JOURNAL OF FUNCTIONAL ANALYSIS **64**, 112–123 (1985)

The Cauchy–Green Formula and Rational Approximation on the Sets with a Finite Perimeter in the Complex Plane

D. KHAVINSON*

Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701

Communicated by the Editors

Received September 1984; revised December 18, 1984

Integral representations of Lipschitz functions on the sets with a finite perimeter in $\mathbb{C}$ are studied. These formulas can be viewed as generalizations of the classical Cauchy–Green theorem. Also, it is shown that those results lead to a convenient approach to certain problems in the theory of rational approximation.

1. INTRODUCTION

Let μ be a complex Baire measure in $\mathbb{C}$. The Cauchy transform $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{\zeta - z}.$$

Let m_i, $i = 1, 2$, denote the i-dimensional Hausdorff measure in $\mathbb{C}$. Let us introduce the following definition (cf. [3, 7, 14, 20]).

DEFINITION. A m_2-measurable set $E \subset \mathbb{C}$ is called a set of *finite perimeter* if there exists a measure μ such that

$$\hat{\mu}(z) = \chi_E \overset{\text{def}}{=} \begin{bmatrix} 1, & z \in E, \\ 0, & z \notin E, \end{bmatrix} \quad m_2\text{-a.e.}$$

Any finitely connected domain G with a rectifiable boundary is obviously a set of finite perimeter with $\mu \equiv (1/2\pi i)\, d\zeta\,|_{\partial G}$. Other less trivial examples due

* This work was supported in part by the National Science Foundation under Grant DMS-8400582.

0022-1236/85 $3.00

to Garnett in [10] are given by the sets of the so-called Swiss cheese type. One defines a Swiss cheese to be a compact set obtained by deleting from the closed unit disk Δ_0 a sequence of Δ_j of pairwise disjoint open disks whose radii have a finite sum and whose union is dense in Δ_0. We denote such a set by K. It is not hard to see (cf. [10, 20]), that K is a set of finite perimeter with the measure μ defined by

$$\mu|_{\partial\Delta_0}=\frac{1}{2\pi i}\,d\zeta, \qquad \mu|_{\partial\Delta_j}=-\frac{1}{2\pi i}\,d\zeta, \quad j=1,2,\ldots .$$

As was noted by Pietz in [20, 21] the results of Federer, De Giorgi, and Fleming [3, 4, 6–9] in geometric measure theory described below give a complete geometric characterization of the sets with a finite perimeter. To state the main theorem combining the results of Federer and De Giorgi and describing the geometrical structure of such sets we have to recall the concept of exterior normal due to Federer (see [6–8]).

DEFINITION. Let $E\subset\mathbb{C}$ be a measurable bounded set. We say that E has an *exterior normal* $n(x)$ at the point x if $|n(x)|=1$ and letting

$$\begin{aligned}
\Delta(x,r)&=\{y\in\mathbb{C}: |y-x|<r\},\\
\Delta_+(x,r)&=\{y\in\Delta(x,r): (\overrightarrow{y-x})\cdot\overrightarrow{n(x)}\geqslant 0\},\\
\Delta_-(x,r)&=\{y\in\Delta(x,r): (\overrightarrow{y-x})\cdot\overrightarrow{n(x)}\leqslant 0\},
\end{aligned}$$

we have

$$\lim_{r\to 0^+}\frac{m_2\{\Delta_-(x,r)\cap E\}}{m_2\{\Delta_-(x,r)\}}=1: \ \lim_{r\to 0^+}\frac{m_2\{\Delta_+(x,r)\cap E\}}{m_2\{\Delta_+(x,r)\}}=0.$$

Let $B_E=B(E)=\{x\in\mathbb{C}: E$ has an exterior normal $n(x)$ at $x\}$. Following Federer we shall call B_E the *reduced boundary* of E.

The following theorem is a "complex" version of the results due to Federer and De Giorgi (see [3, 8, 14, 20]).

