An Integral Transform in $L^2$-Cohomology for the Ladder Representations of $U(p, q)$

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Starting from the realization of the Fock space as $L^2$-cohomology of $C^{p+q}$,
$\mathcal{H}^{0,p}(C^{p+q}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}^{0,p}_m(C^{p+q})$, an integral transform is constructed which is a direct-image mapping from $\mathcal{H}^{0,p}_m(C^{p+q})$ into the space of holomorphic sections of some vector bundle $E_m$ over $M \cong U(p, q)/(U(q) \times U(p))$, $m \geq 0$. The transform intertwines the natural actions of $U(p, q)$ and is injective if $m \geq 0$, so it provides a geometric realization of the ladder representations of $U(p, q)$. The sections in the image of the transform satisfy certain linear differential equations, which are explicitly described. For example, Maxwell's equations are of this form if $p = q = 2$ and $m = 2$. Thus, this transform is analogous to the Penrose correspondence.

INTRODUCTION

This article concerns an integral transform which is a direct-image mapping from the square-integrable Dolbeault cohomology of $C^{p+q}$, $\mathcal{H}^{0,p}(C^{p+q})$, to the space of sections of some holomorphic vector bundle $E_m$ over $M$ ($M$ is a domain in the Grassmannian $G(p, C^{p+q})$ of $p$-planes in $C^{p+q}$). The sections in the image of the transform satisfy certain differential equations on $M$, and the transform intertwines the natural actions of $U(p, q)$ on $L^2$-cohomology and on holomorphic sections of $E_m$. The representations $\{\mathcal{H}^{0,p}_m(C^{p+q})\}_{m \in \mathbb{Z}}$ are called the ladder representations of $U(p, q)$ and are closely related to the metaplectic (or oscillator) representation. Their importance stems partly from their applications in physics. The group $U(2, 2)$ gives a covering of the group of conformal transformations of Lorentzian space-time. It has long been known (see Bateman [3]) that the conformal group preserves the space of solutions of Maxwell's equations. More generally (see [2, 7]), the conformal group preserves the space of solutions of each of the massless field equations, a one-parameter family of linear differential equations indexed by the half-integer "spin" $m/2$, $m \in \mathbb{Z}$ (Maxwell's equations occur here if $m = 2$). Jakobsen and Vergne [8] first
showed that these representations are unitary for any $m$ and identified them as being the above-mentioned ladder representations of $U(2, 2)$. One consequence of the integral transform constructed here is that it provides a geometric realization of the ladder representations in which $U(p, q)$ acts naturally and unitarily. Recent work of Rawnsley et al. [15] uses $L^2$-cohomology to unitarize a broad class of similar representations.

The integral transform constructed here was inspired by the Penrose correspondence ([13]; see also [17]), which is a relation between subsets of complex projective 3-space $\mathbb{C}P^3$ and subsets of the Grassmannian $G(2, \mathbb{C}^4)$. The Penrose correspondence enables one to construct transforms between certain Dolbeault cohomology spaces over a subset of $\mathbb{C}P^3$ and solutions of differential equations over $M \subset G(2, \mathbb{C}^4)$. In the case of the massless field equations on $M$, this transform was first constructed in [14] and [18], and was considered in a more general and invariant context in [6]. The transform constructed below resembles the Penrose transform in that it is a direct-image mapping and its range is the space of solutions of the massless field equations over $M \subset G(p, q)$. Our transform differs from the Penrose correspondence in that its domain is the square-integrable Dolbeault cohomology $\mathcal{H}^{0, \tau}(\mathbb{C}^{p+q})$ rather than the classical Dolbeault cohomology groups. The benefit here is that the $L^2$-cohomology groups are Hilbert spaces where the group actions are unitary. Other analogs of the Penrose correspondence for the metaplectic representation have been obtained by Patton [10, 11] and by Patton and Rossi [12].

In order to explain our results in more detail, we first explicitly describe the $L^2$-cohomology groups mentioned above. As in Carmona [5], let $W = \mathbb{C}^{p+q}$ denote $\mathbb{C}^n$ endowed with a fixed hermitian form $h$ of signature $(p, q)$. We introduce a positive definite hermitian form $g$ on $W$ which we extend to $\wedge^* T^* W$ for any $z$. The inner product of two $(0, \tau)$-forms $\omega, \tau$ with compact support on $W$ is then the integral over $W$, with respect to the measure $\exp(\frac{1}{2} h(z, z)) dm(z)$, of the inner product of $\omega(z)$ and $\tau(z)$ at each point $z \in W$ (see (1.1)). We let $L^2_{\tau}(W; g)$ denote the Hilbert space of all $(0, \tau)$-forms with finite norm under this inner product. We extend $\bar{\partial}$ to operate weakly inside $L^2_{\tau}(W; g)$; $\bar{\partial}$ becomes a closed operator with dense domain $\{ \omega \in L^2_{\tau}(W; g) \mid \bar{\partial}\omega \text{ has finite norm} \}$. As in Blattner and Rawnsley [4], define

$$\mathcal{H}^{0, \tau}(W) = \ker \bar{\partial}/\text{cl}(\text{im} \bar{\partial}).$$

We remark that $\mathcal{H}^{0, \tau}(W)$ is independent of $g$ as a topological vector space, but its Hilbert space structure depends on $g$.

As in the Hodge approach to cohomology, we would now like to find a space of "harmonic" differential forms which consists of representatives of
the $L^2$-cohomology classes in $\mathcal{H}^{0,r}(W)$. Let $\tilde{\partial}_g$ denote the adjoint of $\partial$ with respect to $g$, and define

$$H^{0,r}(W; g) = \ker \partial \cap \ker \tilde{\partial}_g.$$  

$H^{0,r}(W; g)$ is now our space of representatives of the $L^2$-cohomology classes; note that it depends on the choice of $g$. Carmona [5] shows that $H^{0,r}(W; g) = \{0\}$ unless $r = p$, and $H^{0,p}(W; g)$ is isomorphic to the Fock space, studied by Bargmann in [1]. Blattner and Rawnsley [4] show that $U(p, q)$ acts naturally and unitarily on $\mathcal{H}^{0,p}(W)$ by describing $\mathcal{H}^{0,p}(W)$ with a $U(p, q)$-invariant notion of "pseudo-harmonic" differential forms. This action induces a corresponding unitary representation of $U(p, q)$ on $H^{0,p}(W; g)$. When we decompose these spaces into eigenspaces under the action of the center $\mathbb{T}^1 \cong S^1$ of $U(p, q)$, we obtain direct sum decompositions into irreducible subspaces

$$\mathcal{H}^{0,p}(W) = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}^{0,p}_m(W),$$

and

$$H^{0,p}(W; g) = \bigoplus_{m \in \mathbb{Z}} H^{0,p}_m(W; g).$$

Here, we index with respect to $m$ so that $\omega \in H^{0,p}_m(W; g)$ if and only if $\omega(e^{i\theta}z) = e^{-(m+p)\theta} \omega(z)$.

Suppose now that $V$ is a $p$-dimensional subspace of $W$. We say that $V$ is positive if $h | V$ is positive definite (written $h | V > 0$). As in the previous paragraph, we may consider the spaces $H^{0,p}(V; h | V)$ and $H^{0,p}_m(V; h | V)$. The Fock space description of $H^{0,p}(V; h | V)$ (see (II.15)) allows us to conclude that $H^{0,p}_m(V; h | V)$ is parametrized by the space of holomorphic polynomials on $V \cong \mathbb{C}^p$ which are homogeneous of degree $m$. In particular, $H^{0,p}_m(V; h | V)$ is finite-dimensional for any $m$, and if $m < 0$, $H^{0,p}_m(V; h | V) = \{0\}$. The Fock space description of $H^{0,p}(V; h | V)$ also allows us to conclude that the orthogonal projection $P_v : L^2(V; h | V) \to H^{0,p}(V; h | V)$ maps any element of $L^2(V; h | V)$ onto the representative of its cohomology class in $H^{0,p}(V; h | V)$ (see (II.13)). Define $M \subset G(p, W)$ to be the set of all positive $p$-planes in $W$. We will construct a vector bundle $E_m$ over $M$, for any $m \geq 0$, whose fiber over any point $V \in M$ is the space $H^{0,p}_m(V; h | V)$. We will show that $E_m$ is a holomorphic vector bundle which is homogeneous for $U(p, q)$. Thus the space $\mathcal{O}(M, E_m)$ of holomorphic sections of $E_m$ carries a natural $U(p, q)$-action.

We now wish to define the transform

$$\Phi : H^{0,p}_m(W; g) \to \mathcal{O}(M, E_m)$$
by defining the value of the section $\Phi_\omega$ at $V \in M$ to be the representative of the $L^2$-cohomology class of $\omega \mid V$ in $H^0_{m, p}(V; h \mid V)$,

$$\Phi_\omega(V) = P_V(\omega \mid V).$$

The projection $P_V$ is given in coordinates by an integral transform with a quadratic exponential kernel related to the reproducing kernel of the Fock space. In order to show that $\Phi_\omega$ is well defined we must verify that, for any $V \in M$, $\omega \mid V$ determines an $L^2$-cohomology class, i.e., that

$$\omega \mid V \in L^0_{m, p}(V; h \mid V)$$

so that the integral $P_V(\omega \mid V)$ converges. This is not immediately obvious, since $L^2$-cohomology is not "functorial," but it follows from the description of the Fock space given by Bargmann in [1]. Next, we verify that $\Phi_\omega$ is independent of the choice of $g$, so that $\Phi$ is well defined on $\mathcal{H}^0_{m, p}(W)$. At the same time, we obtain the proof that $\Phi$ intertwines the natural actions of $U(p, q)$ on $\mathcal{H}^0_{m, p}(W)$ and on sections of $E_m$. If $m < 0$, $\Phi$ annihilates $\mathcal{H}^0_{m, p}(W)$, and if $m \geq 0$, $\Phi$ is injective on $\mathcal{H}^0_{m, p}(W)$. Finally, we show that the sections in the image of $\Phi$ satisfy certain linear differential equations which we explicitly describe. For example, Maxwell's equations are of this form if $p = 2, q = 2$, and $m = 2$.

This paper is organized as follows. In Section I, after presenting some useful lemmas, we give the definitions and relevant results concerning the square-integrable Dolbeault cohomology that we will use throughout. In Section II we define the space $M$ of positive $p$-planes, $M \subset G(p, \mathbb{C}^{p + q})$, and we construct the vector bundles $E_m \to M$. In Section III we define the transform $\Phi$ on $H^0_{m, 0}(W; g)$, which requires the preliminary verification that $\omega \mid V \in L^0_{2, p}(V; h \mid V)$ for any $\omega \in H^0_{m, p}(W; g)$. We then show that $\Phi$ is well defined on $\mathcal{H}^0_{m, p}(W)$ by showing that $\Phi$ is independent of the choice of $g$. We gather the remaining results into Section IV. Included here are the proofs that the sections $\Phi_\omega$ are holomorphic on $M$, that $\Phi$ is injective on $\mathcal{H}^0_{m, p}(W)$ if $m \geq 0$, and that the sections $\Phi_\omega$ satisfy the differential equations mentioned above.

Some comments about the notation are in order. We denote the transpose of a matrix $A$ by $^tA$, and the complex-conjugate transpose $^\dagger A$ by $^\dagger A$. If $z$ is a complex number, we use the notations $e^z$ and $\exp(z)$ interchangeably. We denote the restriction of a function $f$ to a set $S$ by $f \mid S$. We use $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$ of nonnegative integers. We use the abbreviations det for determinant, cl for closure, ker for kernel, im for image, Im for imaginary part, and Re for real part. Finally, with apologies to semisimple Lie group theorists, since we use the lower case $g$ for a hermitian form, we use $G$ for an element of $U(p, q)$. 

