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Smallest counterexample to the 5-flow conjecture has girth at least eleven ☆

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ABSTRACT

The famous 5-flow conjecture of Tutte is that every bridgeless graph has a nowhere-zero 5-flow. We show that a smallest counterexample to this conjecture must have girth at least eleven.

1. Introduction

A graph admits a *nowhere-zero k-flow* if its edges can be oriented and assigned numbers $\pm 1, \dots, \pm(k-1)$ so that for every vertex, the sum of the values on incoming edges equals the sum on the outgoing ones. It is well known that a graph with a bridge (1-edge-cut) does not have a nowhere-zero k-flow for any $k \geqslant 2$ (see, e.g., [3,6]). The famous 5-*flow conjecture* of Tutte [12] is that every bridgeless graph has a nowhere-zero 5-flow.

Let G be a counterexample to the 5-flow conjecture of the smallest possible order. It is well known (see cf. Jaeger [3]) that G must be a *snark* which is a cyclically 4-edge-connected cubic graph without a 3-edge-coloring and with girth (the length of the shortest cycle) at least 5. (Note that a graph is *cyclically k-edge-connected* if deleting fewer than k edges does not result in a graph having at least two components containing cycles.) By [7], G must be cyclically 6-edge-connected. In [9], we have extended the methods from [7,8] and shown that G has girth at least 9. To prove this, we needed to evaluate ranks of large matrices, which was done by computers.

In this paper we further improve the methods from [7–9] and show that a smallest counterexample to the 5-flow conjecture must have girth at least 11. Similarly as in [9], we also need to use computers to evaluate ranks of matrices. But the most important part of this paper to present a re-

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duction of the size of matrices used in the computation, which is based on the automorphism group of a circuit. Using this reduction we can exclude girths 9 and 10 by a personal computer.

Let us note that it is interesting to find the lower bounds for the girth of a smallest counterexample, because we do not know whether there exists a cyclically 6-edge-connected snark with girth more than 6 (see [2,4–6]). Furthermore, every cubic graph embedded in a surface has bounded girth (see cf. Gross and Tucker [1]), thus every lower bound for the girth of a smallest counterexample verifies the 5-flow conjecture for a class of graphs embeddable in some surfaces.

2. Flows in simple networks

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. By a *multi-terminal network*, briefly a *network*, we mean a pair (G,U) where G is a graph and $U=(u_1,\dots,u_n)$ is an ordered set of pairwise distinct vertices of G. The vertices $u_1,\dots,u_n$ are called the *outer* vertices of (G,U) and the others are called the *inner* vertices of (G,U).

To each edge connecting u and v (including loops) we associate two distinct (directed) arcs, one directed from u to v, the other directed from v to u. If one of these arcs is denoted x then the other is denoted x^{-1}. Let $D(G)$ denote the set of such arcs, so that $|D(G)|=2|E(G)|$. If $v\in V(G)$, then $\omega_G(v)$ denotes the set of arcs of G directed from v to $V(G)\setminus\{v\}$.

If G is a graph and A is an additive Abelian group, then an *A-chain* in G is a mapping $\varphi: D(G)\to A$ such that $\varphi(x^{-1})=-\varphi(x)$ for every $x\in D(G)$. Furthermore, the mapping $\partial\varphi: V(G)\to A$ such that $\partial\varphi(v)=\sum_{x\in\omega_G(v)}\varphi(x)$ $(v\in V(G))$ is called the *boundary* of φ. An A-chain φ in G is called *nowhere-zero* if $\varphi(x)\neq 0$ for every $x\in D(G)$. If (G,U) is a network, then an A-chain φ in G is called an *A-flow* in (G,U) if $\partial\varphi(v)=0$ for every inner vertex v of (G,U).

By a (*nowhere-zero*) *A-flow* in a graph G we mean a (nowhere-zero) A-flow in the network $(G,\emptyset)$. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in Jaeger [3]. By Tutte [12,13], a graph has a nowhere-zero k-flow if and only if it has a nowhere-zero A-flow for any Abelian group A of order k. Thus the study of nowhere-zero 5-flows is, in a certain sense, equivalent to the study of nowhere-zero $\mathbb{Z}_5$-flows. We use this fact and deal only with $\mathbb{Z}_5$-flows because they are easier to handle than integral flows.

A network (G,U), $U=(u_1,\dots,u_n)$, is called *simple* if the vertices $u_1,\dots,u_n$ have degree 1. If φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G,U), then denote by $\partial\varphi(U)$ the n-tuple $(\partial\varphi(u_1),\dots,\partial\varphi(u_n))$. By simple counting, we have $\sum_{i=1}^n\partial\varphi(u_i)=-\sum_{v\in V(G)\setminus U}\partial\varphi(v)=0$ (see [6]). Furthermore, $\partial\varphi(u_i)\neq 0$ because u_i has degree 1 $(i=1,\dots,n)$. Thus $\partial\varphi(U)$ belongs to the set

$$S_n=\{(a_1,\dots,a_n);\ a_1,\dots,a_n\in\mathbb{Z}_5-\{0\},\ a_1+\cdots+a_n=0\}.$$

Let $P=\{Q_1,\dots,Q_r\}$ be a partition of the set $\{1,\dots,n\}$. Denote by G_P the graph having vertices $u_1,\dots,u_n$, $v_1,\dots,v_r$ and edges $e_1,\dots,e_n$ such that $e_i=u_iv_j$ if and only if $i\in Q_j$ $(i=1,\dots,n$, $j=1,\dots,r)$. We say that P is *proper* if $|Q_1|,\dots,|Q_r|\neq 1$. Let $\mathcal{P}_n$ denote the set of proper partitions of $\{1,\dots,n\}$.