THEOREM 1.1. *Let E be a set of finite perimeter in $\mathbb{C}$, i.e., $\exists\mu$: $\hat\mu(z)=\chi_E$ a.e. Let $P(E)=2\pi\,\|\mu\|$. Then, the following hold:*

(i) *B_E is m_1-measurable and $P(E)=m_1(B_E)$.*

(ii) *For any Borel set $A\subset\mathbb{C}$, $\mu(A)=\mu(A\cap B_E)=(1/2\pi)\int_{A\cap B_E} n(x)\cdot dm_1(x)$.*

(iii) $\lim_{r\to 0^+} m_1(B_E\cap\Delta(x,r))/2r=1$ *a.e. on B_E.*

(iv) *B_E except, maybe, a set of m_1-measure zero is contained in a countable union of rectifiable arcs and $d\mu=(1/2\pi i)\,d\zeta|_{B_E}$.*

We refer the reader to the papers [13, 14, 20, 21] for the further discussion concerning the sets of finite perimeter in $\mathbb{C}$.

The following theorem, combining the results of De Giorgi and Fleming (see [3, 9]) describes the sets of finite perimeter as "geometric limits" of finitely connected smoothly bounded regions.

THEOREM 1.2. *Let E be a set of finite perimeter and $\chi_E = \hat{\mu}(z)$ a.e.; then there exists a sequence $\{\Pi_n\}_1^\infty$ of finitely connected smoothly bounded compact sets such that*

(i) $m_2(\Pi_n \setminus E) + m_2(E \setminus \Pi_n) \to 0$ *as* $n \to \infty$ *and, moreover* $\chi_{\Pi_n} \to \chi_E$ *a.e.*

(ii) $\|(1/2\pi i)\, d\zeta\,|_{\partial \Pi_n} - d\mu\| \to 0$ *as* $n \to \infty$.

Let us give a brief description of the contents of this paper.

In Section 2, we establish an analog of the classical Cauchy–Green formula which holds for Lipschitz functions on the sets of finite perimeter. In Section 3, we apply the obtained version of the Cauchy–Green formula to study the problem of approximation of Lipschitz functions on the sets of finite perimeter by rational functions. We show that the function f satisfying the Lipschitz condition is uniformly approximable by rational functions if and only if $\partial f/\partial \bar{z} \equiv 0$ a.e. Also, we obtain the analog of F. and M. Riesz theorem for the Swiss cheeses. We want to mention here the related papers of O'Farrell [15–18] containing many interesting results concerning rational approximation of Lipschitz functions in the Lipschitz norm.

In Section 4, we establish the analog of the Cauchy representation formula for the algebra H^∞ on the sets of finite perimeter, where H^∞ is defined as a closure of rational functions in the weak (*) topology of $L^\infty(m_2)$.

Notation. For a compact set X in $\mathbb{C}$, $R(X)$ denotes the uniform closure of the space of rational functions with poles outside of X. $\mathrm{Lip}(1, \mathbb{C}) = \{\phi\colon \mathbb{C} \to \mathbb{C}\colon |\phi(z) - \phi(w)| \leqslant \text{Const.}\, |z-w|\}$. $(\partial/\partial \bar{z}) = \frac{1}{2}((\partial/\partial x) + i(\partial/\partial y))$, where $z = x + iy$. $C^1 = \{f\colon f$ is continuously differentiable in $\mathbb{C}\}$. $L^1_{\mathrm{loc}} = \{f\colon \int_K |f|\, dm_2 < +\infty$ for all compact sets K in $\mathbb{C}\}$.

2. THE CAUCHY–GREEN FORMULA ON THE SETS OF FINITE PERIMETER

Recall the classical Cauchy–Green formula: Let G be a finitely connected domain with a smooth boundary Γ consisting of finitely many Jordan curves. Let $\phi \in C^1(\mathbb{C})$. Then

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{\pi}\iint_{G}\frac{1}{\zeta-z}\frac{\partial\phi}{\partial\bar{\zeta}}\,dm_2$$
$$=\begin{cases}\phi(z) & \text{if } z\in G,\\ 0 & \text{if } z\notin\bar{G}.\end{cases}\tag{1}$$

THEOREM 2.1. *Let X be a compact set of finite perimeter. Then, for each $\phi\in\mathrm{Lip}(1,\mathbb{C})$, the following holds*

$$\frac{1}{2\pi i}\int_{B_X}\frac{\phi(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{\pi}\iint_{X}\frac{1}{\zeta-z}\frac{\partial\phi}{\partial\bar{\zeta}}\,dm_2$$
$$=\begin{cases}\phi(z), & \text{a.e. on } X,\\ 0, & z\in\mathbb{C}\backslash X.\end{cases}\tag{2}$$