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I. SQUARE-INTEGRABLE DOLBEAULT COHOMOLOGY

In this section we define and explicitly describe certain square-integrable Dolbeault cohomology spaces of \( \mathbb{C}^n \). A hermitian form of signature \((p, q)\), \(p + q = n\), determines the growth condition. We present, following Carmona [5], a set of differential forms as a space of representatives of this \( L^2 \)-cohomology. We describe the natural action of the group \( U(p, q) \) on these spaces, using results of Blattner and Rawnsley [4]. These \( L^2 \)-cohomology spaces will form the fibers of the vector bundles constructed in the following section. The integral transform which we construct in Section III has an \( L^2 \)-cohomology space as its domain.

Ia. Notations and Preliminaries

Let \( W \) denote a vector space of dimension \( n \) over \( \mathbb{C} \). If \( z \in W \), we regard \( z \) as a column vector, so that \( *z = \bar{z}^T \) is a row vector. Let \( C^{r,s}(W) \) denote the space of smooth \((r, s)\)-forms on \( W \). We will use multi-index notation to describe the elements of \( C^{r,s}(W) \). Let \( \mathbb{N} \) denote the set of nonnegative integers. If \( J \in \mathbb{N}^r, 1 \leq j_1 < \cdots < j_r \leq n \), define

\[
z_J = \left( \begin{array}{c} z_{j_1} \\
\vdots \\
z_{j_r} \end{array} \right),
\]

\[
dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_r}, \quad \bar{dz}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_r}, \text{ and } |z_J|^2 = |z_{j_1}|^2 + \cdots + |z_{j_r}|^2.
\]

Fix a choice of a hermitian form \( h \) of signature \((p, q)\) on \( W \), choose a positive definite hermitian form \( g \) on \( W \). The hermitian forms \( h \) and \( g \) can be simultaneously diagonalized, since \( g \) is positive definite. In fact, the choice of \( h \) and \( g \) uniquely determines subspaces \( W^+ \) and \( W^- \) of \( W \) such that \( \dim W^+ = p \), \( \dim W^- = q \), \( h|W^+ \) is positive definite, \( h|W^- \) is negative definite, and \( W^+ \perp W^- \). Any \( z \in W \) can then be written as \( z = z_+ + z_- \), with \( z_+ \in W^+ \) and \( z_- \in W^- \). We define hermitian forms \( h_+ \), \( h_- \), and \( |h| \) on \( W \) by defining \( h_+(z, z) = h(z_+, z_+), h_-(z, z) = h(z_-, z_-) \), and \( |h|(z, z) = h_+(z, z) - h_-(z, z) \).

Given \( h \) and \( g \), we will fix a choice of basis for \( W \) so that we may describe \( h \) and \( g \) using a fixed choice of coordinates on \( W \). Let \( I_r \) denote the \( r \times r \) identity matrix and \( \text{diag}(a_1, \ldots, a_r) \) the \( r \times r \) diagonal matrix with entries \( a_1, \ldots, a_r \) on the main diagonal. Choose a basis \( \{e_1, \ldots, e_n\} \) of \( W \) so that \( \{e_1, \ldots, e_q\} \) is a basis of \( W^- \), \( \{e_{q+1}, \ldots, e_n\} \) is a basis of \( W^+ \), and so that the matrix \( h_0 \) of \( h \) is given by

\[
h_0 = \begin{pmatrix} -I_q & 0 \\ 0 & I_p \end{pmatrix} = I_{p,q}.
\]
in the coordinates z for \( \{e_1, ..., e_n\} \). In these coordinates, the matrix \( g_0 \) of \( g \) will be given by \( g_0 = \text{diag}(a_1, ..., a_n) \) for some positive real numbers \( a_1, ..., a_n \). We will refer to \( \{e_1, ..., e_n\} \) as our preferred choice of basis for \( W \) with respect to \( h \) and \( g \), and \( z \) will be our preferred coordinate on \( W \) with respect to \( h \) and \( g \). Using this basis for \( W \), we see that the matrix for \( |h| \) is \( I_n \). If \( z, w \in W \), and if we set \( S = (1, 2, ..., q) \in \mathbb{N}^q \) and \( T = (q + 1, ..., n) \in \mathbb{N}^p \), then

\[
 h(z, w) = z_q w_q + \cdots + z_1 w_1 = z^T w.
\]

and

\[
 h^{-1}(z, w) = -(z_1, w_1 + \cdots + z_q w_q) = -z^T w.
\]

We will denote by \( U(p, q) \) the group of all complex linear transformations of \( W \) which preserve \( h \). The decomposition \( W = W^- \oplus W^+ \) allows us to decompose any \( n \times n \) matrix \( G \in U(p, q) \) into block form \( G = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \), where \( A: W^- \rightarrow W^- \) is \( q \times q \), \( B: W^+ \rightarrow W^- \) is \( q \times p \), \( C: W^- \rightarrow W^+ \) is \( p \times q \), and \( D: W^+ \rightarrow W^+ \) is \( p \times p \). If \( A \) is any square complex matrix, we denote that \( A \) is positive definite by writing \( A \gg 0 \).

Ib. Integration Formulas

Before defining square-integrable Dolbeault cohomology, we include here some useful integration formulas. For \( u \in \mathbb{C}' \), let \( |u|^2 = |u_1|^2 + \cdots + |u_r|^2 \). We normalize the measure \( dm(u) \) on \( \mathbb{C}' \) so that

\[
 \int_{\mathbb{C}'} \exp\left(-\frac{1}{2} |u|^2\right) dm(u) = 1,
\]

i.e., \( dm(u) = (2\pi)^{-r} dx \, dy \), if \( u = x + iy \), \( x, y \in \mathbb{R}' \).

**Lemma 1.1.** Let \( j, k \) be nonnegative integers and let \( \alpha \in \mathbb{C} \) have positive real part. Then,

\[
 \int_{\mathbb{C}} z^k \bar{z}^j e^{-(1/2)|\alpha|^2|z|^2} dm(z) = 0 \quad \text{if} \quad j \neq k,
\]

and

\[
 \int_{\mathbb{C}} |z|^2 e^{-(1/2)|\alpha|^2|z|^2} dm(z) = \frac{2^k k!}{\alpha^{k+1}}.
\]

**Lemma 1.2.** Let \( z \in \mathbb{C}' \), and let \( U \) be an \( r \times r \) complex matrix such that
(U + *U) \geq 0. \text{ Let } \phi \text{ be a holomorphic function of } z \text{ for which } \\
\int_{C} \phi(z) \exp(-\frac{1}{2}z^*Uz) \, dm(z) \text{ converges absolutely. Then,} \\
\int_{C} \phi(z) e^{-(1/2)z^*Uz} \, dm(z) = \frac{\phi(0)}{\det U}.

\textbf{Proof.} Any complex matrix } U \text{ with } U + *U \geq 0 \text{ can be diagonalized, hence the result follows from Lemma 1.1. } \blacksquare

\textbf{Remark.} Lemma 1.2 is true for } \phi \text{ antiholomorphic, as well.

\textbf{Ic. } L^2 \text{-Cohomology}

As before, } W \text{ is a complex vector space, } h \text{ is a hermitian form on } W \text{ of } \\
signature \langle p, q \rangle, g \text{ is a positive definite hermitian form on } W, C^{r,s}(W) \text{ is the space of smooth } (r, s)-\text{forms on } W, \text{ and } C^r_c(W) \text{ is the space of smooth } (r, s)-\text{forms on } W \text{ with compact support. Using the appropriate identifications, we consider } g \text{ as an inner product on } T_z^*(W)_C, \text{ the complexified cotangent space to } W \text{ at } z, \text{ and we extend } g \text{ to } \bigwedge^k T_z^*(W)_C \text{ in the usual manner. If } \omega, \tau \in C^r_c(W), \text{ we may now define their inner product by} \\
(\omega, \tau)_g = \int_{\mathbb{C}^n} g(\omega(z), \tau(z)) \exp(\frac{1}{2}h(z, z)) \, dm(z). \quad (1.1)

If } \omega(z) = \phi(z) \, dz_L \wedge d\overline{z}_K \text{ and } \tau(z) = \psi(z) \, dz_L \wedge d\overline{z}_M, (1.1) \text{ implies that } \omega \text{ and } \\
\tau \text{ are orthogonal unless } J = L \text{ and } K = M, \text{ and in that case} \\
(\omega, \tau)_g = a(J) a(K) \int_{\mathbb{C}^n} \phi(z) \overline{\psi(z)} \exp(\frac{1}{2}h(z, z)) \, dm(z). \quad (1.2)

Here } a(J) \text{ is defined by } a(J) = a_{ij} \times \cdots \times a_{ij}, \text{ where } g_0 = \text{diag}(a_1, \ldots, a_n) \text{ is the matrix of } g. L^r_c(W; g) \text{ will now denote the completion of } C^r_c(W) \text{ with respect to this inner product.} 

Suppose } g \text{ and } g' \text{ are any two choices of positive definite hermitian forms on } W. \text{ If a differential form } \omega \text{ satisfies } (\omega, \omega)_g < \infty, \text{ then it satisfies} \\
(\omega, \omega)_{g'} < \infty. \text{ Moreover, the identity mapping} \\
L^r_c(W; g) \xrightarrow{\text{b}} L^r_c(W; g')

is bounded, hence continuous. The topology of } L^r_c(W; g) \text{ is thus independent of } g. \text{ When we wish to consider the underlying topological vector space, we will write } L^r_c(W).

The cohomology we will consider has coboundary operator } \tilde{\partial}, \text{ which is well defined on } C^r_c(W). \text{ We extend } \tilde{\partial} \text{ to operate weakly in } L^r_c(W), \text{ so that } \tilde{\partial} \text{ acts in the sense of distributions on the coefficients of any } \omega \in L^r_c(W). \text{ The domain of } \tilde{\partial} \text{ is then the set of all } \omega \in L^r_c(W) \text{ such that } \tilde{\partial} \omega \in L^{r+1}_c(W). \text{ With this definition, } \tilde{\partial} \text{ is a closed, densely defined, unbounded linear
operator. It follows that $\ker \partial$ is a closed subspace of $L^2_s(W)$ and that $\ker \partial$ contains the image of $\partial$ acting on $L^2_s(W)$.

As in Blattner and Rawnsley [4], we now consider the complex

$$\cdots L^0_s(W) \xrightarrow{\partial_{r-1}} L^0_{s-1}(W) \xrightarrow{\partial_r} L^0_{s-1}(W) \cdots$$

and form the resulting cohomology spaces

$$\mathcal{H}^{0,r}(W) = \ker \partial_r / \text{cl(im } \partial_{r-1}).$$

Blattner and Rawnsley [4] show that, in this case, the image of $\partial$ is closed, so we may omit the "cl" in the preceding definition. The group $U(p, q)$ acts continuously on $L^s_{*}(W)$ and commutes with $\partial$, thus $U(p, q)$ acts naturally on $\mathcal{H}^{0,r}(W)$ by a continuous representation. Blattner and Rawnsley [4] also show that there is a canonical choice of inner product on $\mathcal{H}^{0,r}(W)$ so that the action of $U(p, q)$ is unitary.

