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# Flow past an elliptic cylinder

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**Abstract**

A theoretical investigation of the unsteady two-dimensional flow of a viscous, incompressible fluid normal to a thin elliptic cylinder is described. The cylinder, which is started impulsively from rest in an open field, continues to move with uniform velocity for the remainder of the problem. Using a vorticity–streamfunction formulation of the full Navier–Stokes equations, transformation techniques are employed to find the initial flow. Strategies which employ boundary layer theory and series expansions of the flow variables to find flow solutions for small values of time are outlined.

*Keywords:* Two-dimensional flow; Elliptic cylinder

## 1. Introduction

The primary objective of the present investigation is to establish flow past a normal flat plate as a limiting case of flow past thin elliptic cylinders. Chandna [1] studied the problem of unsteady two-dimensional flow of a viscous, incompressible fluid past a normal flat plate, using the careful and rigorous methods established by Collins and Dennis [2] and Dennis and Staniforth [4] for the investigation of initial flows. In their groundbreaking work, Collins and Dennis used boundary layer transformations and series expansions of the dependent variables to obtain the flow past a circular cylinder for small values of time. One notes that a correct solution for the initial problem, and a reasonable grasp of the development of flow at the outset, are crucial elements in the determination of the actual flow, as errors committed in the predictions of initial and small-time flows propagate through the remainder of the work.

In this work, we consider the flow of a viscous, incompressible fluid past a thin elliptic cylinder which, at time $t=0$, is suddenly set in motion with constant velocity 1. The paper is organized as follows. In Section 2, we recapitulate the basic equations governing the unsteady, plane irrotational

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motion of viscous, incompressible fluids. Elliptic cylindrical coordinates are introduced to faciliate a more suitable flow domain, and boundary layer transformations are employed to enable treatment of the time singularity at the outset of the problem. The initial solution is calculated in Section 3, and Section 4 maps out the strategy for finding flow solutions for small values of time.

## 2. Basic equations

We consider the unsteady motion of a viscous, incompressible fluid in the $(x, y)$-plane. The coordinates $y$ and $x$ are taken, respectively, to be measured along the major and minor axes of the elliptical cross section of the cylinder with the origin at the intersection of the major and minor axes of the ellipse. The elliptical cross-section, in Cartesian coordinates, is given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $b>a$. Working in terms of the dimensionless velocity components $(u, v)$ of the velocity vector $\boldsymbol{v}(x, y, t)$, and the dimensionless pressure $p(x, y, t)$, the equations which govern the motion of the fluid express conservation of mass and conservation of linear momentum. They take the familiar form:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}=-\boldsymbol{\nabla}p+\frac{2}{R}\nabla^2\boldsymbol{v}, \tag{2}$$

where, the Reynolds number $R$ is defined as $R=2Ud/v$, where $U$ is the reference velocity, $d$ is the reference length, and $v$ is the coefficient of kinematic viscosity. Taking the curl of Eq. (2), and introducing the vector vorticity function, $\boldsymbol{\omega}$, as the curl of the velocity vector field, we obtain the following system of governing equations:

$$\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{v}, \tag{3}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t}-\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega})=-\frac{2}{R}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\omega}), \tag{4}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0.$$

The above equations together with appropriate boundary conditions on $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are taken to govern the flow.

The continuity Eq. (1) guarantees the existence of the streamfunction, $\psi(x, y, t)$, such that:

$$u(x,y,t)=\frac{\partial\psi}{\partial y},\qquad v(x,y,t)=-\frac{\partial\psi}{\partial x}. \tag{5}$$

We further note that for two-dimensional flow, the vorticity vector function will have only one non-zero component:

