ in I_m^+, t in I_1^+,

$$h_i(x_1+t, ..., x_m+t)\geqslant(\leqslant)\, h_i(x_1, ..., x_m). \tag{4.1}$$

THEOREM 4.1. *The only multivariate distribution that is both* IFR_3 *and* DFR_3 *in* I_m^+ *for all* $m\leqslant n$ *is specified by*

$$R(\mathbf{x})=p_{i_1}^{x_{i_1}}(p_{i_1 i_2}/p_{i_1})^{x_{i_2}} \times(p_{i_1 i_2 i_3}/p_{i_1 i_2})^{x_{i_3}}\cdots(p_{i_1\cdots i_2}/P_{i_1\cdots i_{m-1}})^{x_{i_m}}, \qquad x_{i_1}\geqslant x_{i_2}\geqslant\cdots\geqslant x_{i_m}, \tag{4.2}$$

for each permutation $(i_1, i_2, ..., i_m)$ *of the integers from* 1 *to* m. *The parameters are such that*

$$0<p_{i_1 i_2\cdots i_m}\leqslant\cdots\leqslant p_{i_1, i_2}\leqslant p_1, p_2, ..., p_m<1,$$

$$p_{i_1 i_2\cdots i_j}=p_{123\cdots j} \qquad \text{for} \quad j=2, ..., m$$

and

$$1-\sum_{j=1}^{n} p_j-\sum\sum_{j<k} p_{jk}+\cdots+(-1)^{n-1} p_{123\cdots m}\geqslant 0.$$

Proof. The distribution of $\mathbf{X}$ is both IFR_3 and DFR_3 if and only if

$$h(x_1+t, ..., x_m+t)=h(x_1, ..., x_m) \qquad \text{for all} \quad m\leqslant n. \tag{4.3}$$

First we prove that (4.3) is equivalent to the multivariate lack of memory property (MLMP),

$$R(x_1+t, ..., x_m+t)=R(x_1, ..., x_m)\, R(t, ..., t) \qquad \text{for all} \quad m\leqslant n. \tag{4.4}$$

When (4.3) holds, from (1.2) for $i=1$,

$$\frac{R(x_1+t+1, x_2+t, ..., x_m+t)}{R(x_1+1, x_2, ..., x_m)}=\frac{R(x_1+t, x_2+t, ..., x_m+t)}{R(x_1, ..., x_m)}$$

$$=\frac{R(t, x_2+t, ..., x_m+t)}{R(0, x_2, ..., x_m)}. \tag{4.5}$$

Similarly using the definitions of $h_2(x_1, ..., x_m)$

$$\frac{R(x_1+t, x_2+t, ..., x_m+t)}{R(x_1, x_2, ..., x_m)}=\frac{R(x_1+t, t, ..., x_m+t)}{R(x_1, 0, x_3, ..., x_m)}. \tag{4.6}$$

Setting $x_2=0$ in (4.5) and substituting in (4.6)

$$\frac{R(x_1+t, x_2+t, ..., x_m+t)}{R(x_1, x_2, ..., x_m)}=\frac{R(t, t, x_3+t, ..., x_m+t)}{R(0, 0, x_3, ..., x_m)}.$$

Successively using $h_3, ..., h_m$ and noting $R(0, 0, ..., 0)=1$ we obtain (4.4). The converse is obtained by using (4.4) in the expression for $h(\mathbf{x}+\mathbf{t})$. To complete the proof it remains to establish that the only solution of (4.4) is (4.2).

For $m=1$, the only solution is $R(x_j)=p_1^{x_j}$ for some $0<p_1<1$ and for $m=2$,

$$R(x_1+t, x_2+t)=R(x_1, x_2)\, R(t, t).$$

Setting $x_2=0$,

$$R(x_1+t, t)=p_1^{x_1}R(t, t). \tag{4.7}$$

Further,

$$R(y+t,\ y+t) = R(y,\ y)\ R(t,\ t)$$

gives $R(t, t) = p_{12}^t$ for some $0 < p_{12} < 1$. Thus for $x_1 \geqslant x_2$ from (4.7),

$$R(x_1, x_2) = p_1^{x_1}(p_{12}/p_1)^{x_2}. \tag{4.8}$$

Assuming the solution (4.2) to hold for any m variables in $\mathbf{X}$,

$$R(x_1+t, ..., x_m+t, t) = R(x_1, x_2, ..., x_m)\ R(t, ..., t)$$

specializes to

$$\begin{aligned} R(x_1+t, x_2+t, ..., x_m+t, t) &= R(x_1, x_2, ..., x_m, 0)\ R(t, ..., t) \\ &= p_1^{x_1}(p_{12}/p_1)^{x_2} \cdots (p_{1\cdots m}/P_{12\cdots m-1})^{x_m}\ R(t, ..., t), \quad x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m. \end{aligned} \tag{4.9}$$

Also,

$$R(y+t, ..., y+t) = R(y, ..., y)\ R(t, ..., t)$$

$$R(t, ..., t) = p_{12\cdots m+1}^t, \qquad 0 < p_{12\cdots m+1} < 1.$$

$$R(\mathbf{x}) = p_1^{x_1}(p_{12}/p_1)^{x_2} \cdots (p_{1\cdots m+1}/p_{12\cdots m})^{x_{m+1}}$$

$$x_1 \geqslant x_2 \geqslant \cdots x_{m+1}.$$

By induction we have derived the result for $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{m+1}$. The expression for $R(\mathbf{x})$ in other regions of the sample space are similarly obtained and the conditions on the parameters are obtained as in Theorem 3.1.