There exists a space of representatives for $\mathcal{H}^{0,r}(W)$ which is given by differential forms. Define the closed operator $\star \partial_g$ to be the adjoint of $\partial$ with respect to $(\cdot, \cdot)_g$. The kernel of $\star \partial_g$ is then orthogonal, with respect to $g$, to the image of $\partial$. We define the space of $g$-harmonic $(0, r)$-forms as

$$H^{0,r}(W; g) = \ker \partial_g \cap \ker \star \partial_g.$$

We now have a direct sum decomposition

$$\ker \partial_g = \text{im } \partial_{r-1} \oplus H^{0,r}(W; g).$$

Hence, $H^{0,r}(W; g)$ is naturally isomorphic to $\mathcal{H}^{0,r}(W)$ as a vector space. However, $H^{0,r}(W; g)$ depends crucially on the choice of $g$.

Carmona [5] describes $H^{0,r}(W; g)$ as follows.

**Theorem 1.1** (Carmona). $H^{0,r}(W; g) = \{0\}$ unless $r = p$, and $H^{0,p}(W; g) = \{\omega \in L^0_s(W; g) \mid \omega(z) = \phi(z) \overline{d\tau} \}$ where $\phi$ satisfies:

(i) $\phi(z)$ is holomorphic in $z_s$,

(ii) $\phi(z) \exp(\frac{1}{2} h_+(z, z))$ is holomorphic in $\overline{z}_r$,

(iii) $\int_{\mathbb{C}^n} |\phi(z)|^2 \exp(\frac{1}{2} h(z, z)) \, dm(z) < \infty$. \hspace{1cm} (1.3)

Equivalently,

$$H^{0,p}(W; g) = \{\omega \in L^0_p(W; g) \mid \omega(z) = f(z_s, \overline{z}_r) \exp(\frac{1}{2} h_+(z, z)) \, dz_r \}$$

where $f$ is holomorphic and

$$\int_{\mathbb{C}^n} |f(z)|^2 \exp(-\frac{1}{2} |h_+(z, z)|) \, dm(z) < \infty$. \hspace{1cm} (1.4)
Let $\mathcal{L}^2(W)$ denote the set of all complex-valued functions $f$ on $W$ which satisfy
\[ \int_W |f(w)|^2 \, dm(w) < \infty. \]

In [1] Bargmann studied the subset $\mathcal{F}(W; g)$ of $\mathcal{L}^2(W)$ given by
\[ \mathcal{F}(W; g) = \{ f \in \mathcal{L}^2(W) \mid f(z) \exp(\frac{1}{2} |z|^2) \text{ is analytic in } z_s \text{ and } \overline{z}_T \}. \]

We refer to $\mathcal{F}(W; g)$ as the Bargmann–Segal–Fock space. The space $\mathcal{F}(W; g)$ parametrizes $H^{0,p}(W; g)$, i.e.,
\[ H^{0,p}(W; g) = \{ \omega \in L^0_{2,p}(W; g) \mid \omega(z) = f(z) \exp(-\frac{1}{2} h(z, z)) \, dz_T \}
\]
where $f \in \mathcal{F}(W; g)$.

In particular, the action of $U(p, q)$ on $\mathcal{F}(W; g)$ is the same as the action on $H^{0,p}(W; g)$, which we describe below.

If $\omega \in L^0_{2,p}(W; g)$ is given by $\omega(z) = \phi(z) \, dz_j$, then the natural action of $G \in U(p, q)$ on $\omega$ is given by
\[ (l(G) \omega)(z) = \phi(G^{-1} z) \, d\overline{(G^{-1} z)_j} \]
\[ = \phi(G^{-1} z) \sum_{1 \leq k_1 < \cdots < k_p \leq n} c_{j,k}(G^{-1}) \, dz_K, \]

where $c_{j,k}(G^{-1})$ is the determinant of the $p \times p$ minor of $G^{-1}$ with rows indexed by $J$ and columns indexed by $K$. We recall that $l(G)$ preserves $\ker \tilde{\partial}$ and $\text{im } \tilde{\partial}$. However, $l(G)$ does not preserve $H^{0,p}(W; g)$, since $g$ is not invariant under $U(p, q)$. Let $P_g : \ker \tilde{\partial} \to H^{0,p}(W; g)$ denote the Hilbert space orthogonal projection. The action of $U(p, q)$ on $H^{0,p}(W; g)$ corresponding to the natural action on $\mathcal{F}^{0,p}(W)$ is then given by
\[ \sigma(G) = P_g \circ l(G). \]

**Proposition 1.1.** Let $\omega \in \ker \tilde{\partial}_p$ be given by $\omega(z) = \phi(z) \, dz_T + \sum_{j \neq T} \phi_j(z) \, dz_j$. Then the orthogonal projection $P_g$ satisfies $P_g \omega = P_g(\phi \, dz_T)$ and is given by
\[ (P_g \omega)(z) = \overline{dz_T} \int_{\mathcal{C}_n} \phi(w) \, K(z, w) \, dm(w), \]

where
\[ K(z, w) = \exp(-\frac{1}{2}[h_+ (z - w, z) + h_- (z - w, w)]) \]
\[ = \exp(-\frac{1}{2}[*z_T(z_T - w_T) - *w_s(z_s - w_s)]). \]
**Corollary 1.1.** Let $\omega \in H^{0,p}(W; g)$ be given by $\omega(z) = \phi(z)\overline{dz}$, and let $G^{-1} = (\frac{A}{B}, \frac{C}{D})$, for $G \in U(p, q)$. As in (1.6),

$$
(l(G)\omega)(z) = \phi(G^{-1}z) \left( \det Dz + \sum_{J \neq T} c_{T,J}(G^{-1}) \overline{dz}_T \right).
$$

Therefore, by (1.7),

$$
(s(G)\omega)(z) = P_s(l(G)\omega)(z)
= \det Dz \int_{\mathbb{C}^n} \phi(G^{-1}w) K(z, w) \, dm(w).
$$

The group $U(p, q)$ does not act irreducibly on $\mathcal{H}^{0,p}(W)$ or $H^{0,p}(W; g)$. In fact, if we decompose these spaces into invariant subspaces under the action of $\{e^{i\theta}I_n : \theta \in \mathbb{R}\}$, the center of $U(p, q)$, we obtain subspaces invariant under all of $U(p, q)$.

**Proposition 1.2.** Under the action of $U(p, q)$, $\mathcal{H}^{0,p}(W)$ and $H^{0,p}(W; g)$ decompose into Hilbert space direct sums

$$
\mathcal{H}^{0,p}(W) = \bigoplus_{m=-\infty}^{\infty} \mathcal{H}^{0,p}_m(W)
$$

and

$$
H^{0,p}(W; g) = \bigoplus_{m=-\infty}^{\infty} H^{0,p}_m(W; g)
$$

of invariant subspaces. The definition of $H^{0,p}_m(W; g)$ given by

$$
H^{0,p}_m(W; g) = \{ \omega \in H^{0,p}(W; g) \mid \omega(e^{i\theta}z) = e^{-i(m+\rho)\theta} \omega(z) \} \quad (1.8)
$$

determines the index $m$.

Blattner and Rawnsley [4] identify the representation $s$ as follows. $U(p, q)$ is a subgroup of the symplectic group $Sp(p + q, \mathbb{R})$ since it preserves the imaginary part of $h$, which is an alternating form on $\mathbb{C}^{p+q} \approx \mathbb{R}^{2(p+q)}$. Hence the metaplectic group $Mp(p + q, \mathbb{R})$, the double cover of $Sp(p + q, \mathbb{R})$, induces a double cover $MU(p, q)$ of $U(p, q)$. Let $\mu$ denote the metaplectic representation of $Mp(p + q, \mathbb{R})$, and let $\tilde{\mu}$ denote its restriction to $MU(p, q)$. The representation $\tilde{\mu}$ does not factor through $U(p, q)$. Let $\text{Det}^{1/2}$ denote the unique character of $MU(p, q)$ whose square is the pullback to $MU(p, q)$ of
the determinant $\text{Det}$ on $U(p, q)$. Then, $\tilde{\mu} \otimes \text{Det}^{1/2}$ factors through $U(p, q)$ (see Sternberg and Wolf [16]), and

$$(\tilde{\mu} \otimes \text{Det}^{1/2})|_{U(p, q)} = \sigma.$$  

The restriction of the representation $\sigma$ is irreducible on each $H^0_{m,p}(W; g)$, and these are the so-called “ladder” representations of $U(p, q)$.

II. THE GEOMETRY OF POSITIVE $p$-PLANES IN $W$

In this section we define the space $M$ of all positive $p$-planes in $W$, and for any $m \geq 0$ we construct a vector bundle $E_m$ over $M$ whose fiber $E_m(V)$ over $V \in M$ is the $L^2$-cohomology space $H^0_{m,p}(V; h|V)$. We describe the actions of $U(p, q)$ on $M$, on $E_m$, and on sections of $E_m$. It will be shown in Section IV that the space $L^2(M, E_m)$ of holomorphic sections of $E_m$ contains the representation space for the action of $U(p, q)$ on the solutions of certain differential equations.

A positive $p$-plane in $W$ is a $p$-dimensional subspace $V$ of $W$ such that $h|V$ is positive definite (written $h|V > 0$), where $h$ is the fixed hermitian form of signature $(p, q)$ on $W$. The set $M$ of all such positive $p$-planes is a subset of the Grassmannian $G(p, W)$ of all $p$-planes in $W$. Since $U(p, q)$ preserves $h$, the natural action of $U(p, q)$ on $G(p, W)$ preserves $M$. Recall that we fixed a basis $\{e_1, \ldots, e_n\}$ of $W$ for which the matrix $h_0$ of $h$ is given by

$$h_0 = I_{p,q} = \begin{pmatrix} -I_q & 0 \\ 0 & I_p \end{pmatrix}.$$  

The stabilizer in $U(p, q)$ of any particular point $V_0 \in M$ is $U(q) \times U(p)$, the maximal compact subgroup of $U(p, q)$. Therefore, $M \approx U(p, q)/(U(q) \times U(p))$.

We now consider the $L^2$-cohomology of a positive $p$-plane $V$, as defined in the previous section. We use the positive definite hermitian form $h_V = h|V$ to define the inner product on the cotangent spaces of $V$, so $h_V$ determines the differential forms which represent $L^2$-cohomology classes on $V$. We also use $h_V$ in the exponential weighting factor for integration on $V$. Let $u$ be a coordinate on $V$. If $\omega, \tau \in C_c^\infty(V)$, their inner product is then given as in (1.1) by

$$(\omega, \tau)_V = \int_V h_V(\omega(u), \tau(u)) \exp(\frac{1}{2} h_V(u, u)) \, dm_V(u),$$
where the measure $dm_v(u)$ on $V$ is defined so that

$$
\int_V \exp \left(-\frac{1}{2} h_v(u, u)\right) dm_v(u) = 1. \quad (II.2)
$$

Since $h_v$ is positive definite, $(h_v)_+ = |h_v| = h_v$ and $(h_v)_- \equiv 0$. Therefore, Theorem 1.1 concludes that $H^{0, r}_0(V; h_v) = \{0\}$ unless $r = p$, and, as in (1.4),

$$
H^{0, p}_0(V; h_v) = \left\{ \omega \in L^2(V; h_v) \mid \omega(u) = f(\bar{u}) \exp \left(-\frac{1}{2} h_v(u, u)\right) d\bar{u} \right\},
$$

where $f$ is holomorphic and

$$
\int_V |f(u)|^2 \exp \left(-\frac{1}{2} h_v(u, u)\right) dm_v(u) < \infty \right\}, \quad (II.3)
$$

where we write $d\bar{u} = d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_p$. The space of such differential forms with $f$ a polynomial is dense in $H^{0, p}_0(V; h_v)$.