$$\boldsymbol{\omega}=(0,0,\zeta(x,y,t)). \tag{6}$$

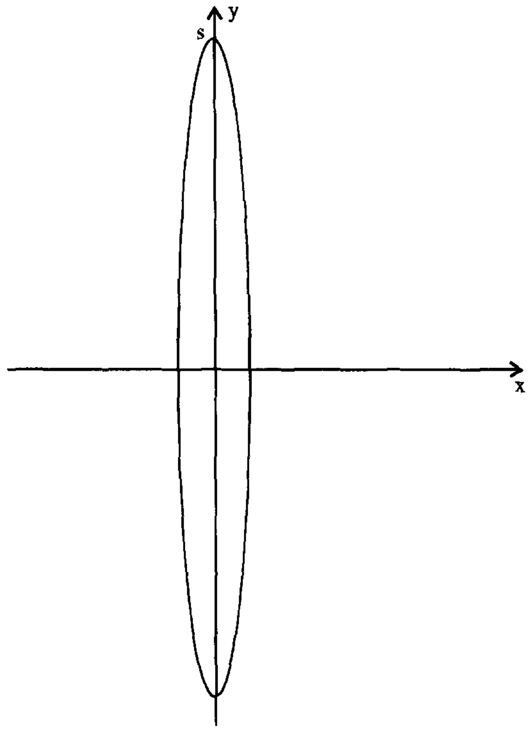


Fig. 1. Flow in an open field.

Employing Eqs. (5) and (6) in Eqs. (3) and (4), we obtain the following formulation of the governing equations in terms of the streamfunction, $\psi(x,y,t)$ and the scalar vorticity function, $\zeta(x,y,t)$:

$$\frac{\partial \zeta}{\partial t} = \frac{2}{R}\nabla^2\zeta + \left[\frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial x}\right], \tag{7}$$

$$\nabla^2\psi + \zeta = 0. \tag{8}$$

Eqs. (7) and (8) govern the motion of a viscous, incompressible fluid past an elliptic cylinder started inpulsively from rest at time $t = 0$. The cylinder continues to move with dimensionless velocity 1 in the direction of the positive $x$-axis, and the fluid at large enough distances from the cylinder is assumed to remain undisturbed.

We adopt the following equivalent formulation of the problem. For $t < 0$, the cylinder and the fluid are moving with velocity 1 in the direction of the positive $x$-axis. At $t = 0$, the cylinder is immediately brought to rest. The fluid at large enough distances from the cylinder is assumed to be moving with uniform velocity, $\boldsymbol{v} = 1\hat{\boldsymbol{\imath}}$ for all time $t$. Since this problem is symmetric with respect to the $x$-axis, we need only consider half the domain (Fig. 1). The boundary conditions for the flow variables are given below:

For $t < 0$: $\psi = y$ throughout the flow field.

For $t \geqslant 0$: (i) $\psi = \dfrac{\partial \psi}{\partial x} = 0$ along the cylinder surface;

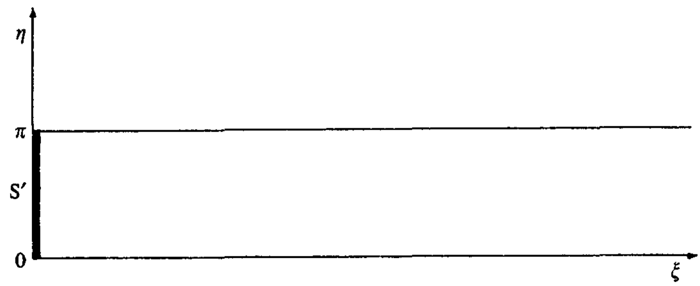


Fig. 2. The transformed flow domain in $(\xi, \eta)$ coordinates.

$$\begin{array}{ll} \text{(ii)} & \dfrac{\partial \psi}{\partial y} \to 1, \ \dfrac{\partial \psi}{\partial x} \to 0 \quad \text{far from the ellipse;} \\ \text{(iii)} & \zeta \to 0 \quad \text{far from the ellipse;} \\ \text{(iv)} & \psi = \zeta = 0 \quad \text{along the } x\text{-axis.} \end{array} \tag{9}$$

In the above, condition (i) expresses the no-slip condition for viscous flow, conditions (ii) and (iii) express uniform flow far from the elliptic cylinder, and condition (iv) expresses the symmetry inherent in the problem.

## 2.1. Introduction of elliptic cylindrical coordinates

The transformation

$$\xi + i\eta = \text{arcsinh}(x + y) - \xi^*, \tag{10}$$

where $\tanh \xi^* = a/b$, is employed to transform the physical domain of the problem from the upper-half of the $xy$-plane to a semi-infinite strip of width $\pi$. The inverse transformation equations of (10) are given by

$$x = \sinh(\xi + \xi^*) \cos \eta, \qquad y = \cosh(\xi + \xi^*) \sin \eta. \tag{11}$$