Proof. At first, let us assume that $\phi\in C^1(\mathbb{C})$. Let $\{\Pi_n\}_1^\infty$ be a sequence of finitely connected compact sets with smooth Jordan boundaries converging to X and satisfying (i)–(ii) of Theorem 1.2. Let $z_0\in\mathbb{C}\backslash X$ be such that $\chi_{\Pi_n}(z_0)\to 0$ as $n\to\infty$. Hence, $\exists n_0\colon\forall n>n_0,\ z_0\notin\Pi_n$. Then, according to (1) we have

$$\frac{1}{2\pi i}\int_{\partial\Pi_n}\frac{\phi(\zeta)\,d\zeta}{\zeta-z_0}-\frac{1}{\pi}\iint_{\Pi_n}\frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta-z_0}\,dm_2=0$$

as $n>n_0$. Since $\phi(\zeta)/(\zeta-z_0)$ is continuous near X, $(1/2\pi i)\,d\zeta\,|_{\partial\Pi_n}\to(1/2\pi i)\,d\zeta\,|_{B_X}$ and $\chi_{\Pi_n}\to\chi_X$ a.e., we obtain

$$\frac{1}{2\pi i}\int_{B_X}\frac{\phi(\zeta)\,d\zeta}{\zeta-z_0}-\frac{1}{\pi}\iint_{\mathbb{C}}\chi_X\frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta-z_0}\,dm_2$$
$$=\lim_{n\to\infty}\left\{\frac{1}{2\pi i}\int_{\partial\Pi_n}\frac{\phi(\zeta)\,d\zeta}{\zeta-z_0}-\frac{1}{\pi}\iint_{\mathbb{C}}\chi_{\Pi_n}\frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta-z_0}\,dm_2\right\}=0.$$

So, (2) holds for almost all $z\in\mathbb{C}\backslash X$. Since $\mathbb{C}\backslash X$ is open and the left-hand side of (2) is continuous on $\mathbb{C}\backslash X$, (2) holds for all $z\in\mathbb{C}\backslash X$. Now, fix $z_0\in X$ satisfying the following conditions:

(a) $\chi_{\Pi_n}(z_0)\to 1$ as $n\to\infty$,

(b) $\int_{B_X}d\,|\zeta|/|\zeta-z_0|<\infty$,

(c) $(1/2\pi i)\int_{B_X}d\zeta/(\zeta-z_0)=1$.

Clearly, (a), (b), and (c) hold a.e. on X. By (a) $\exists n_0\colon\forall n>n_0,\ z_0\in\mathring{\Pi}_n$. Then, applying (1), we have

$$\phi(z_0)=\frac{1}{2\pi i}\int_{\partial\Pi_n}\frac{\phi(\zeta)}{\zeta-z_0}\,d\zeta-\frac{1}{\pi}\iint_{\Pi_n}\frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta-z_0}\,dm_2$$

as $n > n_0$. Since $\chi_{\Pi_n} \to \chi_X$ a.e. and $1/(\zeta - z_0) \in L^1_{\mathrm{loc}}$ we obtain

$$\frac{1}{\pi} \iint_{\Pi_n} \frac{\partial \phi}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 \to \frac{1}{\pi} \iint_X \frac{\partial \phi}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 \tag{3}$$

as $n \to \infty$. Note, that $|(\phi(\zeta) - \phi(z_0))/\zeta - z_0| \leqslant \text{const.}$ for $\zeta \in \mathbb{C} \backslash \{z_0\}$. Then, using properties (b) and (c) of z_0, we obtain for $n > n_0$

$$\begin{aligned}
&\left| \frac{1}{2\pi i} \int_{B_X} \frac{\phi(\zeta)\, d\zeta}{\zeta - z_0} - \frac{1}{2\pi i} \int_{\partial \Pi_n} \frac{\phi(\zeta)\, d\zeta}{\zeta - z_0} \right| \\
&\quad = \left| \frac{1}{2\pi i} \int_{B_X} \frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0} \, d\zeta - \frac{1}{2\pi i} \int_{\partial \Pi_n} \frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0} \, d\zeta \right| \\
&\quad \leqslant \text{const} \, \| d\zeta \,|_{B_X} - d\zeta \,|_{\partial \Pi_n} \| \to 0
\end{aligned}$$