We defined the spaces $H^{a, p}_m(V; h_v)$ in (1.8) for any integer $m$ by

$$
H^{a, p}_m(V; h_v) = \{ \omega \in H^{a, p}_0(V; h_v) \mid \omega(e^{i\theta}u) = e^{-im+q} \omega(u) \}. \quad (II.5)
$$

If $\omega \in H^{0, p}_m(V; h_v)$ is given by $\omega(u) = f(\bar{u}) \exp \left(-\frac{1}{2} h_v(u, u)\right) d\bar{u}$ as in (II.3), we conclude that $f$ satisfies

$$
f(e^{i\theta}v) = e^{im}f(v) \quad (II.4)
$$

for all $v \in \mathbb{C}^p$. Define $\mathcal{P}(m, \mathbb{C}^p)$ to be the space of holomorphic polynomials which are homogeneous of degree $m \geq 0$ on $\mathbb{C}^p$. From (II.4) we deduce the vector space isomorphisms

$$
H^{a, p}_m(V; h_v) \cong \mathcal{P}(m, \mathbb{C}^p) \quad \text{if} \quad m \geq 0,
$$

$$
\cong \{0\} \quad \text{if} \quad m < 0. \quad (II.5)
$$

In particular, $H^{0, p}_m(V; h_v)$ is finite-dimensional for any $m$.

We now wish to construct a vector bundle $E_m$ over $M$, for any $m \geq 0$, whose fiber $E_m(V)$ over any $V \in M$ is given by $E_m(V) = H^{0, p}_m(V; h_v)$. We wish $E_m$ to have the structure of a holomorphic vector bundle which is homogeneous for $U(p, q)$. After describing the action of $U(p, q)$ on $E_m$, we then show that there exists a trivialization

$$
J: M \times \mathcal{P}(m, \mathbb{C}^p) \to E_m
$$

of $E_m$ which satisfies, for each $G \in U(p, q)$, $V \in M$, and $f \in \mathcal{P}(m, \mathbb{C}^p)$,

$$
G \cdot J(V, f) = J(V', f'),
$$

where $V'$ and $f'$ depend holomorphically on $V, f$, and $G$. 

We first discuss a coordinate \( \lambda \) on \( M \). Any \( p \)-plane \( V \subset W \) is spanned by the columns of some \( n \times p \) matrix \( A \), written in coordinates with respect to our fixed basis \( \{ e_1, ..., e_n \} \) of \( W \). The form of \( h_0 \) given in (II.1) requires that the bottom \( p \times p \) minor of \( A \) be nonsingular whenever \( V \in M \). We may thus fix a preferred choice of basis for \( V \) so that \( A \) is given by

\[
A = \begin{pmatrix} \lambda \\ I_p \end{pmatrix}, \quad \text{with } \lambda \text{ a } q \times p \text{ complex matrix.}
\]

We identify \( V = V(\lambda) \) with \( \mathbb{C}^p \) by identifying the column vector \( u \in \mathbb{C}^p \) with the point \( Au \in V(\lambda) \subset W \), and we refer to \( u \) as our preferred coordinate on \( V(\lambda) \). The condition that \( V(\lambda) \in M \) means that \( h_\lambda = h | V(\lambda) \geq 0 \), so

\[
h_\lambda(u, u) = h(Au, Au) = *u(I_p - *\lambda\lambda) u > 0 \quad (\text{II.6})
\]

for all nonzero \( u \in \mathbb{C}^p \). Consequently, \( V(\lambda) \in M \) if and only if

\[
I_p - *\lambda\lambda = I(\lambda) \geq 0.
\]

If we now define

\[
\mathcal{D}_{p,q} = \{ q \times p \text{ complex matrices } \lambda \mid I(\lambda) \geq 0 \},
\]

we see that \( \mathcal{D}_{p,q} \) parametrizes \( M \). Note that \( \mathcal{D}_{1,1} \) is just the unit disk in the complex plane.

We wish to determine the measure \( dm_\lambda(u) = dm_\lambda(u) \) in terms of our preferred coordinate \( u \) on \( V(\lambda) \). Using (II.6) and Lemma 1.2 we compute that

\[
\int_{\mathbb{C}^p} \exp(-\frac{1}{2} h_\lambda(u, u)) \, dm(u) = \int_{\mathbb{C}^p} \exp(-\frac{1}{2} *u I(\lambda) u) \, dm(u) = \det(I(\lambda))^{-1}.
\]

Therefore, definition (II.2) requires that

\[
dm_\lambda(u) = \det(I(\lambda)) \, dm(u). \quad (\text{II.7})
\]

Write \( G \in U(p, q) \) in block form \( G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). The action of \( G \) on \( W \) sends \( Au \in V \subset W \) to \( GAu \). Therefore \( G \) sends the plane \( V(\lambda) \) determined by \( A \) to the plane \( V(\lambda') \) determined by

\[
GA = \begin{pmatrix} A\lambda + B \\ C\lambda + D \end{pmatrix} \sim \begin{pmatrix} (A\lambda + B)(C\lambda + D)^{-1} \\ I_p \end{pmatrix} = A'.
\]

We may thus write \( G.V(\lambda) = V(G.\lambda) = V(\lambda') \) where

\[
\lambda' = G.\lambda = (A\lambda + B)(C\lambda + D)^{-1}. \quad (\text{II.8})
\]

Now let \( u \) and \( u' \) denote the preferred coordinates on \( V = V(\lambda) \) and
$V' = V(\lambda')$, respectively. Since $G \lambda u = A'(C \lambda + D) u$, the map $\pi(G) : V \to V'$ is given in coordinates by

$$u' = \pi(G) u = (C \lambda + D) u. \quad (11.9)$$

The pullback $\tilde{\pi}(G)$ of $\pi(G^{-1})$ is the natural map from $H^0_{\lambda}(V; h_v)$ to $H^0_{\lambda'}(V'; h_{v'})$. If $w \in H^0_{\lambda}(V; h_v)$ is given by $w(u) = \phi(u) d\tilde{u}$,

$$[\tilde{\pi}(G) \omega](u') = \omega(\pi(G^{-1}) u') = \phi((C \lambda + D)^{-1} u') \det(C \lambda + D)^{-1} d\tilde{u}'. \quad (11.10)$$

The map $\tilde{\pi}(G)$ clearly preserves the homogeneity degree of $\omega$, so that $\tilde{\pi}(G) H^0_{\lambda}(V; h_v) \subset H^0_{\lambda'}(V'; h_{v'})$. Thus, $\tilde{\pi}(G)$ describes an action of $U(p, q)$ on $E_m$ which is linear on each fiber and which is compatible with the action of $U(p, q)$ on the base $M$ in the sense that

$$G.E_m(V) = \tilde{\pi}(G) H^0_{\lambda}(V; h_v) \subset H^0_{\lambda'}(V'; h_{v'}) = E_m(G.V).$$

The vector bundle $E_m$ is therefore homogeneous for $U(p, q)$.

Before discussing the holomorphic structure of $E_m$, we record here the transformation properties of the matrix $I(\lambda)$ under the action of $G \in U(p, q)$.

**Lemma 2.1.** Let $\lambda \in \mathcal{D}_{p,q}$, $G = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in U(p, q)$, and write $\lambda' = G \lambda = (A \lambda + B)(C \lambda + D)^{-1}$. Let $I(\lambda) = I_p - *(\lambda \lambda')$. Then,

$$(*(C \lambda + D) I(\lambda'))(C \lambda + D) = I(\lambda).$$

**Corollary 2.1.** For all $u, v \in \mathbb{C}^p$,

$$h_{\lambda'}((C \lambda + D) u, (C \lambda + D) v) = h_{\lambda}(u, v).$$

**Corollary 2.2.** $|\det(C \lambda + D)|^2 \det I(\lambda') = \det I(\lambda)$.

We may now show that $E_m$ has the structure of a holomorphic vector bundle.

**Proposition 2.1.** Define the map $J : \mathcal{D}_{p,q} \times \mathcal{P}(m, \mathbb{C}^p) \to E_m$ by

$$J(\lambda, f)(u) = \det I(\lambda) f(I(\lambda) u) \exp(-\frac{1}{2} h_{\lambda}(u, u)) d\tilde{u}. \quad (11.11)$$

Then if $G = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in U(p, q)$, $J(\lambda, f)$ satisfies

$$\tilde{\pi}(G) J(\lambda, f) = J(\lambda', f').$$
where $\lambda' = (A\lambda + B)(C\lambda + D)^{-1}$ and $f' = \det(C\lambda + D) f \circ t(C\lambda + D)$ depend holomorphically on $G, \lambda,$ and $f$.

**Proof.** It is clear from (II.3) that $J(\lambda, f) \in E_m(V(\lambda)) = H^{0,0}_{m}(V(\lambda); h_{\lambda})$. Writing $\tilde{\pi}(G)$ as in (II.10) and using the above Lemma 2.1 and its corollaries, we compute that

\[
[\tilde{\pi}(G) J(\lambda, f)](u') = J(\lambda', f)((C\lambda + D)^{-1} u')
\]

\[= \det I(\lambda') f'(I(\lambda')(C\lambda + D)^{-1} u')
\]

\[\times \exp(-\frac{1}{2} h_{\lambda'}((C\lambda + D)^{-1} u', (C\lambda + D)^{-1} u')) \det(C\lambda + D)^{-1} d\bar{u}'
\]

\[= \det I(\lambda') \det(C\lambda + D) f'(I(\lambda') u') \exp(-\frac{1}{2} h_{\lambda'}(u', u')) d\bar{u}'.
\]

But, (II.11) implies that

\[J(\lambda', f')(u') = \det I(\lambda') f'(I(\lambda') u') \exp(-\frac{1}{2} h_{\lambda'}(u', u')) d\bar{u}'
\]

Comparing the above expressions, we conclude that

\[\det(C\lambda + D) f \circ t(C\lambda + D) = f'.
\]

The group $U(p, q)$ acts on the base $M$ and the vector bundle $E_m$, therefore it acts on sections of $E_m$. Let $\mathcal{C}^\infty(M, E_m)$ denote the space of smooth sections of $E_m$, and $\mathcal{O}(M, E_m)$ the space of holomorphic sections of $E_m$. Since the group acts by holomorphic transformations, it preserves $\mathcal{O}(M, E_m)$ as a subspace of $\mathcal{C}^\infty(M, E_m)$. The natural action of $G \in U(p, q)$ on $s \in \mathcal{C}^\infty(M, E_m)$ is given by

\[\rho(G) s](A) = Z(G)(s(G^{-1} . A)).
\]

In order to express $\rho(G)$ explicitly, we decompose $G^{-1}$ into block form, $G^{-1} = (\begin{array}{cc} A & B \\ C & D \end{array})$, so that

\[(\rho(G) s)(\lambda)(u) = [s(G^{-1} . \lambda)](\pi(G^{-1}) u)
\]

\[= [s((A\lambda + B)(C\lambda + D)^{-1})][(C\lambda + D) u]. \quad (11.12)
\]