This transformation successfully maps the surface of the elliptic cylinder to $\xi = 0$, with $\eta$ varying from 0 to $\pi$, and also ensures the periodicity of the physical properties of the fluid. The transformed domain is illustrated in Fig. 2. Under transformation (10), Eqs. (7) and (8) become, respectively,

$$M^2 \frac{\partial \zeta}{\partial t} = \frac{2}{R} \left[ \frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} \right] + \left[ \frac{\partial \zeta}{\partial \eta} \frac{\partial \psi}{\partial \xi} - \frac{\partial \zeta}{\partial \xi} \frac{\partial \psi}{\partial \eta} \right], \tag{12}$$

$$M^2 \zeta = - \left[ \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right], \tag{13}$$

where $M^2 = \frac{1}{2}[\cos 2\eta + \cosh 2(\xi + \xi^*)]$. Dennis and Staniforth [4] have noted that for the transformation defined by (10), the following relationships hold as $\xi \to \infty$:

$$x \sim \frac{1}{2} e^{\xi} \cos \eta, \qquad y \sim \frac{1}{2} e^{\xi} \sin \eta. \tag{14}$$

Employing (14) together with the relations:

$$\frac{\partial\psi}{\partial\xi}=\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\xi},\qquad \frac{\partial\psi}{\partial\eta}=\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\eta} \tag{15}$$

we write the transformed formulation of the problem as follows:

For $t<0$: $\zeta=0 \quad \forall(\xi,\eta)\in\mathscr{R}^2$.

For $t\geqslant 0$: $$M^2\frac{\partial\zeta}{\partial t}=\frac{2}{R}\left[\frac{\partial^2\zeta}{\partial\xi^2}+\frac{\partial^2\zeta}{\partial\eta^2}\right]+\frac{\partial(\psi,\zeta)}{\partial(\xi,\eta)}, \tag{16}$$

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}=M^2\zeta, \tag{17}$$

subject to the conditions

(i) $\psi=\dfrac{\partial\psi}{\partial\xi}=0$ on $\xi=0$;

(ii) $\psi=\zeta=0$ on $\eta=0,\pi$;

(iii) $e^{-\xi}\dfrac{\partial\psi}{\partial\xi}\to\frac{1}{2}e^{\xi^*}\sin\eta,\quad e^{-\xi}\dfrac{\partial\psi}{\partial\eta}\to\dfrac{1}{2}e^{\xi^*}\cos\eta$ as $\xi\to\infty$;

(iv) $\zeta\to 0$ as $\xi\to\infty$. (18)

## 2.2. *Formulation of an integral condition: series expansions of the dependent variables*

We assume sine series expansions for the streamfunction and the vorticity:

$$\psi(\xi,\eta,t)=\sum_{n=1}^{\infty}f_n(\xi,t)\sin n\eta, \tag{19}$$

$$\zeta(\xi,\eta,t)=\sum_{n=1}^{\infty}g_n(\xi,t)\sin n\eta. \tag{20}$$

Employing these expansions in Eqs. (16) and (17), multiplying the resultant expressions by $\sin k\eta$, and integrating with respect to $\eta$ from $\eta=0$ to $\eta=\pi$, we have $\forall k\in\mathcal{N}$:

$$\frac{\partial^2 f_k}{\partial\xi^2}-k^2 f_k=-\frac{1}{4}[2(\cosh 2(\xi+\xi^*))g_k+g_{k-2}+g_{k+2}-g_{2-k}] \tag{21}$$

and

$$2\cosh 2(\xi+\xi^*)\frac{\partial g_k}{\partial t}+\frac{\partial g_{k-2}}{\partial t}+\frac{\partial g_{k+2}}{\partial t}-\frac{\partial g_{2-k}}{\partial t}=\frac{8}{R}\left[\frac{\partial^2 g_k}{\partial\xi^2}-k^2 g_k\right]=S_k(\xi,t), \tag{22}$$

where

$$S_k(\xi,t)=2\sum_{m=1}^{\infty}mg_m\left[\frac{\partial f_{k-m}}{\partial\xi}+\frac{\partial f_{k+m}}{\partial\xi}-\frac{\partial f_{m-k}}{\partial\xi}\right]-mf_m\left[\frac{\partial g_{k-m}}{\partial\xi}+\frac{\partial g_{k+m}}{\partial\xi}-\frac{\partial g_{m-k}}{\partial\xi}\right].$$