as $n \to \infty$. Therefore, by (3)

$$\begin{aligned}
\phi(z_0) &= \lim_{n \to \infty} \left\{ \frac{1}{2\pi i} \int_{\partial \Pi_n} \frac{\phi(\zeta)\, d\zeta}{\zeta - z_0} \, d\zeta - \frac{1}{\pi} \iint_{\Pi_n} \frac{\partial \phi}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 \right\} \\
&= \frac{1}{2\pi i} \int_{B_X} \frac{\phi(\zeta)\, d\zeta}{\zeta - z_0} - \frac{1}{\pi} \iint_X \frac{\partial \phi}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 .
\end{aligned}$$

This proves (2) for $\phi \in C^1$. Assume now, that $\phi \in \mathrm{Lip}(1, \mathbb{C})$. As it is well-known there exists a sequence $\{\phi_n\}_1^\infty$, $\phi_n \in C^1$ such that $\|\phi_n - \phi\|_{C(X)} \to 0$ and $\|\partial \phi_n / \partial \bar{z}\|_{L^\infty} \leqslant M < +\infty$ (it suffices to take $\phi_n = \phi * \psi_{\varepsilon_n}$, where ψ_ε is an approximate identity, $\varepsilon_n \downarrow 0$). Taking a subsequence if necessary we can assume that $\partial \phi_n / \partial \bar{z} \to \partial \phi / \partial \bar{z}$ in the weak $(*)$ topology of L^∞. In fact, $\partial \phi_n / \partial \bar{z} \to \partial \phi / \partial \bar{z}$ in the distribution sense, at the same time there exists a subsequence converging weak $(*)$ to a function $\phi_0 \in L^\infty$. Then, ϕ_0 must equal to $\partial \phi / \partial \bar{z}$ a.e. Take $z_0 \in X$ such that the conditions (a)–(c) are satisfied. Then, using the fact that we have proved (2) for C^1-functions, we obtain

$$\begin{aligned}
\phi(z_0) &= \lim_{n \to \infty} \phi_n(z_0) \\
&= \lim_{n \to \infty} \left\{ \frac{1}{2\pi i} \int_{B_X} \frac{\phi_n(\zeta)}{\zeta - z_0} \, d\zeta - \frac{1}{\pi} \iint_X \frac{\partial \phi_n}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 \right\}.
\end{aligned}$$

As $\partial \phi_n / \partial \bar{\zeta} \to \partial \phi / \partial \bar{\zeta}$ weak $(*)$ and $1/(\zeta - z_0) \in L^1_{\mathrm{loc}}$ we have

$$\lim_{n \to \infty} \frac{1}{\pi} \iint_X \frac{\partial \phi_n}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 = \frac{1}{\pi} \iint_X \frac{\partial \phi}{\partial \bar{\zeta}} \frac{1}{\zeta - z_0} \, dm_2 .$$

Also, according to (b) we have

$$\left| \frac{1}{2\pi i} \int_{B_X} \frac{\phi_n(\zeta)}{\zeta - z_0} \, d\zeta - \frac{1}{2\pi i} \int_{B_X} \frac{\phi(\zeta)}{\zeta - z_0} \, d\zeta \right| \leqslant \frac{1}{2\pi} \, \|\phi_n - \phi\|_{C(X)} \int_{B_X} \frac{d\,|\zeta|}{|\zeta - z_0|} \to 0$$

as $n \to \infty$. Therefore, (2) also holds a.e. for $z \in X$. To prove (2) for $z \in \mathbb{C} \setminus X$, $\phi \in \mathrm{Lip}(1, \mathbb{C})$ we have to repeat the same argument and again use the fact that (2) holds for all ϕ_n. Theorem is proved.

3. Rational Approximation of the Lipschitz Functions

The following theorem extends to Lipschitz functions the known result for smooth functions (cf. to [1, Chap. III, Corollary 3.22]).

