Discrete Mathematics 310 (2010) 1491–1494

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Hamiltonian cycles through prescribed edges of 4-connected maximal planar graphs

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ARTICLE INFO

Article history:
Received 9 October 2008
Received in revised form 30 September 2009
Accepted 9 October 2009
Available online 29 October 2009

Keywords:
Maximal planar graph
Hamiltonian cycle
Prescribed edges

ABSTRACT

In 1956, W.T. Tutte proved that every 4-connected planar graph is hamiltonian. Moreover, in 1997, D.P. Sanders extended this to the result that a 4-connected planar graph contains a hamiltonian cycle through any two of its edges. It is shown that Sanders' result is best possible by constructing 4-connected maximal planar graphs with three edges a large distance apart such that any hamiltonian cycle misses one of them. If the maximal planar graph is 5-connected then such a construction is impossible.

1. Introduction and results

We use [3] for terminology and notation not defined here and consider finite simple graphs only.

The first major result on the existence of hamiltonian cycles in graphs embeddable in surfaces was by Whitney [11] in 1931, who proved that 4-connected maximal planar graphs are hamiltonian. In 1956, Tutte [9,10] generalized Whitney's result from maximal planar graphs to arbitrary 4-connected planar graphs. Actually, Tutte proved that a 4-connected planar graph G has a hamiltonian cycle through any two edges of a given face of G. Moreover, in [6,7] it is proved that a 4-connected planar graph G has a hamiltonian cycle through any three edges of a given face of G or that face is a 3-gon.

Improving a result of Thomassen [8], in 1997, Sanders [6] proved the following:

Theorem 1 ([6]). *Every 4-connected planar graph on at least three vertices has a hamiltonian cycle through any two of its edges.*

In [5] the connectivity of a subset X of the vertex set of a graph G is defined as follows. Let G be a graph, $X \subseteq V(G)$, and $G[X]$ be the subgraph of G induced by X. A set $V \subset V(G)$ *splits* X if the graph $G - V$ obtained from G by removing V contains at least two components each containing a vertex of X. Let $\kappa(X)$ be infinity if $G[X]$ is complete, or the minimum cardinality of a set $V \subset V(G)$ splitting X. Let us remark that G is k-connected if and only if $\kappa(V(G)) \geq k$.

Theorem 2 is a local version of Theorem 1 and if $X = V(G)$, Theorem 1 follows from Theorem 2. This is proven in [5].

Theorem 2 ([5]). *If G is a planar graph, $X \subseteq V(G)$, $|X| \geq 3$, $\kappa(X) \geq 4$, $E \subset E(G[X])$, and $|E| \leq 2$, then G contains a cycle C with $X \subseteq V(C)$ and $E \subset E(C)$.*

The following theorem is proven in [4] and, unlike Theorem 2, it is appropriable if $|E| \geq 3$.

Theorem 3 ([4], Theorem 6). *If G is a graph, $X \subseteq V(G)$, $E \neq \emptyset$ is a set of independent edges of $G[X]$, $|X| \geq 2|E| + 1$, and $\kappa(X) \geq |X| - |E|$, then G contains a cycle C with $X \subseteq V(C)$ and $E \subset E(C)$.*

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doi:10.1016/j.disc.2009.10.005

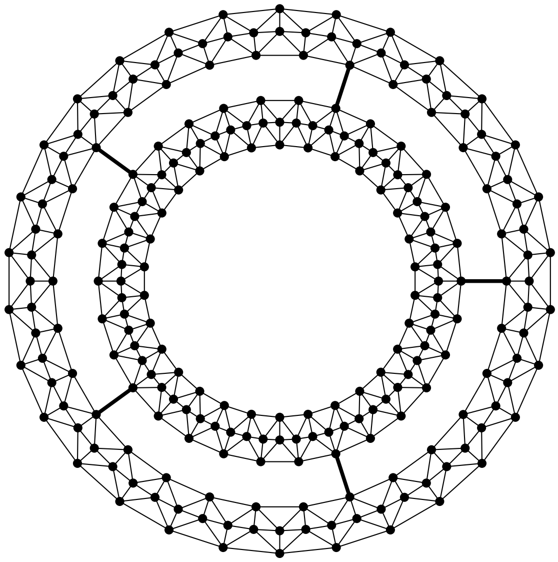

Fig. 1.

Note that Theorem 3 holds for arbitrary graphs. Obviously $|X| \geq 2|E|$ since E is a set of independent edges of $G[X]$.[1] If $X = V(G)$, $|X| - |E| \geq 6$, and G is planar then Theorem 3 cannot be used since a planar graph is at most 5-connected.