In the above, functions with negative subscripts are taken to be zero. The boundary conditions for (21) and (22) follow from those given in (18):

$$\text{(i)} \quad f_k(0,t) = \frac{\partial f_k}{\partial \xi}(0,t) = 0;$$

$$\text{(ii)} \quad 2e^{-\xi} f_k \to e^{\xi^*} \delta_{k,1}, \qquad 2e^{-\xi} \frac{\partial f_k}{\partial \xi} \to e^{\xi^*} \delta_{k,1} \quad \text{as } \xi \to \infty;$$

$$\text{(iii)} \quad g_k \to 0 \quad \text{as } \xi \to \infty, \tag{23}$$

where

$$\delta_{k,1} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \neq 1. \end{cases} \tag{24}$$

Examining the conditions in (23), we note that there is only one condition applied to the vorticity function coefficients, and a relative surplus of conditions applied to the streamfunction coefficients. A solution to this disparity is suggested by Dennis and Quartapelle [3] in the form of globally applied integral conditions on the vorticity. We multiply Eq. (21) by $e^{-k\xi}$, integrate the resulting expression with respect to $\xi$ from $\xi = 0$ to $\xi \to \infty$, and employ conditions (23)(ii) to obtain an integral condition on the vorticity coefficient functions:

$$\int_0^\infty e^{-k\xi}[(2\cosh(\xi + \xi^*))g_k + g_{k-2} + g_{k+2} - g_{2-k}]\, d\xi = 4e^{\xi^*}\delta_{k,1}. \tag{25}$$

From (20), we deduce that $\forall k \in \mathcal{N}$:

$$g_k = \frac{2}{\pi}\int_0^\pi \zeta(\xi,\eta,t)\sin k\eta\, d\eta.$$

Employing the above in (24), we generate the following integral condition on the vorticity function itself:

$$\int_0^\pi \int_0^\infty e^{-k\xi}[2\cosh(\xi + \xi^*)\sin k\eta + \sin(k-2)\eta + \sin(k-2)\eta$$
$$- \sin(2-k)\eta]\zeta\, d\xi\, d\eta = 2\pi e^{\xi^*}\delta_{k,1}. \tag{26}$$

Thus, the unsteady two-dimensional flow of a viscous, incompressible fluid past an impulsively started thin ellipse placed normal to the direction of flow is governed by Eqs. (16) and (17) subject to the local boundary conditions (18)(i), (ii), (iv) and the global integral condition (26).

### 2.3. Flow in the boundary-layer

It is known from boundary-layer that in the initial boundary layer after an impulsive start, the boundary-layer thickness is proportional to $\sqrt{t/R}$. Furthermore, the vorticity $\zeta$ and the streamfunction $\psi$ are proportional to $\sqrt{R/t}$ and $\sqrt{t/R}$, respectively. Thus, we introduce the scaled coefficient functions of the vorticity and streamfunction:

$$\Psi(z,\eta,t) = \frac{1}{\lambda}\psi(\xi,\eta,t), \qquad \Omega(z,\eta,t) = \lambda\zeta(\xi,\eta,t), \tag{27}$$

where $\xi = \lambda z$, and $\lambda$ is the boundary-layer thickness defined to be

$$\lambda = 2\sqrt{\frac{2t}{R}}. \tag{28}$$

The coefficient functions are scaled in a like manner:

$$F_n(x,t) = \frac{1}{\lambda} f_n(\xi,t), \qquad G_n(x,t) = \lambda g_n(\xi,t). \tag{29}$$

Employing (28) and (29) in Eqs. (21) and (22), we obtain the following expansions for the scaled streamfunction and vorticity:

$$\frac{\partial^2 F_k}{\partial z^2} - k^2\lambda^2 F_k = -\frac{1}{4}[2\cosh(2\lambda z + \xi^*)G_k + G_{k-2} + G_{k+2} - G_{2-k}], \tag{30}$$

$$\begin{aligned} &2\cosh(2\lambda z + \xi^*)\left[2t\frac{\partial}{\partial t} - z\frac{\partial}{\partial z} - 1\right][G_k + G_{k-2} + G_{k+2} - G_{2-k}] \\ &= 2\left[\frac{\partial G_k}{\partial x^2} - k^2\frac{8t}{R}G_k\right] + 4t\sum_{m=1}^{\infty} mG_m\frac{\partial}{\partial z}[F_{k-m} + F_{k+m} - F_{m-k}] \\ &\quad -mF_m\frac{\partial}{\partial z}[G_{k-m} + G_{k+m} - G_{m-k}]. \end{aligned} \tag{31}$$