In closing, we record the result of Proposition 1.1 as it applies to a positive $p$-plane $V = V(\lambda)$. Since a $(0, p)$-form on $V$ is top-dimensional, $\ker \partial_p = L^0_{2,p}(V; h_{\nu})$. If $\omega \in L^0_{2,p}(V; h_{\nu})$ is given by $\omega(u) = \phi(u) d\bar{u}$, then the orthogonal projection

\[P_{V} : L^0_{2,p}(V; h_{\nu}) \rightarrow H^0_{0,p}(V; h_{\nu})
\]
is given as in Proposition 1.1 by

\[(P_v \omega)(u) = d\bar{u} \int_{C_\rho} \phi(v) K_\lambda(u, v) \det I(\lambda) \, dm(v), \quad (II.13)\]

where \(K_\lambda(u, v) = \exp(-\frac{1}{2} h_\lambda(u - v, u)).\)

**Corollary 2.3.** If \(\lambda \in \mathbb{D}_{p,q}, \; G = (A C \; B) \in U(p, q), \) and if \(\lambda' = G \cdot \lambda = (A\lambda + B)(C\lambda + D)^{-1},\) then for any \(u, v \in \mathbb{C}^p,\)

\[K_\lambda((C\lambda + D)u, (C\lambda + D)v) = K_\lambda(u, v).\]

**III. The Construction of \(\Phi\)**

This section contains the construction of the transform \(\Phi\) on \(\mathcal{R}_m^{0,p}(W)\). Given a choice of a positive definite hermitian form \(g\) on \(W\), and an element \(\omega \in H_\omega^{0,p}(W; g)\), we define \(\Phi_\omega\) by

\[\Phi_\omega(V) = P_v(\omega \, | \, V)\]

for any positive \(p\)-plane \(V \in M\). This requires the preliminary verification that \(\omega \, | \, V \in L_2^{0,p}(V; h \, | \, V)\), so that the integral \(P_v(\omega \, | \, V)\) converges. We easily see that \(\Phi_\omega\) depends smoothly on \(V\) and that \(\Phi_\omega(V) \in H_\omega^{0,p}(V; h \, | \, V)\) if \(\omega \in H_\omega^{0,p}(W; g)\)—i.e., \(\Phi\) preserves homogeneity degree. Therefore, (II.5) implies that \(\Phi_\omega = 0\) if \(\omega \in \bigoplus_{m \leq 0} H_\omega^{0,p}(W; g)\). If \(m \geq 0\), \(\Phi\) maps \(H_\omega^{0,p}(W; g)\) into \(\mathcal{C}_\infty(M, E_m)\), the space of smooth sections of \(E_m\). We next show that \(\Phi\) is independent of the choice of \(g\) and is therefore well defined on \(\mathcal{R}_m^{0,p}(W)\).

This amounts to showing that, for any two choices \(g\) and \(g'\) of positive definite hermitian forms on \(W\), the following diagram commutes:

\[
\begin{array}{ccc}
\ker \tilde{\delta} & \xrightarrow{P_v} & H_\omega^{0,p}(W; g) \\
\downarrow & & \downarrow P_v \\
H_\omega^{0,p}(W; g') & \xrightarrow{\phi} & \mathcal{C}_\infty(M, E_m)
\end{array}
\]

We hereby accomplish the dual purpose of showing that the mapping \(\Phi\) intertwines the respective actions of \(U(p, q)\) on \(\mathcal{R}_m^{0,p}(W)\) and \(\mathcal{C}_\infty(M, E_m)\).
We remark that the image functions $\Phi_\omega$ actually depend holomorphically on $V \in M$. We could thus write

$$\Phi: \mathcal{H}_m^{0,\rho}(W) \to \mathcal{O}(M, E_m).$$

The proof of this statement will, however, be delayed until the following section.

We now proceed to the first step in the definition of $\Phi$, the proof that $\omega \mid V \in L^0_{\mathcal{F}}(V; h \mid V)$ for any $\omega \in H^{0,\rho}(W; g)$ and any $V \in M$. Let $\omega \in H^{0,\rho}(W; g)$ be given as in (1.5) by $\omega(z) = f(z) \exp(-\frac{1}{2} h(z, z)) \overline{dz}$, where $f \in \mathcal{F}(W; g)$. Suppose $V$ is determined by the columns of $A = (v_i)$ with $\lambda \in \mathcal{D}_{p,q}$. Since we identify $u \in C^p$ with $Au \in V \subset W$, the restriction $\omega \mid V$ is given by

$$(\omega \mid V)(u) = \omega(\lambda u, u) = f(\lambda u, u) \exp(-\frac{1}{2} h(\lambda u, u)) du,$$

where we recall from (11.6) that $h(Au, Au) = *u * u = h(\lambda u, u)$. By (1.3), $\omega \mid V \in L^0_{\mathcal{F}}(V; h_V)$ if and only if

$$\int_{C^p} |f(\lambda u, u)|^2 \det I(\lambda) \, dm(u) < \infty.$$

We may ignore the constant $\det I(\lambda)$. We introduce the notation $(R_\lambda f)(u) = (f \mid V(\lambda))(u)$. It now remains to prove the following.

**Proposition 3.1.** For all $f \in \mathcal{F}(W; g)$ and all $\lambda \in \mathcal{D}_{p,q}$, $R_\lambda f = f \mid V(\lambda) \in L^2(C^p)$, i.e.,

$$\int_{C^p} |f(\lambda u, u)|^2 \, dm(u) < \infty.$$

We present the following lemmas before beginning the proof of Proposition 3.1.

**Lemma 3.1.** For every $\lambda \in \mathcal{D}_{p,q}$, there exist matrices $A \in U(q)$ and $B \in U(p)$ such that $v = (v_i)$, $\lambda = A\lambda B^{-1}$ is real and diagonal. Specifically, $v_{ij} = 0$ if $i \neq j$ and $v_{ii} \geq 0$ for all $i$.

**Proof:** This is a consequence of the Cartan decomposition $U(p, q) = KA^+K$, but it also follows from an easy calculation in linear algebra.

**Lemma 3.2.** It suffices to consider only real, diagonal $\lambda \in \mathcal{D}_{p,q}$.

**Proof:** Let $f \in \mathcal{F}(W; g)$ and $\lambda \in \mathcal{D}_{p,q}$. Suppose $v = A\lambda B^{-1}$ is given as in Lemma 3.1. We obtain the restriction $R_\lambda f$ of $f$ to $V(\lambda)$ by applying to $f$ the
left translation by \((\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \in U(p, q)\) followed by the restriction \(R_v\) to \(V(v)\) followed by the left translation by \(B^{-1} \in U(p)\). Since left translation by a nonsingular matrix preserves square-integrability, we conclude that \(R_v f \in L^2(C^p)\) whenever \(R_v f \in L^2(C^p)\).

The following integral formula will also be useful:

\[
\int_{\mathbb{R}} \exp(-at^2 + 2\beta t) \, dt = \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left(\frac{B^2}{\alpha}\right). \tag{III.1}
\]

**Proof of Proposition 3.1.** Let \(f \in \mathcal{F}(W; g)\) and let \(\lambda \in \mathbb{R}_{p,q}\) be real and diagonal. Bargmann’s isomorphism of \(\mathcal{F}(W; g)\) with \(L^2(\mathbb{R}^n)\) (see [1]) shows that there exists \(\phi \in L^2(\mathbb{R}^n)\) such that

\[
f(z) = \exp \left( \left(\frac{i}{2}\right) \left[ (\text{Im} z_s) z_s - (\text{Im} z_T) z_T^* \right] \right) \\
\times \int_{\mathbb{R}^p} \int_{\mathbb{R}^q} \exp(-i[\xi x + \xi_T y]) \\
\times \exp \left( -\frac{1}{2} \left[ |x|^2 + |y|^2 \right] \right) \phi(x, y) \, dx \, dy,
\]

where \(x \in \mathbb{R}^q, y \in \mathbb{R}^p\).

Accordingly, if \(u = \xi + i\eta \in C^p, \xi, \eta \in \mathbb{R}^p\), then

\[
|(R_\lambda f)(\xi + i\eta)| = |f(\lambda \xi + i\lambda \eta, \xi + i\eta)| \\
= \left| \exp \left( -\frac{1}{2} \eta(I_p + \lambda \lambda^T) \right) \exp \left( \frac{i}{2} \eta(\lambda \lambda - I_p) \xi \right) \right| \\
\times \left| \int_{\mathbb{R}^p} \int_{\mathbb{R}^q} \exp(-i[\xi x + \lambda x]) + \eta(\lambda x - y) \\
- \frac{1}{2} \left[ |x|^2 + |y|^2 \right] \phi(x, y) \, dx \, dy \right| \\
= \exp \left( -\frac{1}{2} \eta(I_p + \lambda \lambda^T) \right) \\
\times \left| \int_{\mathbb{R}^p} \int_{\mathbb{R}^q} \exp(-i[\xi y + \eta(\lambda x - y)) \\
- \frac{1}{2} \left[ |x|^2 + |y - \lambda x|^2 \right] \phi(x, y - \lambda x) \, dx \, dy \right|,
\]
after replacing \( y \) by \( y - \lambda x \). We wish to evaluate

\[
\int_{\mathbb{R}^2} \left| R_{\lambda} f(\xi + i\eta) \right|^2 \frac{d\xi d\eta}{(2\pi)^p}.
\]

Since \( \lambda \) is diagonal, the exponentials break up into a product in such a way that the integral can be iterated in any order. Accordingly, the general result follows from repeated application of the case \( p = q = 1 \). Henceforth, let \( p = q = 1 \). If we set

\[
F_n(y) = \exp(-\eta y - \frac{1}{2} y^2) \int_{\mathbb{R}} \exp(x\lambda(2\eta + y) - \frac{1}{2} x^2(1 + \lambda^2)) \phi(x, y - \lambda x)\,dx
\]

then we see that

\[
\int_{\mathbb{R}^2} \left| R_{\lambda} f(\xi + i\eta) \right|^2 \frac{d\xi d\eta}{2\pi}
= \int_{\mathbb{R}} \exp(-\eta^2(1 + \lambda^2)) \int_{\mathbb{R}} \left| \hat{F}_n(\xi) \right|^2 d\xi d\eta
= \int_{\mathbb{R}} \exp(-\eta^2(1 + \lambda^2)) \int_{\mathbb{R}} |F_n(y)|^2 dy d\eta
= \int_{\mathbb{R}^2} \exp(-\eta^2(1 + \lambda^2) - 2\eta y - y^2)
\times \left( \int_{\mathbb{R}} \exp(x\lambda(2\eta + y) - \frac{1}{2} x^2(1 + \lambda^2)) \phi(x, y - \lambda x)\,dx \right)^2 dy d\eta,
\]

by the Plancherel Theorem. We now use the Cauchy–Schwarz Inequality to compute that

\[
\int_{\mathbb{R}^2} \left| R_{\lambda} f(\xi + i\eta) \right|^2 \frac{d\xi d\eta}{2\pi}
\leq \int_{\mathbb{R}^2} \exp(-\eta^2(1 + \lambda^2) - 2\eta y - y^2) \left( \int_{\mathbb{R}} \exp(2x\lambda(2\eta + y) - x^2(1 + \lambda^2))\,dx \right)
\times \left( \int_{\mathbb{R}} |\phi(x, y - \lambda x)|^2\,dx \right) dy d\eta.
\]
Integral formula (III.1) allows us to evaluate the first integral in \(x\), therefore

\[
\int_{\mathbb{R}^2} |R_A f(\xi + i\eta)|^2 \frac{d\xi d\eta}{2\pi} \leq \int_{\mathbb{R}^2} \exp(-\eta^2(1 + \lambda^2) - 2\eta y - y^2) \frac{\sqrt{\pi}}{\sqrt{1 + \lambda^2}} \exp \left( \frac{\lambda^2(2\eta + y)^2}{1 + \lambda^2} \right) \times \left( \int_{\mathbb{R}^2} |\phi(x, y - \lambda x)|^2 \, dx \right) \, dy \, d\eta
\]

\[
= \frac{\sqrt{\pi}}{\sqrt{1 + \lambda^2}} \int_{\mathbb{R}^2} \exp \left( \frac{-1}{(1 + \lambda^2)} (|\eta(1 - \lambda^2) - y|^2) \right) \times \left( \int_{\mathbb{R}^2} |\phi(x, y - \lambda x)|^2 \, dx \right) \, dy \, d\eta.
\]