We call a maximal planar graph G a plane triangulation if G is embedded into the plane. In [1] it was shown that there are 4-connected plane triangulations containing seven faces an arbitrary distance apart such that each hamiltonian cycle of that graph misses at least one of these faces, i.e. seven edges cannot be guaranteed to belong to a hamiltonian cycle of a 4-connected planar graph even if their pairwise distance is large. We will show that Theorem 1 is best possible in the sense that even three prescribed edges need not belong to a hamiltonian cycle of a 4-connected maximal planar graph. Given two edges xy and uv of a graph G, the number of edges of a shortest path in G connecting a vertex of $\{x, y\}$ and a vertex of $\{u, v\}$ is called the *distance* of xy and uv.

Theorem 4. *There is a 4-connected plane triangulation G containing $E \subseteq E(G)$ with $3|E| = |E(G)|$ such that each hamiltonian cycle of G contains exactly two edges of E. Moreover, for given integer $k \geq 1$, G and E can be chosen such that E contains three edges of pairwise distance at least k.*

The situation changes in comparison with Theorem 4, if the connectivity of G is increased and the pairwise distance of the edges in the set E is at least 3. In this case it is even possible to forbid edges of E from belonging to a hamiltonian cycle as described in the following Theorem 5. A proof is given in [2].

Theorem 5 ([2]). *Let G be a 5-connected plane triangulation and E be a set of edges of G such that the distance between any two edges of E is at least three. Furthermore, let $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$. Then G has a hamiltonian cycle C with $E_1 \subset E(C)$ and $E_2 \cap E(C) = \emptyset$.*

Theorem 5 does not hold if the 5-connected plane graph G is not a triangulation. Moreover, the existence of a cycle satisfying the assertion of Theorem 5 cannot be guaranteed in the case $|E| \geq 5$ and $E_2 = \emptyset$, because for each integer k there is a 5-connected plane graph G containing a set $E \subset E(G)$ with $|E| = 5$ such that any two edges of E have distance at least k and there is no cycle of G containing E. Fig. 1 shows G in the case $k = 6$, where E is the set of the five bold edges forming an odd edge-cut of G.

We want to remark that, for any $k \geq 1$, there is a 5-connected planar graph G containing a set E of seven edges such that any hamiltonian cycle misses one of them, the edges of E have mutually distance at least k, and E does not contain an odd edge-cut.[2]

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1 The inequality $|X| \geq 2|E| + 1$ is needed in the proof of Theorem 3 for technical reasons. Probably Theorem 3 also holds if $|X| \geq 2|E|$. If in Theorem 3, additionally, G is assumed to be planar, then it remains an open problem whether the inequality $\kappa(X) \geq |X| - |E|$ can be weakened.

2 For the construction of such a graph consider a perfect matching M of the graph C of the three-dimensional cube such that M does not form an edge-cut of C. Let H be obtained from C by subdividing each edge of M by a single vertex and connecting two of these four new vertices by a path on three vertices iff they belong to a common face. The graph H is planar, has twelve vertices of degree 3, two vertices of degree 2 and eight 5-gons. Let E be a set of seven edges of a perfect matching of H. Now consider a component F of the graph obtained by the graph of Fig. 1 by deleting the five bold edges and let V_F be the set of the five vertices of F being incident with a bold edge in the graph of Fig. 1. For each face f of H insert a copy of F into f by identifying V_F with the

For three edges of a 5-connected plane triangulation the distance condition in Theorem 5 can be omitted as follows.

Theorem 6. *Let G be a 5-connected plane triangulation and E be a set of three edges of G such that E does not form a facial cycle and there is no vertex incident with all edges of E. Then G has a hamiltonian cycle containing E.*

Considering 5-connected maximal planar graphs, the following theorem is an analogue to Theorem 4.

Theorem 7. *Let G be a 5-connected plane triangulation containing an independent set V of vertices such that each face of G is incident with exactly one vertex of V and $H = G - V$ is 3-regular. Then each hamiltonian cycle of G contains exactly $\frac{1}{3}|E(H)| - 2 = \frac{1}{3}(|V(G)| - 8)$ edges of H.*

Let e_k be the smallest integer l such that there is a 5-connected plane triangulation G containing l edges of pairwise distance at least k such that there is no hamiltonian cycle of G containing all of these l edges. If e_k does not exist then we write $e_k = \infty$.

Theorem 5 implies that $e_k = \infty$ if $k \geq 3$.