Note. It is possible to have a stronger class than that provided by Eq. (4.1) by defining it as $\mathbf{X}$ belongs to IFR_4 (DFR_4) class if for all $m \leqslant n$ and

$$h(x+t, ..., x+t) \geqslant (\leqslant)\ h(x, ..., x), \tag{4.10}$$

but there is no characterization of the corresponding boundary class.

5. INTER RELATIONSHIPS

From the definitions given above it is clear that

(i) $\text{IFR}_2 \Leftarrow \text{IFR}_1 \Rightarrow \text{IFR}_3 \Rightarrow \text{IFR}_4$ and

(ii) $\text{DFR}_2 \Leftarrow \text{DFR}_1 \Rightarrow \text{DFR}_3 \Rightarrow \text{DFR}_4$.

TABLE I

Bivariate distribution	$(h_1(x), h_2(x))$	Example for
(i) Bivariate geometric [7] $p_1^{x_1}p_2^{x_2}\theta^{x_1x_2}$ $0<p_i<1,\ 1+p_1p_2\theta\geqslant p_1+p_2$. $x_i=0, 1, ..., i=1, 2$	$(1-p_1\theta^{x_2}, 1-p_2\theta^{x_1})$	$IFR_2 \not\Rightarrow DFR_4$ $\mathrm{DFR}_2 \not\Rightarrow DFR_4$
(ii) Bivariate Waring $\dfrac{(m)_{x_1+x_2}}{(m+n)_{x_1+x_2}}$ $m, n>0;\ x_i=0, 1, ..., i=1, 2$	$\left(\dfrac{n}{m+n+x_1+x_2}, \dfrac{n}{m+n+x_1+x_2}\right)$	$DFR_2 \not\Rightarrow IFR_2$ $DFR_2 \not\Rightarrow IFR_4$
(iii) Bivariate geometric [6] $p^{x_2}p_1^{x_1-x_2},\ x_1\geqslant x_2$ $p^{x_1}p_2^{x_2x_1},\ x_1\leqslant x_2$ $0<p\leqslant p_i<1,\ x_i=0, 1, 2, ...$ $1+p\geqslant p_1+p_2,\ i=1, 2$	$(1-p_1, 1-p/p_1)x_1>x_2$ $(1-p/p_2, 1-p_2)x_1<x_2$ $(1-p_1, 1-p_2)x_1=x_2$	When $p\geqslant p_1p_2$ $DFR_3 \not\Rightarrow DFR_2$ When $p\leqslant p_1p_2$ $IFR_3 \not\Rightarrow IFR_2$
(iv) Bivariate negative hypergeometric $\binom{k+n-x_1-x_2}{n-x_1-x_2}\Big/\binom{k+n}{n}$ $x_1+x_2\leqslant n.\ x_i=0, 1, 2, ..., n$	$\left(\dfrac{k}{k+n-x_1-x_2}, \dfrac{k}{k+n-x_1-x_2}\right)$	$IFR_1 \not\Rightarrow DFR_2$ $IFR_1 \not\Rightarrow DFR_4$
(v) Bivariate geometric mixture $\alpha p^{x_1+x_2}+(1-\alpha)p^{x_1},\ x_1\geqslant x_2$ $\alpha p^{x_1+x_2}+(1-\alpha)p^{x_2},\ x_1\leqslant x_2$ $0<p, \alpha<1,\ x_i=0, 1, 2, ..., i=1, 2$	$\left(1-p, 1-\dfrac{\alpha p^{x_2+1}+(1-\alpha)}{\alpha p^{x_2}+(1-\alpha)}\right)$ $x_1\geqslant x_2+1$ $\left(1-\dfrac{\alpha p^{x_1+1}+(1-\alpha)}{\alpha p^{x_1}+(1-\alpha)}, 1-p\right)$ $x_1+1\leqslant x_2$ $(1-p, 1-p),\ x_1=x_2$	$IFR_4 \not\Rightarrow IFR_3$ $DFR_4 \not\Rightarrow DFR_3$
(vi) Bivariate geometric $p_1^{x_1},\ p_1^{x_1}\leqslant p_2^{x_2}$ $p_2^{x_2},\ p_1^{x_1}\geqslant p_2^{x_2}$ $0<p_i<1,\ x_i=0, 1, 2, ..., i=1, 2$	$(1-p_1, 0),\ p_1^{x_1}\leqslant p_2^{x_2+1}$ $(0, 1-p_2),\ p_1^{x_1+1}\geqslant p_2^{x_2}$ $\left(1-p_1, 1-\dfrac{p_2^{x_2+1}}{p_1^{x_1}}\right)$ $p_1^{x_1}\leqslant p_2^{x_2},\ p_1^{x_1}\geqslant p_2^{x_2+1}$ $\left(1-\dfrac{p_1^{x_1+1}}{p_2^{x_2}}, 1-p^2\right)$ $p_1^{x_1}\geqslant p_2^{x_2},\ p_1^{x_1+1}\leqslant p_2^{x_2}$	$IFR_2 \not\Rightarrow IFR_4$

Further the counterexamples provided in Table I illustrate that there exist no other implications between the different classes.

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