## 3. The initial solution

While the introduction of sine series expansions for the vorticity and stream-functions facilitated the development of the global integral condition on the vorticity function, it is useful to again consider the equations governing $\zeta(\xi,\eta,t)$ and $\psi(\xi,\eta,t)$. We apply the boundary-layer transformations (27) to Eqs. (16), (17) and obtain:

$$\frac{\partial^2 \Omega}{\partial z^2} + 2M^2 z\frac{\partial \Omega}{\partial z} + 2M^2\Omega = 4t\left[M^2\frac{\partial \Omega}{\partial t} + \frac{\partial(\Psi,\Omega)}{\partial(\eta,z)}\right] - \lambda^2\frac{\partial^2\Omega}{\partial \eta^2}, \tag{32}$$

$$\frac{\partial^2 \Psi}{\partial z^2} + \lambda^2\frac{\partial^2 \Psi}{\partial \eta^2} = M^2\Omega. \tag{33}$$

Taking $t = 0$, and hence $\lambda = 0$, in the above equations, we arrive at the equations which govern the initial vorticity and streamfunction in the boundary layer:

$$\frac{\partial^2 \Omega}{\partial x^2} + [\cosh 2\xi^* + \cos 2\eta]\left[z\frac{\partial \Omega}{\partial z} + \Omega\right] = 0, \tag{34}$$

$$\frac{\partial^2 \Psi}{\partial z^2} = [\cosh 2\xi^* + \cos 2\eta]\Omega. \tag{35}$$

The boundary conditions to be satisfied by $\Psi(z,\eta,t=0)$ and $\Omega(z,\eta,t=0)$ are easily derived from conditions (18)(i),(iv) and (26), and are given here:

$$\begin{aligned} &\text{(i)} \quad \Psi = \frac{\partial \Psi}{\partial z} = 0 \quad \text{along } z = 0; \\ &\text{(ii)} \quad \Omega \to 0 \quad \text{as } z \to \infty; \\ &\text{(iii)} \quad \int_0^{\pi}\int_0^{\infty} (\cosh 2\xi^* + \cos 2\eta)\sin n\eta\,\Omega\,\mathrm{d}z\,\mathrm{d}\eta = \pi \mathrm{e}^{\xi^*}\delta_{n,1}. \end{aligned} \tag{36}$$

Applying the transformation $z = f(\eta)u$ to Eq. (34), we get

$$\frac{\partial^2 \Omega}{\partial u^2} + f^2(\eta)[\cosh 2\xi^* + \cos 2\eta]\left[u\frac{\partial \Omega}{\partial u} + \Omega\right] = 0.$$

Taking

$$f^2(\eta) = \frac{1}{[\cosh 2\xi^* + \cos 2\eta]},$$

we have the simplified differential equation:

$$\frac{\partial^2 \Omega}{\partial u^2} + u\frac{\partial \Omega}{\partial u} + \Omega = 0. \tag{37}$$

Likewise, Eq. (35) is transformed to yield

$$\frac{\partial^2 \Psi}{\partial u^2} = \frac{1}{2}\Omega. \tag{38}$$

Two linearly independent solutions of Eq. (37) which satisfy $\Omega \to 0$ as $u \to \infty$ are given by

$$\Omega_1(u,\eta,t=0) = g(\eta)\mathrm{e}^{-1/2u^2}, \qquad \Omega_2(u,\eta,t=0) = h(\eta)\mathrm{e}^{-1/2u^2}\int_0^{u/2} \mathrm{e}^{v^2}\,\mathrm{d}v.$$

Noting that the substitution of $\Omega_2$ into the integral condition in (36) yields a divergent integral, $h(\eta)$ is forced to zero, and the general solution of (37) is

$$\Omega(z,\eta,t=0) = g(\eta)\exp\left\{-\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta]z^2\right\}. \tag{39}$$

Substituting the above expression into the integral condition (36)(iii), we find $g(\eta)$:

$$g(\eta) = 2\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{[\cosh 2\xi^* + \cos 2\eta]}}\mathrm{e}^{\xi^*}\sin\eta.$$

Hence, the initial vorticity is given by

$$\Omega(z,\eta,t=0) = 2\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{[\cosh 2\xi^* + \cos 2\eta]}}\mathrm{e}^{\xi^*}\sin\eta\exp\left\{-\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta]z^2\right\}. \tag{40}$$