Theorem 8. *For e_1 the inequalities $4 \leq e_1 \leq 9$ hold.*

Moreover, there are infinitely many 5-connected maximal planar graphs G containing a set E of $\frac{1}{3}|E(G)|$ independent edges such that each hamiltonian cycle of G misses two edges of E.

It remains open whether e_2 is finite or not.

Let $\hat{e}_k$ be defined similarly to e_k but for the class of arbitrary 5-connected planar graphs. Then, obviously, $\hat{e}_k \leq e_k$ for all $k \geq 0$ and it follows $\hat{e}_0 = 3$ and $3 \leq \hat{e}_k \leq 5$ for all $k \geq 1$ (see Theorem 1 and Fig. 1).

2. Proofs

Lemma 1. *Let G be a plane triangulation with an independent set $V \subseteq V(G)$ of vertices of degree 4 such that each face of G contains a vertex of V at its boundary. Then $|E(G-V)| = 2|V|$ and each hamiltonian cycle C of G satisfies $|E(G-V) \cap E(C)| = 2$.*

Proof. Obviously G has $4|V|$ faces and, therefore, $\frac{4\cdot 3}{2}|V| = 6|V|$ edges. By the Euler formula, G has $2|V| + 2$ vertices. Exactly $4|V|$ edges of G are incident with a vertex in V; the remaining $2|V|$ edges are the edges of $G - V$. A hamiltonian cycle of G consists of $|V|$ internally disjoint paths; each of them connects two vertices of V and all of its inner vertices do not belong to V. All these paths have at least two edges. Exactly two of the edges of such a path are incident with vertices of V. Hence, C has exactly $2|V|$ edges incident with a vertex in V and, thus, exactly two edges in $G - V$. □

Let uv with $u, v \in V(G)$ be an edge of a graph G. We say that uv is *subdivided* if a new vertex w is added to G and uv is replaced by the new edges uw and vw.

For Lemma 2, recall that the *barycentric subdivision* $B(G)$ of a plane multigraph G is obtained by first subdividing every edge of G, and then adding a new vertex v_f for every face f and joining v_f with all (including the new vertices added by subdividing) vertices at the boundary of f by an edge. Note that $B(G)$ is a maximal planar graph.

Lemma 2. *Let H be a 2-connected plane multigraph without loops and $G = B(H)$. Let V be the set of vertices of G subdividing the edges of H. Then G is a 4-connected maximal planar graph and V is a set of independent vertices of degree 4 such that each face of (a plane embedding of) G contains a vertex of V at its boundary.*

Proof. Obviously, G is a maximal planar graph without double edges. Hence, every minimum separating set of vertices of G induces a cycle on at least three vertices. Furthermore, V is a set of independent vertices of degree 4 such that each face of (a plane embedding of) G contains a vertex of V at its boundary. $G - V$ is bipartite because each face of $G - V$ is a 4-gon. Hence, each cycle of length 3 in G contains a vertex of V and, therefore, forms the boundary of a face of G. Consequently, G is 4-connected. □

Proof of Theorem 4. Let H be an arbitrary 2-connected plane triangulation without loops with three vertices v_1, v_2 and v_3 such that any two of them have distance at least $k + 2$. Let $G = B(H)$. Then G is a 4-connected maximal planar graph by Lemma 2. Let V be the set of vertices of G subdividing the edges of H. Obviously V satisfies the assumptions of Lemma 1. Let $E = E(G - V)$, and we have $|E| = 2|V|$ by Lemma 1 and $|E(G)| = 6|V|$ by the proof of Lemma 1. Let e_1, e_2 and e_3 be edges of E incident with v_1, v_2 and v_3, respectively, and the distance condition of Theorem 4 is fulfilled. Thus, the use of Lemma 1 completes the proof. □

Proof of Theorem 6. If E contains two edges incident with the same vertex u, say xu and yu, then let G' be the graph obtained from G by deleting u and adding the edge xy if it is not in G. Since G is 5-connected and $G - u$ spans G', G' is 4-connected. Because the edges of E neither form a triangle nor are incident with the same vertex, G' contains the third edge e_3 of E (different from xu and yu) and this edge is not identical to xy. Hence, by Theorem 1, G' contains a hamiltonian cycle C' covering xy and e_3. By insertion of u between x and y in C' we obtain a hamiltonian cycle of G containing E.