Let $s=(a_1,\dots,a_n)\in S_n$ and $P=\{Q_1,\dots,Q_r\}\in\mathcal{P}_n$. We define $\chi(s,P)=1$ if $\sum_{i\in Q_j}a_i=0$ for $j=1,\dots,r$, and $\chi(s,P)=0$ otherwise. For example, if $P=\{\{1,2\},\{3,4,5\}\}\in\mathcal{P}_5$, then $\chi((1,4,1,1,3),P)=1$ and $\chi((1,2,2,3,2),P)=0$. $\mathcal{P}_n$ is considered as an ordered p_n-tuple $(P_1,\dots,P_{p_n})$, $p_n=|\mathcal{P}_n|$. Denote by $\boldsymbol{\chi}_n(s)$ the integral vector $(\chi(s,P_1),\dots,\chi(s,P_{p_n}))$.

If (G,U), $U=(u_1,\dots,u_n)$, is a simple network, then for every $s\in S_n$, denote by $\Phi_{G,U}(s)$ the set of nowhere-zero $\mathbb{Z}_5$-flows φ in (G,U) satisfying $\partial\varphi(U)=s$ and define $F_{G,U}(s)=|\Phi_{G,U}(s)|$. In [7] is proved the following statement.

Lemma 1. *Let (G,U), $U=(u_1,\dots,u_n)$, be a simple network and $\mathcal{P}_n=(P_1,\dots,P_{p_n})$. Then there exists an integral vector $\mathbf{x}_{G,U}=(x_1,\dots,x_{p_n})$ such that for every $s\in S_n$, $F_{G,U}(s)=\boldsymbol{\chi}_n(s)\cdot\mathbf{x}_{G,U}$, that means $F_{G,U}(s)=\sum_{i=1}^{p_n}\chi(s,P_i)x_i$.*

By a permutation group Γ on $\{1,\dots,n\}$ we mean any subgroup of the symmetric group on $\{1,\dots,n\}$, i.e., the group of all permutations of elements $1,\dots,n$. If $\gamma\in\Gamma$ and $Q\subseteq\{1,\dots,n\}$, then

$\gamma(Q) = \{\gamma(i);\ i \in Q\}$. If $P = \{Q_1, \dots, Q_r\} \in \mathcal{P}_n$, then define $\gamma(P) = \{\gamma(Q_1), \dots, \gamma(Q_r)\}$. We say that P and $\gamma(P)$ are *Γ-equivalent*. Let $\mathcal{P}_{\Gamma,n}$ be the set of Γ-equivalence classes and $p_{\Gamma,n} = |\mathcal{P}_{\Gamma,n}|$. For each $P_\Gamma \in \mathcal{P}_{\Gamma,n}$ and $s \in S_n$, define

$$\chi(s, P_\Gamma) = \sum_{P \in P_\Gamma} \chi(s, P). \tag{1}$$

$\mathcal{P}_{\Gamma,n}$ is considered as an ordered $p_{\Gamma,n}$-tuple $(P_{\Gamma,1}, \dots, P_{\Gamma,p_{\Gamma,n}})$. Then denote by $\boldsymbol{\chi}_{\Gamma,n}(s)$ the integral vector $(\chi(s, P_{\Gamma,1}), \dots, \chi(s, P_{\Gamma,p_{\Gamma,n}}))$.

If $\gamma \in \Gamma$ and $s = (a_1, \dots, a_n) \in S_n$, then denote by $\gamma(s) = (a_{\gamma(1)}, \dots, a_{\gamma(n)})$. Thus $\chi(s, P) = \chi(\gamma(s), \gamma^{-1}(P))$ for each $P \in \mathcal{P}_n$, whence $\chi(s, P_\Gamma) = \chi(\gamma(s), P_\Gamma)$ for each $P_\Gamma \in \mathcal{P}_{\Gamma,n}$, and therefore for each $\gamma \in \Gamma$ we have

$$\boldsymbol{\chi}_{\Gamma,n}(s) = \boldsymbol{\chi}_{\Gamma,n}\bigl(\gamma(s)\bigr). \tag{2}$$

It is natural to call s and $\gamma(s)$ to be *Γ-equivalent* (this equivalence is formally different from that ones defined on $\mathcal{P}_n$, but from the context it will be always clear which operation we have in mind). Denote by $S_{\Gamma,n}$ the set of Γ-equivalence classes (a partition of S_n). For each $s_\Gamma \in S_{\Gamma,n}$, define

$$\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) = \boldsymbol{\chi}_{\Gamma,n}(s), \tag{3}$$

where $s \in s_\Gamma$. This is well defined by (2). Furthermore, for each $s_\Gamma \in S_{\Gamma,n}$, define

$$F_{\Gamma,G,U}(s_\Gamma) = \sum_{\gamma \in \Gamma} F_{G,U}\bigl(\gamma(s)\bigr), \tag{4}$$

where $s \in s_\Gamma$. This is also well defined because Γ is a group.