We now replace \(\eta\) by \((\eta + y)/(1 - \lambda^2)\) and use formula (III.1) again to compute that

\[
\int_{\mathbb{R}^2} |R_A f(\xi + i\eta)|^2 \frac{d\xi d\eta}{2\pi} \leq (1 - \lambda^2) \int_{\mathbb{R}^2} \exp \left( \frac{-1}{1 + \lambda^2} \eta^2 \right) \eta \int_{\mathbb{R}^2} |\phi(x, y - \lambda x)|^2 \, dx \, dy
\]

\[
= \frac{\pi}{1 - \lambda^2} \int_{\mathbb{R}^2} |\phi(x, y - \lambda x)|^2 \, dx \, dy
\]

\[
= \frac{\pi}{1 - \lambda^2} \int_{\mathbb{R}^2} |\phi(x, y)|^2 \, dx \, dy < \infty.
\]

For any \(\omega \in H^{0,p}(W; g)\) and for any positive \(p\)-plane \(V \in M\), Proposition 3.1 shows that \(\omega | V \in L^{0,p}_2(V; h | V)\), so the projection

\[P_V(\omega | V) \in H^{0,p}(V; h | V)\]

is well defined. If \(\omega \in H^{0,p}(W; g)\) is given as in (I.3) by \(\omega(z) = \phi(z) \overline{dz_T}\), and if \(V = V(\lambda)\) with \(\lambda \in \mathcal{D}_{p,q}\), then \(\omega | V(\lambda) = \phi(\lambda v, v) \overline{d\bar{v}}\). Therefore, by (II.13),

\[P_V(\omega | V)(u) = d\bar{u} \int_{\mathcal{C}_\nu} \phi(\lambda v, v) K_A(u, v) \det I(\lambda) \, dm(v),\]

where \(K_A(u, v) = \exp(-\frac{1}{4} h_A(u - v, u))\).

**Lemma 3.3.** If \(\omega \in H^{0,p}_m(W; g)\), then \(P_V(\omega | V) \in H^{0,p}_m(V; h | V)\).
Proof. We consider the expression for $P_v(\omega \mid V)(e^{i\theta}u)$ using (III.2). After replacing $v$ in the integral by $e^{i\theta}v$ and using definition (I.8), it follows that

$$P_v(\omega \mid V)(e^{i\theta}u) = e^{-i(m+p)\theta}P_v(\omega \mid V)(u).$$

We may now define the transform $\Phi$ on $H_{m,0}^{\rho}(W; g)$ by defining

$$\Phi(\omega)(V) = P_v(\omega \mid V)$$

for any $\omega \in H_{m,0}^{\rho}(W; g)$ and any $V \in M$.

Lemma 3.4. If $\omega \in H_{m,0}^{\rho}(W; g)$ and $V \in M$, then $\Phi(\omega)(V) \in H_{m,0}^{\rho}(V; h \mid V)$ and $\Phi$ depends smoothly on $V \in M$. Therefore, if $m \geq 0$, we may write

$$\Phi: H_{m,0}^{\rho}(W; g) \rightarrow \mathcal{C}^\infty(M, E_m)$$

and if $m < 0$, $\Phi(\omega) = 0$.

Proof. Lemma 3.3 shows that $\Phi(\omega)(V) \in H_{m,0}^{\rho}(V; h \mid V)$. If $m < 0$, we know from (II.5) that $H_{m,0}^{\rho}(V; h \mid V) = \{0\}$, therefore $\Phi(\omega)(V) = 0$. If $m \geq 0$, $\Phi(\omega)$ is a section of the vector bundle $E_m$ over $M$. If $V = V(\lambda)$ with $\lambda \in \mathcal{O}_{B,\omega}$, we may use (III.2) and (II.6) to express $\Phi(\omega)$ as

$$\Phi(\omega)(V)(u) = d\lambda \int_{C^p} \phi(t_u, \nu) \exp\left(-\frac{1}{2} uI(\lambda)(u - \nu)\right) dm(\nu),$$

where $I(\lambda) = I_\rho - ^*\lambda \lambda$. Since the integral converges absolutely, the dependence on $\lambda$ is clearly smooth.

We now wish to show that $\Phi$ is independent of $g$ and is therefore well defined on $H_{m,0}^{\rho}(W; g)$. This is equivalent to showing that the following diagram commutes, for two choices $g$ and $g'$ of a positive definite hermitian form on $W$:

Here $\iota: H_{m,0}^{\rho}(W; g) \rightarrow \ker \tilde{\partial}$ denotes the natural injection. The top half of the diagram commutes. Therefore, it suffices to show that the bottom half of the diagram commutes.
PROPOSITION 3.2. The following diagram commutes:

\[
\begin{array}{ccc}
H^0_m(W; g) & \overset{P^*_{z-p}}{\longrightarrow} & H^0_m(W; g') \\
\phi \downarrow & & \phi \\
\mathcal{C}^\infty(M, E_m) & \overset{\iota}{\longrightarrow} & \mathcal{C}^\infty(M, E_m)
\end{array}
\]

Therefore the mapping \(\Phi\) is independent of \(g\).

Proof. We first remark that the descriptions of the spaces \(H^0_m(W; g)\) and \(H^0_m(W; g')\) relied on the choice of coordinates. Suppose \(z\) denotes the preferred coordinate on \(W\) for \(h\) and \(g\), and \(z'\) the preferred coordinate for \(h\) and \(g'\). In order to apply the projection \(P_z\) to a differential form \(\omega\), \(\omega\) must be expressed in the coordinate \(z'\). Thus the map \(\iota\) will be interpreted as the change of coordinates from \(z\) to \(z'\). This change of coordinates \(z' = Gz\) is given by a matrix \(G\) which preserves \(h\); hence, \(G \in U(p, q)\). Let \(\omega \in H^0_m(W; g)\) be given by \(\omega(z) = \phi(z) \, dz\) and let \(G^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\). As in (I.6), \(\iota\) is given by

\[\iota(\omega)(z') = \omega(G^{-1}z')\]

Similarly, the map \(\iota\) must be interpreted as a change of coordinates given by the action of \(G \in U(p, q)\) on \(s \in \mathcal{C}^\infty(M, E_m)\). As in (II.12), \(\iota\) is given by

\[\iota(s)(\lambda)(u) = [s((A\lambda + B)(C\lambda + D)^{-1})][(C\lambda + D)u].\] (III.4)

As in (III.2), \(\Phi_\omega(V(\lambda))(u) = \Phi_\omega(\lambda, u)\) is given by

\[\Phi_\omega(\lambda, u) = d\bar{u} \int_{C_p} \phi(\lambda v, v) K_\lambda(u, v) \det I(\lambda) \, dm(v),\] (III.5)

where \(K_\lambda(u, v) = \exp(-\frac{1}{2} h_\lambda(u - v, u))\). Thus, using (III.4) and writing \(\lambda' = (A\lambda + B)(C\lambda + D)^{-1}\), we see that

\[\iota(\Phi_\omega)(\lambda, u) = \Phi_\omega(\lambda', (C\lambda + D)u)\]

\[= \det((C\lambda + D)) d\bar{u} \int_{C_p} \phi(\lambda' v, v) \times K_\lambda((C\lambda + D)u, v) \det I(\lambda') \, dm(v) \times K_\lambda(u, v) \det I(\lambda) \, dm(v),\] (III.6)

by Corollaries 2.2 and 2.3, after replacing \(v\) by \((C\lambda + D)v\) in the integral.
For the same $\omega$, we now compute $(\Phi \circ P_{\omega} \circ \iota) \omega$. Using (III.3) and Proposition 1.1, we see that

$$(\Phi \circ P_{\omega} \circ \iota) \omega(z') = \det D z' \int_{C^*} \phi(G^{-1}w) K(z', w) \, dm(w),$$

where

$$K(z', w) = \exp(-\frac{1}{2}[*z'_r(z'_r - w_r) - *w_s(z'_s - w_s)]).$$

Therefore, using (III.5),

$$(\Phi \circ P_{\omega} \circ \iota) (\lambda, u) = \det D u \int_{C^*} \phi(G^{-1}w) \int_{C^*} K((\lambda, v), w)$$

$$\times K_1(u, v) \det I(\lambda) \, dm(\lambda) \, dm(w). \quad (III.7)$$

By Lemma 1.2 we compute that

$$\int_{C^*} K((\lambda, v), w) K_1(u, v) \, dm(v)$$

$$= \exp(-\frac{1}{2}[*uI(\lambda)(u - w_T) - *w_s(\lambda w_T - w_s)]).$$

Therefore, inserting the above into (III.7),

$$(\Phi \circ P_{\omega} \circ \iota) (\lambda, u) = \det D u \int_{C^*} \phi(Aw_s + Bw_T, Cw_s + Dw_T)$$

$$\times \exp(-\frac{1}{2}[*uI(\lambda)(u - w_T) - *w_s(\lambda w_T - w_s)])$$

$$\times \det I(\lambda) \, dm(w_s) \, dm(w_T).$$

We first replace $w_s$ by $y + \lambda w_T$, so that

$$(\Phi \circ P_{\omega} \circ \iota) (\lambda, u)$$

$$= \det D u \int_{C^*} \phi(Ay + (A\lambda + B) w_T, Cy + (C\lambda + D) w_T)$$

$$\times \exp(-\frac{1}{2}[*uI(\lambda)(u - w_T) + *(y + \lambda w_T) y])$$

$$\times \det I(\lambda) \, dm(y) \, dm(w_T).$$

We now replace $w_T$ by $x - (C\lambda + D)^{-1}Cy$, consequently

$$(\Phi \circ P_{\omega} \circ \iota) (\lambda, u)$$

$$= \det D u \int_{C^*} \phi((A - \lambda C)y + (A\lambda + B) x, (C\lambda + D) x)$$

$$\times \exp(-\frac{1}{2}[*uI(\lambda)(u - x + (C\lambda + D)^{-1}(Cy)) + *(\lambda x) y])$$

$$\times \exp(-\frac{1}{2}[*(I_q - \lambda (C\lambda + D)^{-1} C) y]) \det I(\lambda) \, dm(y) \, dm(x).$$
By (I.3), \( \phi \) is holomorphic in \( z_8 \). Thus the integrand, except for the quadratic term in \( y \), is holomorphic in \( y \). Therefore, by Lemma 1.2,

\[
(\Phi \circ P_{g^{'}} \circ \iota)(\omega)(\lambda, u) = \frac{\det D}{\det(I_q - \lambda(C\lambda + D)^{-1}C)} \int_{\Omega} \phi((A\lambda + B)x, (C\lambda + D)x) \times \exp\left(-\frac{1}{4} |uI(\lambda)(u - x)| \right) \det I(\lambda) \, dm(x).
\]

We remark that for any \( q \times p \) matrix \( \alpha \) and any \( p \times q \) matrix \( \beta \),

\[
\det(I_q - \alpha\beta) = \det(I_p - \beta\alpha).
\]

Therefore,

\[
\frac{\det D}{\det(I_q - \lambda(C\lambda + D)^{-1}C)} = \det(C\lambda + D).
\]

The above identity implies that

\[
(\Phi \circ P_{g^{'}} \circ \iota)(\omega)(\lambda, u) = \det(C\lambda + D) \int_{\Omega} \phi((A\lambda + B)x, (C\lambda + D)x) \times K_\lambda(u, x) \det Z(A) \, dm(x).
\]

Comparing the above to (III.6), we conclude that

\[
\Phi \circ P_{g^{'}} \circ \iota = \iota \circ \Phi,
\]

and so the diagram commutes. \( \Box \)

**Corollary.** The following diagram commutes:

\[ H^0_{m,p}(W; g) \xrightarrow{\sigma(G) = P_{g^{'}} \circ \iota(G)} H^0_{m,p}(W; g) \]

\[
\Phi \downarrow \quad \Phi
\]

\[ \mathcal{H}^\infty(M, E_m) \xrightarrow{\rho(G)} \mathcal{H}^\infty(M, E_m) \]

and therefore the map \( \Phi \) intertwines the actions of \( U(p, q) \).