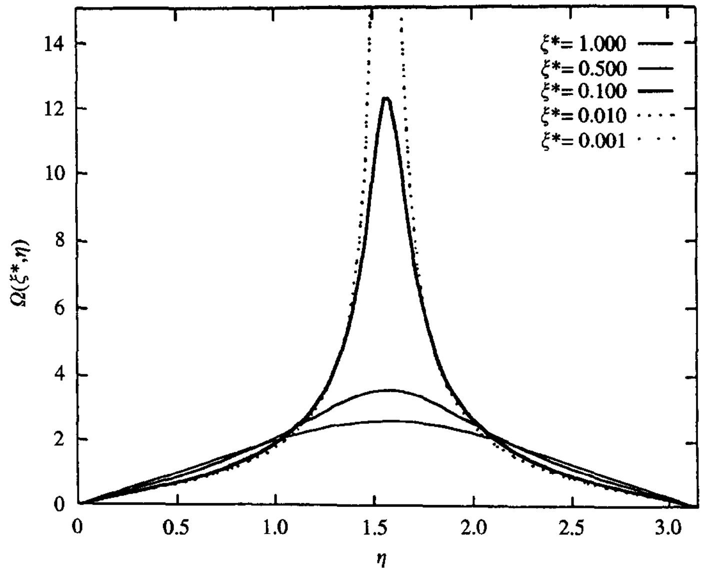


Fig. 3. The initial vorticity profile along the surface of the ellipse for different values of $\xi^*$.

Employing this expression in (38), integrating twice with respect to $u$, and expressing in terms of $z$, we obtain the expression for the initial streamfunction:

$$\Psi(z,\eta,t=0) = e^{\xi^*} \sin\eta z \operatorname{erf}\left(\sqrt{\frac{[\cosh 2\xi^* + \cos 2\eta]}{2}}z\right)$$

$$+ e^{\xi^*}\sqrt{\frac{2}{\pi}}\frac{\sin\eta}{\sqrt{[\cosh 2\xi^* + \cos 2\eta]}}\left[\exp\left\{-\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta]z^2\right\} - 1\right], \tag{41}$$

where

$$\operatorname{erf}(\sqrt{k}x) = 2\sqrt{\frac{k}{\pi}}\int_0^x e^{-ku^2}\,du.$$

The initial vorticity profile along the surface of the elliptic cylinder is obtained by taking $z = 0$ in (39) and it is depicted in Fig. 3 for a range of values of $\xi^*$. We note that as $\xi \to \infty$, the vorticity profile along the surface of the cylinder approaches that predicted for the case of normal flat plate [1].

## 4. Flow for small values of time

The unsteady flow of a viscous, incompressible fluid past an implusively started elliptical cylinder is governed by Eqs. (32) and (33), subject to the boundary conditions given in (36)(i), (ii) and (26), and the initial conditions defined by (40) and (41). We now embark on a study of the behaviour of the fluid for small values of time $t$. We assume that

$$\Omega(z,\eta,t)=\Omega_0(z,\eta,t)+\lambda\Omega_1(z,\eta,t)+\lambda^2\Omega_2(z,\eta,t)+\cdots, \tag{42}$$

$$\Psi(z,\eta,t)=\Psi_0(z,\eta,t)+\lambda\Psi_1(z,\eta,t)+\lambda^2\Psi_2(z,\eta,t)+\cdots. \tag{43}$$

These expansions are employed in the governing Eqs. (32), (33), and the coefficients of successive powers of $\lambda$ are equated to zero to obtain, $\forall k\in\mathcal{N}_0$:

$$\begin{aligned}\frac{\partial^2\Psi_k}{\partial z^2}+\frac{\partial^2\Psi_{k-2}}{\partial\eta^2}&=\frac{1}{2}(\cosh 2\xi^*+\cos 2\eta)\Omega_k+\frac{1}{2}[2z\sinh 2\xi^*]\Omega_{k-1}\\&\quad+\frac{1}{2}\left[\frac{(2z)^2}{2!}\cosh 2\xi^*\right]\Omega_{k-2}+\cdots+\frac{1}{2}\left[\frac{(2z)^{k-1}}{(k-1)!}\operatorname{cs}(k-1)\right]\Omega_1\\&\quad+\frac{1}{2}\left[\frac{(2z)^k}{k!}\operatorname{cs}(k)\right]\Omega_0,\end{aligned} \tag{44}$$