Lemma 2. *Let (G, U), $U = (u_1, \dots, u_n)$, be a simple network and Γ be a permutation group on $\{1, \dots, n\}$. Then there exists a vector $\mathbf{y}_{\Gamma,G,U} \in \mathbb{Z}^{p_{\Gamma,n}}$ such that for every $s_\Gamma \in S_{\Gamma,n}$, $F_{\Gamma,G,U}(s_\Gamma) = \boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \cdot \mathbf{y}_{\Gamma,G,U}$.*

Proof. If $P_i \in \mathcal{P}_n$, then there exists $f(i) \in \{1, \dots, p_{\Gamma,n}\}$ such that $P_i \in P_{\Gamma,f(i)}$ (f is a surjective mapping from $\{1, \dots, p_n\}$ to $\{1, \dots, p_{\Gamma,n}\}$). Define $\Gamma_i = \{\gamma \in \Gamma; \gamma(P_i) = P_i\}$. Γ_i is a subgroup of Γ. For each $P' \in P_{\Gamma,f(i)}$, there exists $\gamma' \in \Gamma$ such that $P' = \gamma'(P_i)$, and $\{\gamma \in \Gamma; \gamma(P_i) = P'\} = \{\gamma'\gamma;\ \gamma \in \Gamma_i\}$, whence $|\Gamma| = |\Gamma_i| \cdot |P_{\Gamma,f(i)}|$. Thus $|\Gamma|/|P_{\Gamma,f(i)}|$ is an integer and for each $s \in S_n$, we have

$$\sum_{\gamma \in \Gamma} \chi\bigl(s, \gamma(P_i)\bigr) = |\Gamma_i| \sum_{P' \in P_{\Gamma,f(i)}} \chi\bigl(s, P'\bigr) = \frac{|\Gamma|}{|P_{\Gamma,f(i)}|} \sum_{P' \in P_{\Gamma,f(i)}} \chi\bigl(s, P'\bigr),$$

whence by (1),

$$\sum_{\gamma \in \Gamma} \chi\bigl(s, \gamma(P_i)\bigr) = \frac{|\Gamma|}{|P_{\Gamma,f(i)}|} \chi(s, P_{\Gamma,f(i)}). \tag{5}$$

By the definitions of $\chi(s, P_i)$, $\gamma(s)$, and $\gamma(P_i)$, we have $\chi(\gamma(s), P_i) = \chi(s, \gamma(P_i))$ for each $s \in S_n$ and $\gamma \in \Gamma$. By Lemma 1, there exists an integral vector $(x_1, \dots, x_{p_n})$ such that for every $s \in S_n$, $F_{G,U}(s) = \sum_{i=1}^{p_n} \chi(s, P_i) x_i$. If $s_\Gamma \in S_{\Gamma,n}$ and $s \in s_\Gamma$, then by (4) and (5),

$$\begin{aligned}
F_{\Gamma,G,U}(s_\Gamma) &= \sum_{\gamma \in \Gamma} F_{G,U}\bigl(\gamma(s)\bigr) = \sum_{\gamma \in \Gamma} \left(\sum_{i=1}^{p_n} \chi\bigl(\gamma(s), P_i\bigr) x_i \right) \\
&= \sum_{i=1}^{p_n} \left(\sum_{\gamma \in \Gamma} \chi\bigl(\gamma(s), P_i\bigr) x_i \right) = \sum_{i=1}^{p_n} \left(\sum_{\gamma \in \Gamma} \chi\bigl(s, \gamma(P_i)\bigr) x_i \right) \\
&= \sum_{i=1}^{p_n} \frac{|\Gamma|}{|P_{\Gamma,f(i)}|} \chi(s, P_{\Gamma,f(i)}) x_i = \sum_{j=1}^{p_{\Gamma,n}} \chi(s, P_{\Gamma,j}) \frac{|\Gamma|}{|P_{\Gamma,j}|} \left(\sum_{i \in f^{-1}(j)} x_i \right).
\end{aligned}$$

Thus setting $\mathbf{y}_{\Gamma,G,U} = (y_1, \dots, y_{p_{\Gamma,n}}) \in \mathbb{Z}^{p_{\Gamma,n}}$ such that for $j = 1, \dots, p_{\Gamma,n}$,

$$y_j = \frac{|\Gamma|}{|P_{\Gamma,j}|}\left(\sum_{i \in f^{-1}(j)} x_i\right), \tag{6}$$

we have by (3), that $F_{\Gamma,G,U}(s_\Gamma) = \boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \cdot \mathbf{y}_{\Gamma,G,U}$ for each $s_\Gamma \in S_{\Gamma,n}$. $\square$

3. Forbidden networks

Let (G, U), $U = (u_1, \dots, u_n)$, be a simple network and Γ be a permutation group on $\{1, \dots, n\}$. Denote by

$$S_{\Gamma,G,U} = \left\{s_\Gamma \in S_{\Gamma,n};\ F_{\Gamma,G,U}(s_\Gamma) > 0\right\}$$

and by $V_{\Gamma,G,U}$ the linear hull of $\{\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma);\ s_\Gamma \in S_{\Gamma,G,U}\}$ in $\mathbb{Q}^{p_{\Gamma,n}}$.

We say that Γ *acts regularly* on (G, U) if for each $\gamma \in \Gamma$, there exists an automorphism of G which maps u_i to $u_{\gamma(i)}$ $(i = 1, \dots, n)$.

Lemma 3. *If a permutation group Γ acts regularly on a simple network (G, U), $U = (u_1, \dots, u_n)$, then $F_{G,U}(s) = F_{G,U}(\gamma(s))$ for each $s \in S_n$ and $\gamma \in \Gamma$.*

Proof. By the definition, for each $\gamma \in \Gamma$, there exists an automorphism $\tilde{\gamma}$ of G such that $\tilde{\gamma}(u_i) = u_{\gamma(i)}$, whence $F_{G,U}(s) = F_{G,U}(\gamma(s))$ for each $s \in S_n$. $\square$

A simple network (H, U) is called *quasicubic*, if every vertex of H has degree at most 3. By the *cubic order* of (H, U), denoted by $\nu_3(H, U)$, we mean the number of the vertices of H of degree 3. We say that (H, U) is a *forbidden network* if H cannot be a subgraph of a graph homeomorphic to a smallest counterexample to the 5-flow conjecture.