**Proof:** The integral formulas involved are just those of the Proposition, suitably reinterpreted. \( \Box \)

The results of this section may be summarized as follows.

**Theorem 3.1.** There exists a well-defined mapping

\[
\Phi: P^0_{m,p}(W) \rightarrow \mathcal{H}^\infty(M, E_m),
\]
for all \( m \geq 0 \), which intertwines the natural actions of \( U(p, q) \) on \( \mathcal{H}_m^{0, p}(W) \) and \( \mathcal{F}_\infty(M, E_m) \). The mapping \( \Phi \) is defined as follows. Let \( \tilde{\omega} \in \mathcal{H}_m^{0, p}(W) \) and let \( \omega \) be a differential form which represents the cohomology class \( \tilde{\omega} \). Let \( V \in M \). The section \( \Phi_{\tilde{\omega}} \) of \( E_m \) then assigns to \( V \) the representative of the cohomology class of \( \omega \mid V \) in \( H_m^{0, p}(V; h \mid V) \).

IV. PROPERTIES OF \( \Phi \)

This section contains the discussion of several properties of the transform \( \Phi \). We first demonstrate that the image functions \( \Phi_{\omega} \) are holomorphic, for each \( \omega \in H_m^{0, p}(W; g) \), \( m \geq 0 \). We next show that the transform \( \Phi \) is injective on \( H_m^{0, p}(W; g) \) for any \( m \geq 0 \). Finally, we discuss the differential equations which are satisfied by the functions in the image of \( \Phi \). If \( p \geq 2 \) and \( q \geq 2 \), then there is a family of second-order differential operators which annihilate each component of \( \Phi_{\omega} \), for any \( \omega \in \oplus_{m \geq 0} H_m^{0, p}(W; g) \). The wave operator is of this form, for \( p = q = 2 \). If \( p \geq 2 \), \( q \geq 1 \), and \( m \geq 1 \), then there exist systems of differential equations satisfied by \( \Phi_{\omega} \), for any \( \omega \in H_m^{0, p}(W; g) \). When \( p = q = 2 \), the Dirac–Weyl equations of a neutrino and Maxwell's equations are of this form, for \( m = 1 \) and \( m = 2 \), respectively.

The trivialization \( J: \mathcal{D}_{p, q} \times \mathcal{P}(m, \mathbb{C}^p) \rightarrow E_m \) was given in Proposition 2.1 by

\[
J(\lambda, f)(u) = \det I(\lambda) f(I(\lambda) u) \exp(-\frac{1}{2} h_A(u, u)) \, du,
\]

where \( \lambda \in \mathcal{D}_{p, q} \), \( f \in \mathcal{P}(m, \mathbb{C}^p) \), and \( I(\lambda) = I_p - * \lambda \lambda \) is the matrix of \( h_A \) on \( V(\lambda) \approx \mathbb{C}^p \).

**Lemma 4.1.** Let \( \omega \in H_m^{0, p}(W; g) \) be given as in (1.3) by \( \omega(z) = \phi(z) \, dz \), and let \( \lambda \in \mathcal{D}_{p, q} \). Define the function \( \psi_\lambda \) on \( \mathbb{C}^p \) by

\[
\psi_\lambda(s) = \int_{\mathbb{C}^p} \phi(\lambda v, v) \exp\left(\frac{i}{2} t v s\right) \, dm(v).
\]

Then, the section \( \Phi_{\omega} \) of \( E_m \) over \( M \) is given in the trivialization of \( E_m \) by the map \( \lambda \mapsto \psi_\lambda \), i.e.,

\[
\Phi_{\omega}(\lambda) = J(\lambda, \psi_\lambda)
\]

for all \( \lambda \in \mathcal{D}_{p, q} \).
Proof: Recall from (111.5) that
\[
\Phi_\omega(\lambda, u) = \det I(\lambda) \int_{\mathbb{C}^p} \phi(\lambda v, v) \exp\left(-\frac{1}{2} h_\lambda(u - v, u)\right) dm(v)
\]
\[
- \det I(\lambda) \exp\left(-\frac{1}{2} h_\lambda(u, u)\right) d\bar{u}
\]
\[
\times \int_{\mathbb{C}^p} \phi(\lambda v, v) \exp\left(\frac{1}{2} v \bar{I}(\lambda) v\right) dm(v).
\]
Comparing this to \(J(\lambda, \psi_\lambda)(u)\), using (IV.1) and (IV.2), we see that they are equal.  

**Theorem 4.1.** If \(\omega \in H^{0,p}_m(W; g)\) with \(m \geq 0\), then \(\Phi_\omega\) depends holomorphically on \(\lambda \in \mathbb{D}_{p,q}\). Hence, we may write
\[
\Phi: H^{0,p}_m(W; g) \to \mathcal{O}(M, E_m).
\]

**Proof.** Let \(\omega \in H^{0,p}_m(W; g)\) be given as in (I.3) by \(\omega(z) = \phi(z) \overline{dz_T}\). We use the trivialization (IV.1) of \(E_m\) to write \(\Phi_\omega(\lambda) = J(\lambda, \psi_\lambda)\), with \(\psi_\lambda\) given as in (IV.2) by
\[
\psi_\lambda(s) = \int_{\mathbb{C}^p} \phi(\lambda v, v) \exp\left(\frac{1}{2} vs\right) dm(v).
\]
We know from (I.3) that \(\phi(z)\) is holomorphic in \(z_s\), hence \(\phi(\lambda v, v)\) is holomorphic in \(\lambda\). Since the above integral converges absolutely, we conclude that \(\psi_\lambda\) depends holomorphically on \(\lambda\), hence \(\Phi_\omega \in \mathcal{O}(M, E_m)\).  

In order to demonstrate the injectivity of \(\Phi\) or the existence of the differential equations, we require some additional notation. Let \(\mathbb{N}\) denote the nonnegative integers, and let \(\alpha \in \mathbb{N}^r\), \(s \in \mathbb{C}^r\). We define \(|\alpha| = \alpha_1 + \cdots + \alpha_r\), \(\alpha! = \alpha_1! \cdots \alpha_r!\), \(s^\alpha = s_1^{\alpha_1} \cdots s_r^{\alpha_r}\), we analogously define \(s^\alpha\), and we define \((\partial/\partial s)^\alpha\) by
\[
\left(\frac{\partial}{\partial s}\right)^\alpha = \left(\frac{\partial}{\partial s_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial s_r}\right)^{\alpha_r}
\]
\[
= \frac{\partial^{|\alpha|}}{\partial s_1^{\alpha_1} \cdots \partial s_r^{\alpha_r}}.
\]

**Lemma 4.2.** Let \(\delta \in \mathbb{N}^p\) and let \(\psi_\lambda(s)\) be given by
\[
\psi_\lambda(s) = \int_{\mathbb{C}^p} \phi(\lambda v, v) \exp\left(\frac{1}{2} vs\right) dm(v).
\]
Then
\[
\left( \frac{\partial}{\partial s_j} \right)^s \psi_\lambda(s) = 2^{-|s|} \int_{C^p} v^s \phi(\lambda v, v) \exp(\frac{1}{2} 'uv) \, dm(v).
\]

Proof:
\[
\frac{\partial \psi_\lambda}{\partial s_j}(s) = \int_{C^p} \phi(\lambda v, v) \frac{\partial}{\partial s_j} \exp \left( \frac{1}{2} \{ v, s_1 + \cdots + v, s_p \} \right) \, dm(v)
\]
\[
= \int_{C^p} \frac{v_j}{2} \phi(\lambda v, v) \exp \left( \frac{1}{2} 'uv \right) \, dm(v).
\]

The statement of the lemma follows by applying this equality repeatedly. Since the above integrals converge absolutely, interchanging the order of integration and differentiation was justified.

We now turn to the proof of the injectivity of \( \Phi \).

**Theorem 4.2.** (a) The mapping \( \Phi: H^{0,\rho}_m(W; g) \rightarrow \mathcal{C}(M, E_m) \) is injective, for any \( m \geq 0 \).

(b) If \( r > 0 \), \( \Phi \) is identically zero on \( H^{0,\rho}_m(W; g) \).

Proof: We recall that part (b) was proved in Lemma 3.4 in the preceding section. In order to prove part (a), suppose \( \omega \in H^{0,\rho}_m(W; g) \) is given as in (1.4) by
\[
\omega(z) = f(z_S, z_T) \exp(-\frac{1}{2} |z_T|^2) \, dz_T,
\]
where \( f \) is holomorphic and satisfies
\[
f(e^{i\theta} z_S, e^{i\theta} z_T) = e^{-i m \theta} f(z_S, z_T).
\]

We may thus expand \( f \) in a power series of the form
\[
f(z_S, z_T) = \sum_{|\alpha| \leq N} \sum_{|\beta| = |\alpha| + m} c_{\alpha, \beta} z_S^\alpha \overline{z_T}^\beta. \tag{IV.3}
\]

Suppose now that \( 0 = \Phi_\omega(\lambda) = J(\lambda, \psi_\lambda) \), so that
\[
0 = \psi_\lambda(s) = \int_{C^p} f(\lambda v, \bar{v}) \exp(-\frac{1}{2} |v|^2) \exp(\frac{1}{2} 'uv) \, dm(v) \tag{IV.4}
\]
for all \( \lambda \in \mathbb{D}_{p,q} \) and all \( s \in C^p \). We wish to conclude that \( \omega = 0 \) by showing
that all the coefficients $c_{\alpha,\beta}$ for $f$ are zero. After substituting $f(\lambda v, \bar{v})$ into (IV.3), we see that

$$f(\lambda v, \bar{v}) = \sum_{\alpha \in \mathbb{N}^q} \sum_{\beta \in \mathbb{N}^p} c_{\alpha,\beta}(\lambda v)^\alpha \bar{v}^\beta. \quad (IV.5)$$

We expand and rearrange (IV.5) to write it as a power series in $\lambda$, which we then insert into the integral (IV.4). The resulting expression is the zero power series in $\lambda$, hence all of its coefficients are zero, i.e.,

$$0 = \sum_{\beta \in \mathbb{N}^p} c_{\alpha,\beta} \int_{C^p} v^{\beta} \exp(-\frac{1}{2} v(\bar{v} - s)) \, dm(v), \quad (IV.6)$$

for all $\alpha \in \mathbb{N}^q$, for all $\epsilon \in \mathbb{N}^p$ such that $|\epsilon| = |\alpha|$, and for all $s \in \mathbb{C}^p$. (We omit the combinatorial argument leading from (IV.5) and (IV.4) to (IV.6).) If $s = 0$, the integration Lemma 1.1 allows us to evaluate such an integral. In particular, it will be nonzero if and only if $\epsilon = \beta$, which need not occur.