where

$$\operatorname{cs}(k)=\begin{cases}\cosh(2\xi^*) & \text{if } k \text{ is even},\\ \sinh(2\xi^*) & \text{if } k \text{ is odd},\end{cases}$$

and

$$\begin{aligned}&\frac{\partial^2\Omega_k}{\partial z^2}+\frac{\partial^2\Omega_{k-2}}{\partial\eta^2}+[\cosh 2\xi^*+\cos 2\eta]\left[-2t\frac{\partial\Omega_k}{\partial t}+z\frac{\partial\Omega_k}{\partial z}+(1-k)\Omega_k\right]\\&+(2z\sinh 2\xi^*)\left[-2t\frac{\partial\Omega_{k-1}}{\partial t}+z\frac{\partial\Omega_{k-1}}{\partial z}+(2-k)\Omega_{k-1}\right]\\&+\left(\frac{1}{2!}(2z)^2\cosh 2\xi^*\right)\left[-2t\frac{\partial\Omega_{k-2}}{\partial t}+z\frac{\partial\Omega_{k-2}}{\partial z}+(3-k)\Omega_{k-2}\right]\\&+\cdots+\left(\frac{1}{(k-1)!}(2z)^{k-1}\operatorname{cs}(k-1)\right)\left[-2t\frac{\partial\Omega_1}{\partial t}+z\frac{\partial\Omega_1}{\partial z}\right]\\&+\left(\frac{1}{k!}(2z)^k\operatorname{cs}(k)\right)\left[-2t\frac{\partial\Omega_0}{\partial t}+z\frac{\partial\Omega_0}{\partial z}+\Omega_0\right]\\&=4t\sum_{p=0}^{k}\left[\frac{\partial\Psi_p}{\partial\eta}\frac{\partial\Omega_{k-p}}{\partial z}-\frac{\partial\Psi_p}{\partial z}\frac{\partial\Omega_{k-p}}{\partial\eta}\right].\end{aligned} \tag{45}$$

In the above, functions with negative subscripts are taken to be zero. The boundary conditions to be applied on the coefficient functions, $\Psi_k(z,\eta,t)$ and $\Omega_k(z,\eta,t)$, are easily derived by employing expansions (42) and (43) in conditions (36). We have:

$$\text{(i)}\qquad \Psi_k = \frac{\partial \Psi_k}{\partial z} = 0 \quad \text{on } z = 0,$$
$$\text{(ii)}\qquad \Omega_k \to 0 \quad \text{as } z \to \infty \tag{46}$$

as locally applied conditions, and the global condition:

$$\int_0^{\pi}\int_0^{\infty} [\cosh 2\xi^* + \cos 2\eta]\sin n\eta\, \Omega_k(z,\eta,t)\,\mathrm{d}z\,\mathrm{d}\eta$$
$$= -\int_0^{\pi} \sin n\eta \int_0^{\infty} [[\cosh 2\xi^* + \cos 2\eta](-nz) + (2z\sinh 2\xi^*)]\Omega_{k-1} + \cdots$$
$$+ \left[[\cosh 2\xi^* + \cos 2\eta]\left(\frac{(-nz)^k}{k!}\right) + (2z\sinh 2\xi^*)\left(\frac{(-nz)^{k-1}}{(k-1)!}\right) + \cdots + \left(\frac{(2z)^k}{k!}\,\mathrm{cs}(k)\right)\right]$$
$$\times\, \Omega_0\,\mathrm{d}z\,\mathrm{d}\eta + \pi\, \mathrm{e}^{\xi^*}\delta_{n,1}. \tag{47}$$

*4.1. The boundary-layer expansion*

In the boundary layer ($\lambda = 0$), it is known that for small values of time $t$, the functions $\Psi_0(z,\eta,t)$ and $\Omega_0(z,\eta,t)$ can be expressed as series of powers of time, with functional coefficients depending on $x$ and $\eta$.