Assume that H is a subgraph of a graph G such that the vertices from $V(H) \setminus U$ have the same degrees in G and H. Suppose that (H', U'), $U' = (u'_1, \dots, u'_n)$, is a simple network. Let G' arises from G after deleting the vertices from $V(H) \setminus U$ and identifying u_i with u'_i for $i = 1, \dots, n$. We say that G' arises from G after *replacing* (H, U) by (H', U'). We say that (H, U) can be *regularly replaced* by (H', U') in a class of graphs $\mathcal{C}$, if for every graph G of $\mathcal{C}$, the graph G' arising from G after replacing (H, U) by (H', U') is always bridgeless.

Lemma 4. *Let (H, U), $U = (u_1, \dots, u_n)$, $n \geqslant 2$, be a quasicubic network and Γ be a permutation group on $\{1, \dots, n\}$ such that $F_{H,U}(s) > 0$ for every $s \in \bigcup\{s_\Gamma;\ s_\Gamma \in S_{\Gamma,H,U}\}$. Suppose there exists a quasicubic network (H', U'), $U' = (u'_1, \dots, u'_n)$, such that $\nu_3(H, U) > \nu_3(H', U')$, $V_{\Gamma,H',U'} \subseteq V_{\Gamma,H,U}$, and (H, U) can be regularly replaced by (H', U') in the class of cyclically* 6*-edge connected quasicubic graphs. Then (H, U) is a forbidden network.*

Proof. Let J be a counterexample to the 5-flow conjecture of the smallest possible order. Then by [7], J is a cyclically 6-edge-connected cubic graph. Suppose that G is homeomorphic with J and H is a subgraph of G. Without loss of generality we can assume that $u_1, \dots, u_n$ have in G degrees 2. Let G' be the graph arising from G after replacing (H, U) by (H', U'). By assumptions, G' is bridgeless and homeomorphic with a cubic graph J'. Since $\nu_3(H, U) > \nu_3(H', U')$, the order of J' is smaller than the order of J, therefore J' and G' admit nowhere-zero 5-flows.

Let I (I') be the graph arising from G (G') after deleting the vertices from $V(H) \setminus U$ ($V(H') \setminus U'$). Then (I, U) and (I', U') are simple networks, and there is an isomorphism of I and I' which maps $u_1, \dots, u_n$ to $u'_1, \dots, u'_n$, respectively. Thus $F_{I,U}(s) = F_{I',U'}(s)$ for every $s \in S_n$.

If there exists $s \in S_n$ such that $F_{H,U}(s)$, $F_{I,U}(s) > 0$, then (H, U) and (I, U) have nowhere-zero $\mathbb{Z}_5$-flows φ_1 and φ_2, respectively, such that $\partial\varphi_1(U) = \partial\varphi_2(U) = s$ and the flows φ_1 and $-\varphi_2$ can be “pieced together” into a nowhere-zero $\mathbb{Z}_5$-flow in G, a contradiction. Thus $F_{H,U}(s)F_{I,U}(s) = 0$ for every $s \in S_n$.

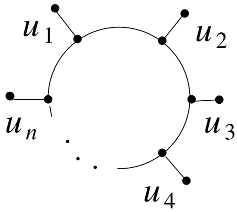

Fig. 1. Graph H_n.

By Lemma 3, $F_{H,U}(s) = F_{H,U}(\gamma(s))$ for each $s \in S_n$ and $\gamma \in \Gamma$. Therefore $s_\Gamma \in S_{\Gamma,H,U}$ if and only if $F_{H,U}(s) > 0$ for each $s \in s_\Gamma$. Hence $F_{I,U}(s) = 0$ for every $s \in \bigcup\{s_\Gamma;\ s_\Gamma \in S_{\Gamma,H,U}\}$, thus $F_{\Gamma,I,U}(s_\Gamma) = 0$ for every $s_\Gamma \in S_{\Gamma,H,U}$.

By Lemma 2, there exists a vector $\mathbf{y}_{\Gamma,I,U} \in \mathbb{Q}^{p_{\Gamma,n}}$, such that for every $s_\Gamma \in S_{\Gamma,n}$, $F_{\Gamma,I,U}(s_\Gamma) = \boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \cdot \mathbf{y}_{\Gamma,I,U}$. Choose $t_{\Gamma,1}, \dots, t_{\Gamma,r} \in S_{\Gamma,H,U}$ so that $\boldsymbol{\chi}_{\Gamma,n}(t_{\Gamma,1}), \dots, \boldsymbol{\chi}_{\Gamma,n}(t_{\Gamma,r})$ form a basis in $V_{\Gamma,H,U}$. We know that $F_{\Gamma,I,U}(t_{\Gamma,i}) = 0$ for $i = 1, \dots, r$. Suppose $s_\Gamma \in S_{\Gamma,n}$ such that $\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \in V_{\Gamma,H,U}$. Then there are numbers $z_1, \dots, z_r$ such that $\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) = \sum_{i=1}^r z_i \boldsymbol{\chi}_{\Gamma,n}(t_{\Gamma,i})$ whence

$$F_{\Gamma,I,U}(s_\Gamma) = \boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \cdot \mathbf{y}_{\Gamma,I,U} = \left(\sum_{i=1}^r z_i \boldsymbol{\chi}_{\Gamma,n}(t_{\Gamma,i})\right) \cdot \mathbf{y}_{\Gamma,I,U} = \sum_{i=1}^r z_i \big(\boldsymbol{\chi}_{\Gamma,n}(t_{\Gamma,i}) \cdot \mathbf{y}_{\Gamma,I,U}\big)$$
$$= \sum_{i=1}^r z_i F_{\Gamma,I,U}(t_{\Gamma,i}) = \sum_{i=1}^r z_i 0 = 0.$$

Thus if $s_\Gamma \in S_{\Gamma,n}$ such that $\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \in V_{\Gamma,H,U}$, then $F_{\Gamma,I,U}(s_\Gamma) = 0$.