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vertices of f such that the resulting graph is planar. Let G be the graph obtained from that graph by deleting all edges in $E(H) \setminus E$. It is easy to see that G is planar and 5-connected and that the edges of E have pairwise distance 6 in G. If G were to have a hamiltonian cycle containing all edges of E then the graph G' obtained from G by subdividing each edge of E by a single vertex would also have a hamiltonian cycle. However, this is impossible because deleting the 14 vertices of H in G' results in a graph of 15 components. This construction can be generalized for any k.

In the remaining case the edges of E form a matching. Let $E = \{e_1 = uv, e_2, e_3\}$. There are exactly two common neighbours x and y of u and v since G is a triangulation with more than 3 edges. Let $u' \notin \{x, y, v\}$ and $v' \notin \{x, y, u\}$ be a neighbour of u and v, respectively. Because the minimum degree of G is at least 5 this choice is possible. Note that x, y, u', v' are vertices at the boundary of a face f_{uv} in $G - \{u, v\}$.

Let $O(uv)$ be the following operation: Delete the vertices u and v and add a new vertex w and the new edges wx, wy, wu', and wv'. In [2], it is proved that the graph G' obtained from G by applying the operation $O(uv)$ is a 4-connected planar graph. Because E forms a matching of G, e_2 and e_3 are contained in $G - u - v = G' - w$.

By Theorem 1, G' has a hamiltonian cycle C' through e_2' and e_3'. Since w is contained in C', it is easy to construct the desired hamiltonian cycle of G through E. □

Proof of Theorem 7. Because H is 3-regular and plane, we obtain by the Euler formula $|F(H)| = 2 + \frac{1}{3}|E(H)|$ and $|V(H)| = \frac{2}{3}|E(H)|$, where $F(H)$ is the set of faces of H. Since V is independent, each hamiltonian cycle of G contains exactly $2|V|$ edges incident with a vertex of V and exactly $|V(G)| - 2|V|$ edges of H. Furthermore, we obtain $|V| = |F(H)| = 2 + \frac{1}{3}|E(H)|$ and $|V(G)| = |V(H)| + |V| = |E(H)| + 2$. Hence, the assertion of Theorem 7 follows. □

Lemma 3. *Let H be a cubic 3-connected plane graph of girth 5. Let G be obtained from H by inserting a new vertex into each face of H and connecting it to each vertex of that face by an edge. Then G is a 5-connected maximal planar graph.*

Proof. Obviously, G is maximal planar. Let $V = V(G) \setminus V(H)$. Then V is independent in G and each vertex of V has degree at least 5 in G.

Since H has neither loops nor double edges, G is a simple plane triangulation; hence, G is at least 3-connected.

Assume there is a separating 3-cycle D of G. Because H has girth 5 there is at least one vertex of D in V, and because V is independent there is exactly one vertex of D in V. Consider a component F of $G - D$. If F does not contain a vertex of H then each vertex of F belongs to V, and because V is independent the neighbours of such a vertex of V are vertices of D; hence, its degree is too small—a contradiction. Thus, $V(D) \setminus V$ is a cut set of H contradicting the 3-connectedness of H.

Assume there is a separating 4-cycle D of G. By the same arguments D contains at least one vertex of V and each component of $G - D$ contains a vertex of H. Hence, $V(D) \setminus V$ is a cut set of H and because H is 3-connected there is exactly one vertex of V at D. Consider the three vertices of $V(D) \setminus V$. Each of them has a neighbour in each component of $H - (V(D) \setminus V)$; hence, one of them has degree at least 4 in H—a contradiction. □

Proof of Theorem 8. Clearly, $e_1 \geq 4$ by Theorem 6.

Let H be the graph of the dodecahedron. G is obtained from H by inserting a vertex of degree 5 into each face of H. Using Lemma 3, G is a 5-connected maximal planar graph. Let V be the set of vertices of G of degree 5. Furthermore, let E be a set of nine edges of a perfect matching of H. Therefore, E is independent also in G. Each hamiltonian cycle of G misses an edge of E by Theorem 7; therefore, $e_1 \leq 9$.

Now consider a cubic 3-connected plane graph H of girth 5. Furthermore, let E be a perfect matching of H; hence, $|E| = \frac{1}{3}|E(H)|$.

Let G be obtained from H as described in Lemma 3 and $V = V(G) \setminus V(H)$. By Lemma 3, G is 5-connected. By Theorem 7 each hamiltonian cycle of G contains exactly $\frac{1}{3}|E(H)| - 2$ edges of H; hence, each hamiltonian cycle misses two edges of E. □

Acknowledgements

The authors thank two anonymous referees for their valuable hints.

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