That is,

$$\Psi_0(z,\eta,t) = \Psi_{00}(z,\eta) + t\Psi_{01}(z,\eta) + t^2\Psi_{02}(z,\eta) + \cdots, \tag{48}$$
$$\Omega_0(z,\eta,t) = \Omega_{00}(z,\eta) + t\Omega_{01}(z,\eta) + t^2\Omega_{02}(z,\eta) + \cdots. \tag{49}$$

Taking $k = 0$ in Eqs. (44) and (45), we obtain equations which govern the boundary-layer coefficient functions, $\Psi_0(z,\eta,t)$ and $\Omega_0(z,\eta,t)$:

$$\frac{\partial^2 \Psi_0}{\partial z^2} = -\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta]\Omega_0, \tag{50}$$
$$\frac{\partial^2 \Omega_0}{\partial z^2} + [\cosh 2\xi^* + \cos 2\eta]\left[z\frac{\partial \Omega_0}{\partial z} + \Omega_0\right]. \tag{51}$$

Employing expansions (48), (49) in the above equations, and equating the coefficients of like powers of $t$ in the resultant equations to zero, we have $\forall p \in \mathcal{N}_0$:

$$\frac{\partial^2 \Psi_{0p}}{\partial z^2} = -\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta]\Omega_{0p}, \tag{52}$$

$$\frac{\partial^2 \Omega_{0p}}{\partial z^2} + [\cosh 2\xi^* + \cos 2\eta]\left[z\frac{\partial \Omega_{0p}}{\partial z} + (1-2p)\Omega_{0p}\right]$$
$$= 4\sum_{q=0}^{p-1}\left(\frac{\partial \Omega_{0,p-q}}{\partial z}\frac{\partial \Psi_{0,q}}{\partial \eta} - \frac{\partial \Omega_{0,p-q}}{\partial \eta}\frac{\partial \Psi_{0,q}}{\partial z}\right). \tag{53}$$

Thus, Eqs. (52) and (53) govern the behaviour of the coefficient functions $\Psi_{0p}(z,\eta)$ and $\Omega_{0p}(z,\eta)$. The associated boundary conditions obtained from (46) are that, for $p \in \mathcal{N}_0$:

$$\begin{aligned} &\text{(i)} \quad \Psi_{0p} = \frac{\partial \Psi_{0p}}{\partial z} = 0 \quad \text{when } z = 0; \\ &\text{(ii)} \quad \Omega_{0p} \to 0 \quad \text{as } z \to \infty. \end{aligned} \tag{54}$$

Likewise, the integral conditions on the coefficient functions $\Omega_{0p}(z,\eta)$ are obtained from (47). We have:

$$\int_0^{\pi} \int_0^{\infty} [\cosh 2\xi^* + \cos 2\eta] \sin n\eta \, \Omega_{00}(x,\eta) \, dz \, d\eta = \pi \, e^{\xi^*} \delta_{n,1} \tag{55}$$

and for $p \in \mathcal{N}$:

$$\int_0^{\pi} \int_0^{\infty} [\cosh 2\xi^* + \cos 2\eta] \sin n\eta \, \Omega_{0p}(z,\eta) \, dz \, d\eta = 0. \tag{56}$$

From the preceding analysis, it is clear that the zeroth order coefficient functions, $\Psi_{00}(z,\eta)$ and $\Omega_{00}(z,\eta)$, are governed by the same system of governing equations and boundary conditions as that which governs the initial solution. Thus,

$$\Omega_{00}(z,\eta) = 2\sqrt{\frac{2}{\pi}} \, e^{\xi^*} \frac{\sin \eta}{\sqrt{[\cosh 2\xi^* + \cos 2\eta]}} \exp\left\{ -\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta] z^2 \right\}, \tag{57}$$

$$\begin{aligned} \Psi_{00}(z,\eta) = {} & e^{\xi^*} \sin \eta z \, \mathrm{erf}\left( \sqrt{\frac{[\cosh 2\xi^* + \cos 2\eta]}{2}} z \right) \\ & + e^{\xi^*} \sqrt{\frac{2}{\pi}} \frac{\sin \eta}{\sqrt{[\cosh 2\xi^* + \cos 2\eta]}} \left[ \exp\left\{ -\frac{1}{2}[\cosh 2\xi^* + \cos 2\eta] z^2 \right\} - 1 \right]. \end{aligned} \tag{58}$$

From Eq. (53), it is clear that subsequent terms in the series expansion of $\Omega_0(z,\eta,t)$ are obtained by employing earlier terms of the expansion into the right-hand side of (53). This equation is solved easily, the homogenous part of the solution being an exponential function, and the particular solution being generated by the method of undetermined coefficients. One then employs the solution for $\Omega_{0k}$, $k \in \mathcal{N}$, in Eq. (52) to determine $\Psi_{0k}$.

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