Since G' has a nowhere-zero $\mathbb{Z}_5$-flow, there exists $s \in S_n$ such that $F_{H',U'}(s), F_{I',U'}(s) > 0$. Thus $s \in s_\Gamma \in S_{\Gamma,H',U'} \cap S_{\Gamma,I',U'}$, i.e., $F_{\Gamma,I',U'}(s_\Gamma) > 0$. By assumptions, $V_{\Gamma,H',U'} \subseteq V_{\Gamma,H,U}$, whence $\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma) \in V_{\Gamma,H,U}$, and we have proved that then $F_{\Gamma,I,U}(s_\Gamma) = 0$, thus also $F_{\Gamma,I',U'}(s_\Gamma) = 0$, a contradiction with the fact that $F_{\Gamma,I',U'}(s_\Gamma) > 0$. This proves the statement. $\square$

Note that in Lemma 4 we do not need that Γ acts regularly on (G, U). It suffices that $F_{G,U}(\gamma(s)) = 0$ for each $\gamma \in \Gamma$ whenever $F_{G',U'}(s) > 0$ and $F_{G,U}(s) = 0$.

Let C_n be the circuit of order n, i.e., the graph having vertices $v_1, \dots, v_n$ and edges $v_1v_2, v_2v_3, \dots, v_nv_1$. Let H_n arises from C_n after adding new vertices $u_1, \dots, u_n$ and edges $u_1v_1, \dots, u_nv_n$ (see Fig. 1). Then (H_n, U_n), $U_n = (u_1, \dots, u_n)$, is a simple network. For $i = 1, \dots, n$, let x_i denote the arc of H_n directed from u_i to v_i and y_i denote the arc directed from v_i to v_{i+1} (considering the indices mod n).

Consider a graph H_{n-2} and change the notation of its vertices by adding primes, i.e., denote them by $v'_1, \dots, v'_{n-2}, u'_1, \dots, u'_{n-2}$. Similarly change the notation of the arcs. Add new vertices $v'_{n-1}, v'_n, u'_{n-1}, u'_n$ and edges $v'_{n-1}u'_{n-1}$, $v'_nu'_n$, $v'_{n-1}v'_n$, and denote the resulting graph by H'_n (see Fig. 2). Furthermore, let x'_{n-1}, x'_n, and z'_n denote the arcs of H'_n directed from u'_{n-1} to v'_{n-1}, from u'_n to v'_n, and from v'_{n-1} to v'_n, respectively. Then (H'_n, U'_n), $U'_n = (u'_1, \dots, u'_n)$, is a simple network.

Lemma 5. *For $n \geqslant 6$, $\nu_3(H_n, U_n) > \nu_3(H'_n, U'_n)$ and (H_n, U_n) can be replaced by (H'_n, U'_n) regularly in the class of cyclically 6-edge connected cubic graphs.*

Proof. $\nu_3(H_n, U_n) = n > n - 2 = \nu_3(H'_n, U'_n)$. Let G' arises from G after replacing (H_n, U_n) by (H'_n, U'_n). If G' has a bridge, then G is not cyclically 6-edge-connected. $\square$

By Lemmas 4 and 5, to show that a smallest counterexample to the 5-flow conjecture has no circuit of length n, it suffices to prove that $V_{\Gamma,H'_n,U'_n} \subseteq V_{\Gamma,H_n,U_n}$ for a permutation subgroup Γ on (H_n, U_n).

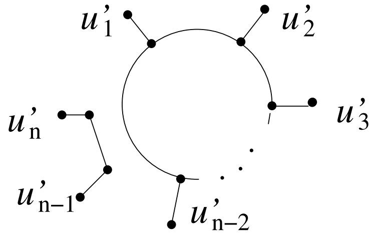

Fig. 2. Graph H'_n.

Let Γ_n denote the dihedral group on $\{1,\dots,n\}$, i.e., the permutation group on $\{1,\dots,n\}$ generated by permutations γ_n and γ'_n, where γ_n maps i to $i+1$ mod n and γ'_n maps i to $n+1-i$ $(i=1,\dots,n)$. Clearly, Γ_n has $2n$ elements of the form $(\gamma_n)^i(\gamma'_n)^j$, where $i=0,1,\dots,n-1$, $j=0,1$, and Γ_n acts regularly on (H_n,U_n). It corresponds with the group of automorphisms of C_n and H_n.

By using computers, we have proved the following statement. More details about the proof we discuss in the last section.