Choose a particular coefficient $c_{\alpha,\gamma}$ with $\alpha \in \mathbb{N}^q$, $\gamma \in \mathbb{N}^p$, and $|\gamma| = |\alpha| + m$. We wish to show that $c_{\alpha,\gamma} = 0$. Since $\epsilon \in \mathbb{N}^p$ satisfies $|\epsilon| = |\alpha| \leq |\gamma|$, we may choose $\epsilon$ so that $\epsilon_j \leq \gamma_j$ for all $j$, $1 \leq j \leq p$. We may then define $\delta \in \mathbb{N}^p$ by $\delta = \gamma - \epsilon$. The right side of (IV.6) is the zero function of $s \in \mathbb{C}^p$; this remains true after applying to it the differential operator $(\partial/\partial s)^\delta$. By Lemma 4.2, differentiating both sides of (IV.6) by $(\partial/\partial s)^\delta$ yields the equality

$$0 = \sum_{\beta \in \mathbb{N}^p} c_{\alpha,\beta} \int_{C^p} v^{\beta + \delta} \exp(-\frac{1}{2} v(\bar{v} - s)) \, dm(v). \quad (IV.7)$$

Setting $s = 0$ in (IV.7) and using Lemma 1.1 to evaluate the integral, we conclude that

$$0 = \sum_{\beta \in \mathbb{N}^p} c_{\alpha,\beta} \int_{C^p} v^{\beta} \exp(-\frac{1}{2} |v|^2) \, dm(v)$$

$$= c_{\alpha,\gamma}(2^{|\gamma|} \gamma!),$$

hence $c_{\alpha,\gamma} = 0$.

We now consider the differential equations which are satisfied by
functions in the image of $\Phi$. Let $\partial/\partial \lambda$ denote the $q \times p$ matrix of differential operators

$$
\frac{\partial}{\partial \lambda} = \begin{pmatrix}
\frac{\partial}{\partial \lambda_{11}} & \cdots & \frac{\partial}{\partial \lambda_{1p}} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial \lambda_{q1}} & \cdots & \frac{\partial}{\partial \lambda_{qp}}
\end{pmatrix}.
$$

**Lemma 4.3.** If $\psi_\lambda$ is given as in (IV.2) by

$$
\psi_\lambda(s) = \int_{C^p} \phi(\lambda v, v) \exp(\frac{1}{2} \lambda s) \, dm(v),
$$

then

$$
\frac{\partial \psi_\lambda}{\partial \lambda_{ij}}(s) = \int_{C^p} v_j \frac{\partial \phi}{\partial z_i} (\lambda v, v) \exp(\frac{1}{2} \lambda s) \, dm(v).
$$

**Proof.**

$$
\frac{\partial \psi_\lambda}{\partial \lambda_{ij}}(s) = \int_{C^p} \frac{\partial}{\partial \lambda_{ij}} \phi(\lambda_{11} v_1 + \cdots + \lambda_{1p} v_p, \ldots, \lambda_{q1} v_1 \\
+ \cdots + \lambda_{qp} v_p, v_1, \ldots, v_p) \exp(\frac{1}{2} \lambda s) \, dm(v)
$$

$$
= \int_{C^p} v_j \frac{\partial \phi}{\partial z_i} (\lambda v, v) \exp(\frac{1}{2} \lambda s) \, dm(v).
$$

Since the above integrals converge absolutely, interchanging the order of integration and differentiation was justified.

We now define a family of second-order differential operators, as follows. If $p \geq 2$ and $q \geq 2$, consider the $2 \times 2$ minor of $\partial/\partial \lambda$ given by the intersection of its $i$th row, $j$th column, $k$th row, and $l$th column. We denote by $D_{ijkl}$ its determinant,

$$
D_{ijkl} = \frac{\partial}{\partial \lambda_{ij}} \frac{\partial}{\partial \lambda_{kl}} - \frac{\partial}{\partial \lambda_{kl}} \frac{\partial}{\partial \lambda_{ij}},
$$

which is a nontrivial differential operator whenever $i \neq k$ and $j \neq l$. 
THEOREM 4.3. Let $p \geq 2$, $q \geq 2$, $m \geq 0$, and let $\omega \in H^0_m(W; g)$. Suppose that $\Phi_\omega(\lambda) = J(\lambda, \psi_\lambda)$, where $\psi_\lambda$ is given as in (IV.2). Then,

$$D_{ijkl}\psi_\lambda \equiv 0$$

whenever $1 \leq i, k \leq q$ and $1 \leq j, l \leq p$.

Proof. Lemma 4.3 allows us to compute that

$$D_{ijkl}\psi_\lambda(s) = \frac{\partial}{\partial \lambda_{ij}} \int_{C^p} v_i \frac{\partial \phi}{\partial z_k} (\lambda v, v) \exp\left(\frac{1}{2} \psi s\right) dm(v)$$

$$- \frac{\partial}{\partial \lambda_{kj}} \int_{C^p} v_i \frac{\partial \phi}{\partial z_i} (\lambda v, v) \exp\left(\frac{1}{2} \psi s\right) dm(v)$$

$$= \int_{C^p} v_j v_l \left( \frac{\partial^2 \phi}{\partial z_k \partial z_l} (\lambda v, v) - \frac{\partial^2 \phi}{\partial z_l \partial z_k} (\lambda v, v) \right)$$

$$\times \exp\left(\frac{1}{2} \psi s\right) dm(v)$$

$$= 0. \quad \blacksquare$$

As above, let $\omega = \phi d\zeta_T \in H^0_m(W; g)$, with $\psi_\lambda$ defined as in (IV.2), so that the section $\Phi_\omega$ is given in the trivialization of $E_m$ by the function $\lambda \mapsto \psi_\lambda$. Recall that $\psi_\lambda \in \mathcal{P}(m, C^p)$ is a homogeneous polynomial of degree $m$ on $C^p$, so $\psi_\lambda$ is a vector-valued function of $\lambda$ whenever $m \geq 1$. The systems of differential equations that we next describe involve the components of the vector $\psi_\lambda$, so we must choose a basis of the vector space $\mathcal{P}(m, C^p)$ in order to describe the components of $\psi_\lambda$.

A convenient basis of $\mathcal{P}(m, C^p)$ will be the set of monomials $\{\tau^\beta\}_{\beta \in \mathbb{N}^p}$, for any $s \in C^p$, where $\beta \in \mathbb{N}^p$ and $|\beta| = m$. The $\beta$th component of $\psi_\lambda$, denoted $\psi_\lambda^\beta$, is just the coefficient of $s^\beta / \beta!$ in the expansion of $\psi_\lambda(s)$. This coefficient can be found easily using the relation

$$\psi_\lambda^\beta = \left( \frac{\partial}{\partial s} \right)^\beta \psi_\lambda. \quad (IV.8)$$

We will represent $\psi_\lambda$ as a vector by listing its components as

$$\psi_\lambda = (\psi_\lambda^\beta)_{|\beta| = m}.$$

Finally, if $\gamma \in \mathbb{N}^p$, we introduce the notation $(\partial / \partial \lambda_i)^\gamma$ for the differential operator

$$\left( \frac{\partial}{\partial \lambda_i} \right)^\gamma \left( \frac{\partial}{\partial \lambda_{i1}} \right)^n \cdots \left( \frac{\partial}{\partial \lambda_{ip}} \right)^\eta.$$
THEOREM 4.4. Let $p \geq 2$, $q \geq 1$, and $m \geq 1$. Let $\omega \in H_{m,p}^{\infty}(W; g)$, $\omega(z) = \phi(z) \, dz$, and let $\psi_\lambda$ be given as in (IV.2) so that $\Phi_\omega(\lambda) = J(\lambda, \psi_\lambda)$. Let $\psi_\lambda = (\psi_\lambda^1)_{|_{\beta_1 = m}}$. If $\alpha, \beta \in \mathbb{N}^p$ are such that $|\alpha| = |\beta| = m$, then

$$\left( \frac{\partial}{\partial \lambda_i} \right)^\gamma \psi_\lambda^\alpha - \left( \frac{\partial}{\partial \lambda_i} \right)^\delta \psi_\lambda^\beta = 0, \quad 1 \leq i \leq q,$$

whenever $\gamma, \delta \in \mathbb{N}^p$ satisfy $\alpha + \gamma = \beta + \delta$.

Proof. First we note that

$$\psi_\lambda^\alpha = 2^{-m} \int_{\mathcal{C}_p} \psi^\beta(\lambda v, v) \exp(\frac{1}{2} i \langle s, v \rangle) \, dm(v),$$

by applying Lemma 4.2 to (IV.8). Therefore,

$$\left( \frac{\partial}{\partial \lambda_i} \right)^\gamma \psi_\lambda^\alpha = 2^{-m} \left( \frac{\partial}{\partial \lambda_i} \right)^\gamma \int_{\mathcal{C}_p} \psi^\beta(\lambda v, v) \exp(\frac{1}{2} i \langle s, v \rangle) \, dm(v).$$

Applying Lemma 4.3 repeatedly allows us to conclude that

$$\left( \frac{\partial}{\partial \lambda_i} \right)^\gamma \psi_\lambda^\alpha = 2^{-m} \int_{\mathcal{C}_p} \psi^\beta(\lambda v, v) \left( \frac{\partial}{\partial z_i} \right)^\gamma \phi(\lambda v, v) \exp(\frac{1}{2} i \langle s, v \rangle) \, dm(v).$$

Similarly,

$$\left( \frac{\partial}{\partial \lambda_i} \right)^\delta \psi_\lambda^\beta = 2^{-m} \int_{\mathcal{C}_p} \psi^\alpha \left( \frac{\partial}{\partial z_i} \right)^\delta \phi(\lambda v, v) \exp(\frac{1}{2} i \langle s, v \rangle) \, dm(v).$$

Since $\alpha + \gamma = \beta + \delta$ and $|\alpha| = |\beta|$, we conclude that $|\gamma| = |\delta|$, and therefore

$$\left( \frac{\partial}{\partial \lambda_i} \right)^\gamma \psi_\lambda^\alpha - \left( \frac{\partial}{\partial \lambda_i} \right)^\delta \psi_\lambda^\beta = 0.$$

Remark. We obtain first-order equations above if $|\gamma| = |\delta| = 1$. Define elements $e_1, \ldots, e_p \in \mathbb{N}^p$ so that $e_j$ has a 1 in the $j$th place, 0's elsewhere. Note that

$$\left( \frac{\partial}{\partial \lambda_i} \right)^e = \frac{\partial}{\partial \lambda_{ij}}.$$ 

If $\alpha \in \mathbb{N}^p$ satisfies $|\alpha| = m - 1$, the conclusion of Theorem 4.4 then states that

$$\frac{\partial}{\partial \lambda_{ij}} \psi_\lambda^\alpha + e_k - \frac{\partial}{\partial \lambda_{ik}} \psi_\lambda^\alpha + e_l = 0, \quad 1 \leq t \leq q.$$
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REFERENCES