Lemma 6. *$V_{\Gamma_9,H'_9,U'_9} \subseteq V_{\Gamma_9,H_9,U_9}$ and $V_{\Gamma_{10},H'_{10},U'_{10}} \subseteq V_{\Gamma_{10},H_{10},U_{10}}$.*

Theorem 1. *A smallest counterexample to the 5-flow conjecture has girth at least 11.*

Proof. Let G be a smallest counterexample to the 5-flow conjecture. By [7,8], G is a cyclically 6-edge-connected cubic graph with girth at least 9. By Lemmas 4, 5, and 6, G cannot have circuits of orders 9 and 10. Thus G has girth at least 11. $\square$

4. Open problems

If (G,U), $U=(u_1,\dots,u_n)$, is a simple network, then denote by

$$S_{G,U} = \{s \in S_n;\ F_{G,U}(s) > 0\}$$

and by $V_{G,U}$ the linear hull of $\{\boldsymbol{\chi}_n(s);\ s \in S_{G,U}\}$ in $\mathbb{Q}^{p_n}$. Let V_n denotes the linear hull of $\{\boldsymbol{\chi}_n(s);\ s \in S_n\}$. In [9] we mentioned that the affirmative solution of the following problem implies the 5-flow conjecture.

Conjecture 1. *$V_{H_n,U_n} = V_n$ for every $n \geqslant 2$.*

We have verified Conjecture 1 for $n \leqslant 8$ in [9]. With respect to results of this paper, it is enough to consider weaker conjectures.

Conjecture 2. *$V_{\Gamma_n,H'_n,U'_n} \subseteq V_{\Gamma_n,H_n,U_n}$ for every $n \geqslant 2$.*

Conjecture 3. *For each $n \geqslant 2$, there exists a quasicubic network (H'',U''), $U''=(u''_1,\dots,u''_n)$, such that $\nu_3(H,U) > \nu_3(H'',U'')$, (H,U) can be regularly replaced by (H'',U'') in the class of cyclically 6-edge connected quasicubic graphs, and $V_{\Gamma_n,H'',U''} \subseteq V_{\Gamma_n,H,U}$.*

Really, by Lemmas 4 and 5, if one of the conjectures is true, then the smallest counterexample to the 5-flow conjecture cannot have a circuit of any order, whence the 5-flow conjecture holds true.

In fact, it suffices to consider any subgroup of Γ_n, because each such a subgroup acts regularly on (H_n,U_n) and, furthermore, the following statement holds true.

Lemma 7. *Let (H,U), $U=(u_1,\dots,u_n)$, $n\geqslant 2$, be a quasicubic network and Γ be a permutation group on $\{1,\dots,n\}$. Suppose there exists a quasicubic network (H',U'), $U'=(u'_1,\dots,u'_n)$, and a subgroup Γ' of Γ such that $V_{\Gamma',H',U'}\subseteq V_{\Gamma',H,U}$. Then $V_{\Gamma,H',U'}\subseteq V_{\Gamma,H,U}$.*

Proof. Since Γ' is a subgroup of Γ, then for each $P_{\Gamma',i}\in\mathcal{P}_{\Gamma',n}$, there exists $g(i)\in\{1,\dots,p_{\Gamma,n}\}$ such that $P_{\Gamma',i}\subseteq P_{\Gamma,g(i)}$. Clearly g is a surjective mapping form $\{1,\dots,p_{\Gamma',n}\}$ to $\{1,\dots,p_{\Gamma,n}\}$. For each $s\in S_n$, $\chi(s,P_{\Gamma',i})=\sum_{P\in P_{\Gamma',i}}\chi(s,P)$ and

$$\chi(s,P_{\Gamma,j})=\sum_{P\in P_{\Gamma,j}}\chi(s,P)=\sum_{i\in g^{-1}(j)}\chi(s,P_{\Gamma',i}).\tag{7}$$

Choose $t_{\Gamma',1},\dots,t_{\Gamma',r}\in S_{\Gamma',H,U}$ so that $\boldsymbol{\chi}_{\Gamma',n}(t_{\Gamma',1}),\dots,\boldsymbol{\chi}_{\Gamma',n}(t_{\Gamma',r})$ form a basis in $V_{\Gamma',H,U}$. Choose $t_i\in t_{\Gamma',i}$ for $i=1,\dots,r$. Suppose $s_\Gamma\in S_{\Gamma,H',U'}$. Then there exists $s\in s_\Gamma$ such that $F_{H',U'}(s)>0$, thus $s\in s_{\Gamma'}\in S_{\Gamma',H',U'}$, whence, by assumptions and (3), $\boldsymbol{\chi}_{\Gamma',n}(s)=\boldsymbol{\chi}_{\Gamma',n}(s_{\Gamma'})$ is a linear combination of vectors $\boldsymbol{\chi}_{\Gamma',n}(t_1),\dots,\boldsymbol{\chi}_{\Gamma',n}(t_r)$. But by (7), we get that then $\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma)=\boldsymbol{\chi}_{\Gamma,n}(s)$ is a linear combination of vectors $\boldsymbol{\chi}_{\Gamma,n}(t_1),\dots,\boldsymbol{\chi}_{\Gamma,n}(t_r)$. Thus $V_{\Gamma,H',U'}\subseteq V_{\Gamma,H,U}$. $\square$

In view of Lemma 7, if we prove that $V_{\Gamma',H'_n,U'_n}\subseteq V_{\Gamma',H_n,U_n}$, where Γ' is a subgroup of Γ_n, then it implies that $V_{\Gamma_n,H'_n,U'_n}\subseteq V_{\Gamma_n,H_n,U_n}$. But in order to prove the first formula we need to deal with matrices of larger size then in the second formula. Furthermore, the second formula can be true though the first one could be false. Thus the most suitable choice is to deal with group Γ_n, that is the maximal permutation group that acts regularly on (H_n,U_n).

5. Computations

Let $\mathcal{A}$ denote the automorphism group of $\mathbb{Z}_5$. The elements of $\mathcal{A}$ are $\alpha_0=\mathrm{id}$, $\alpha_1=(1,2,4,3)$, $\alpha_2=(1,4)(2,3)$ and $\alpha_3=(1,3,4,2)$. If $s=(s_1,\dots,s_n)\in S_n$ and $\alpha\in\mathcal{A}$, then denote $\alpha(s)=(\alpha(s_1),\dots,\alpha(s_n))\in S_n$. We say that s and $\alpha(s)$ are σ_n-equivalent. Clearly, $\chi_n(s)=\chi_n(\alpha(s))$ and $F_{U,G}(s)=F_{U,G}(\alpha(s))$ for any simple network (G,U) with n outer vertices (because φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G,U) if and only if $\alpha(\varphi)$ is so). Thus it suffices to consider only elements from $S'_n=\{(1,s_2,\dots,s_n)\in S_n\}$ instead of S_n. This reduction we made in [9,11].

Let Γ be a permutation group on $\{1,\dots,n\}$. Then $\boldsymbol{\chi}_{\Gamma,n}(s)=\boldsymbol{\chi}_{\Gamma,n}(\alpha(s))$ and $\alpha(\gamma(s))=\gamma(\alpha(s))$ for every $s\in S_n$, $\alpha\in\mathcal{A}$, and $\gamma\in\Gamma$. Thus, if $s_\Gamma\in S_{\Gamma,n}$, then $\alpha(s_\Gamma)=\{\alpha(s);\ s\in s_\Gamma\}$ either coincides with s_Γ, or these two sets are distinct. But $\boldsymbol{\chi}_{\Gamma,n}(s_\Gamma)=\boldsymbol{\chi}_{\Gamma,n}(\alpha(s_\Gamma))$. Therefore, it suffices to consider a minimal set $S'_{\Gamma,n}$ so that $S_{\Gamma,n}=\{\alpha(s); s\in S'_{\Gamma,n}\}$ instead of $S_{\Gamma,n}$. Note that $S'_{\Gamma,n}$ is not uniquely defined, but every set with this property has the same number of elements.

For any network (G,U), $U=(u_1,\dots,u_n)$, we can consider $S'_{G,U}=S_{G,U}\cap S'_n$ and $S'_{\Gamma,G,U}=S_{\Gamma,G,U}\cap S'_{\Gamma,n}$ instead of $S_{G,U}$ and $S_{\Gamma,G,U}$.

For example we discuss the case when $n=5$ and $\Gamma=\Gamma_5$ (the dihedral group on $\{1,\dots,5\}$). By [7,8], $p_5=11$ and $\mathcal{P}_5$ contains the following partitions

$$\begin{aligned}
&P_i=\bigl\{\{i,i+1\},\{i+2,i+3,i+4\}\bigr\},\quad (i=1,\dots,5),\\
&P_{5+i}=\bigl\{\{i,i+2\},\{i+1,i+3,i+4\}\bigr\},\quad (i=1,\dots,5),\\
&P_{11}=\bigl\{\{1,2,3,4,5\}\bigr\}
\end{aligned}$$

(considering the sums mod 5). We evaluate $\chi_5(t_i)$, $i=1,\dots,7$, where

$$\begin{aligned}
t_1&=(1,1,1,1,1),\quad &\chi_5(t_1)&=(0,0,0,0,0,0,0,0,0,0,1),\\
t_2&=(1,1,2,4,2),\quad &\chi_5(t_2)&=(0,0,0,0,0,0,1,0,1,0,1),\\
t_3&=(1,4,1,2,2),\quad &\chi_5(t_3)&=(1,1,0,0,0,0,0,0,0,0,1),\\
t_4&=(1,1,4,2,2),\quad &\chi_5(t_4)&=(0,1,0,0,0,1,0,0,0,0,1),
\end{aligned}$$

$$t_5 = (1, 2, 1, 4, 2), \quad \chi_5(t_5) = (0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1),$$
$$t_6 = (1, 1, 1, 3, 4), \quad \chi_5(t_6) = (0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 1),$$
$$t_7 = (1, 1, 3, 1, 4), \quad \chi_5(t_7) = (0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1).$$

Furthermore, $\mathcal{P}_{\Gamma_5,5} = (P_{\Gamma_5,1}, P_{\Gamma_5,2}, P_{\Gamma_5,3})$, where

$$P_{\Gamma_5,1} = \{P_i;\ i = 1, \dots, 5\},$$
$$P_{\Gamma_5,2} = \{P_i;\ i = 6, \dots, 10\},$$
$$P_{\Gamma_5,3} = \{P_{11}\}.$$

Thus $p_{\Gamma_5,5} = 3$. We get $\boldsymbol{\chi}_{\Gamma_5,5}(t_i)$ from $\chi_5(t_i)$ (see (1), (3)), and setting $t_{\Gamma_5,i} = \{\gamma(t_i); \gamma \in \Gamma_5\}$ we can evaluate $|t_{\Gamma_5,i}|$ for $i = 1, \dots, 7$,

$$t_1 = (1, 1, 1, 1, 1), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_1) = (0, 0, 1), \quad |t_{\Gamma_5,1}| = 1,$$
$$t_2 = (1, 1, 2, 4, 2), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_2) = (0, 2, 1), \quad |t_{\Gamma_5,2}| = 5,$$
$$t_3 = (1, 4, 1, 2, 2), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_3) = (2, 0, 1), \quad |t_{\Gamma_5,3}| = 5,$$
$$t_4 = (1, 1, 4, 2, 2), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_4) = (1, 1, 1), \quad |t_{\Gamma_5,4}| = 10,$$
$$t_5 = (1, 2, 1, 4, 2), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_5) = (1, 1, 1), \quad |t_{\Gamma_5,5}| = 10,$$
$$t_6 = (1, 1, 1, 3, 4), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_6) = (1, 2, 1), \quad |t_{\Gamma_5,6}| = 10,$$
$$t_7 = (1, 1, 3, 1, 4), \quad \boldsymbol{\chi}_{\Gamma_5,5}(t_7) = (2, 1, 1), \quad |t_{\Gamma_5,7}| = 10.$$

Hence $S'' = \bigcup_{i=1}^{7} t_{\Gamma_5,i}$ is a set of cardinality 51 containing pairwise non-σ_5-equivalent elements from S_5. By [9], $|S'_5| = 51$. Thus every element of S'' is σ_5-equivalent with exactly one element from S'_5 and vice versa. Therefore we can choose $S'_{\Gamma_5,5} = \{t_{\Gamma_5,i};\ i = 1, \dots, 7\}$. We can check that $t_1, t_2 \notin S_{H_5,U_5}$ and $t_3, \dots, t_7 \in S_{H_5,U_5}$. Hence $S'_{\Gamma_5,H_5,U_5} = \{t_{\Gamma_5,i};\ i = 3, \dots, 7\}$ and V_{Γ_5,H_5,U_5} is the linear hull of vectors $\boldsymbol{\chi}_{\Gamma_5,5}(t_i)$, $i = 3, \dots, 7$, i.e., $V_{\Gamma_5,H_5,U_5} = \mathbb{Q}^3$.

Now we discuss the computations implying Lemma 6. Let M'_n be a matrix of size $|S'_{\Gamma_n,H_n,U_n}| \times p_{\Gamma_n,n}$, whose rows are vectors of the form $\boldsymbol{\chi}_{\Gamma_n,n}(s_\Gamma)$ where $s_\Gamma \in S'_{\Gamma_n,H_n,U_n}$. Adding to that matrix rows of the form $\boldsymbol{\chi}_{\Gamma_n,n}(s_\Gamma)$ where $s_\Gamma \in S'_{\Gamma_n,H'_n,U'_n} \setminus S'_{\Gamma_n,H_n,U_n}$, we get a matrix M_n. The rank of M'_n is the dimension of V_{Γ_n,H_n,U_n}, and if M_n and M'_n have the same rank, then $V_{\Gamma_n,H'_n,U'_n} \subseteq V_{\Gamma_n,H_n,U_n}$.

By using computers we have checked that $p_{\Gamma_9,9} = 238$, $|S'_{\Gamma_9,H_9,U_9}| = 262$, and $|S'_{\Gamma_9,H'_9,U'_9} \setminus S'_{\Gamma_9,H_9,U_9}| = 168$. Similarly, for parameter 10 we have checked that $p_{\Gamma_{10},10} = 1079$, $|S'_{\Gamma_{10},H_{10},U_{10}}| = 792$, and $|S'_{\Gamma_{10},H'_{10},U'_{10}} \setminus S'_{\Gamma_{10},H_{10},U_{10}}| = 623$.

Thus M'_9, M_9, M'_{10}, and M_{10} have size 262×238, 430×238, 792×1079, and 1415×1079, respectively. By using computers we have verified that M_9 and M'_9 have rank 151. We used Maple programming language to evaluate them and the program runs few minutes on a personal computer. In a similar way we have verified that M_{10} and M'_{10} have rank 539. This program runs one day. The facts that M'_n and M_n have the same rank for $n = 9, 10$ implies Lemma 6.

In order to stress how much we can reduce the size of matrices using Lemma 2, we discuss what happens if we use the trivial permutation group (containing only identical permutation) instead of Γ_n. Then the matrix corresponding to M_n has p_n columns. Clearly, $p_1 = 0$ and $p_2 = p_3 = 1$. By [8,9], for each $n \geqslant 2$, we have

$$p_n = 1 + \sum_{i=2}^{n-2} \binom{n-1}{i-1} p_{n-i}.$$

Thus $p_9 = 3425$ and $p_{10} = 17\,722$.

Let $c(n) = |S'_{H_n,U_n}|$, and $c_i(n) = |\{s \in S'_n;\ F_{H_n,U_n}(s) = i\}|$ where i is a positive integer. By [10], $c(1) = c_1(1) = c_2(1) = c_3(1) = c_1(2) = c_2(2) = 0$, $c(2) = c_3(2) = 1$, and for every $n \geqslant 3$,

$$
\begin{aligned}
&c(n) = c_1(n) + c_2(n) + c_3(n),\\
&c_1(n) = 3c_1(n-2) + 2c_2(n-2) + 2c_1(n-1) + 2c_2(n-1),\\
&c_2(n) = 2c_2(n-2) + 3c_3(n-2) + c_2(n-1) + 3c_3(n-1),\\
&c_3(n) = c_3(n-2).
\end{aligned}
$$

Thus $|S'_{H_9,U_9}| = 4665$ and $|S'_{H_{10},U_{10}}| = 14\,251$. In [11] was proved that $|S'_{H'_n,U'_n} \setminus S'_{H_n,U_n}| = c_1(n-2)$, whence $|S'_{H'_9,U'_9} \setminus S'_{H_9,U_9}| = c_1(7) = 420$ and $|S'_{H'_{10},U'_{10}} \setminus S'_{H_{10},U_{10}}| = c_1(8) = 1386$.

Therefore, replacing Γ_n by the trivial permutation group on $\{1,\dots,n\}$, instead of matrices M'_9, M_9, M'_{10}, and M_{10} we must deal with matrices of size 4665×3425, 5085×3425, $14\,251 \times 17\,722$, and $15\,637 \times 17\,722$, respectively. We were not able to evaluate ranks of these matrices by personal computer.

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