THEOREM 3.1. *Let X be a compact set in $\mathbb{C}$. If $f \in \mathrm{Lip}(1, \mathbb{C})$ and $\partial f / \partial \bar{z} \equiv 0$ a.e. on X, then $f \in R(X)$. Moreover, if X has a finite perimeter then the converse statement is also true. Namely, if $f \in R(X) \cap \mathrm{Lip}(1, \mathbb{C})$, then $\partial f / \partial \bar{z} \equiv 0$ a.e. on X.*

Proof. Let X be an arbitrary compact set. Take any measure $\mu \perp R(X)$. Then, according to Green's formula we have

$$\int_X f \, d\mu = -\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \, \hat{\mu}(z) \, dm_2 = 0$$

since $\partial f / \partial \bar{z} \equiv 0$ on X and $\hat{\mu}(z) \equiv 0$ on $\mathbb{C} \setminus X$. Applying the Hahn–Banach theorem, we obtain that $f \in R(X)$.

Now, suppose that X has a finite perimeter. At first, let us note that for each $z_0 \in X$ and satisfying the conditions (b) and (c) from the proof of Theorem 2.1, the measure

$$\frac{1}{2\pi i} \frac{1}{\zeta - z_0} \, d\zeta \,|_{B_X}$$

is a representing measure for $R(X)$. In fact, let $\phi \in R(X)$ be analytic in the neighborhood of X. Then, $(\phi(\zeta) - \phi(z_0))/(\zeta - z_0) \in R(X)$. Since $(1/2\pi i) \, d\zeta \,|_{B_X}$ is orthogonal to $R(X)$, in accordance with our choice of z_0 we obtain

$$\frac{1}{2\pi i} \int_{B_X} \frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0} \, d\zeta = 0$$

and, hence,

$$\phi(z_0)=\frac{1}{2\pi i}\int_{B_X}\frac{\phi(\zeta)}{\zeta-z_0}\,d\zeta. \tag{4}$$

In view of (c), we can take uniform limits and, therefore, (4) holds for all $\phi\in R(X)$. Let $f\in R(X)\cap \mathrm{Lip}(1,\mathbb{C})$. Then, by (4) we have for almost all $z_0\in X$

$$f(z_0)=\frac{1}{2\pi i}\int_{B_X}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta$$

Since $f\in \mathrm{Lip}(1,\mathbb{C})$ for almost all $z_0\in X$ according to Theorem 2.1, we also have

$$f(z_0)=\frac{1}{2\pi i}\int_{B_X}\frac{f(\zeta)\,d\zeta}{\zeta-z_0}-\frac{1}{\pi}\iint_X\frac{\partial f}{\partial\bar\zeta}\frac{1}{\zeta-z_0}\,dm_2.$$

Hence,

$$\iint_X\frac{\partial f}{\partial\bar\zeta}\frac{1}{\zeta-z_0}\,dm_2\equiv 0 \qquad \text{a.e. on } X.$$

For any $z_1\in\mathbb{C}\backslash X$, $f\cdot(1/(\zeta-z_1))\in R(X)$. As $(1/2\pi i)\,d\zeta\,|_{B_X}\perp R(X)$, we have

$$\frac{1}{2\pi i}\int_{B_X}\frac{f(\zeta)\,d\zeta}{\zeta-z}\equiv 0 \qquad \text{on} \quad \mathbb{C}\backslash X.$$

Then, according to Theorem 2.1, we obtain

$$\iint_X\frac{\partial f}{\partial\bar\zeta}\frac{1}{\zeta-z}\,dm_2\equiv 0 \qquad \text{on} \quad \mathbb{C}\backslash X.$$

Thus, $\widehat{(\partial f/\partial\bar\zeta)\,\chi_X\,dm_2}\equiv 0$ a.e. in $\mathbb{C}$. So, $(\partial f/\partial\bar\zeta)\,|_X\equiv 0$ a.e. Theorem is proved.

Remark. In [19] A. O'Farell using different techniques has generalized the above theorem in the following way. For any compact set K and any $f\in \mathrm{Lip}\,1$, $f\in R(K)$ if and only if $\partial f/\partial\bar z=0$ at almost all nonpeak points of $R(K)$. (According to Theorem 4 in [14], the set of peak points of $R(K)$ on the set K with a finite perimeter has area zero.)

The following statement can be considered as a version of F. and M. Riesz theorem (cf. to [5, 11, 12]).

THEOREM 3.2. *Let X be a nowhere dense set of finite perimeter. Let $f \in \mathrm{Lip}(1, \mathbb{C})$. Then, $f \in R(X)$ if and only if*

$$\int_{B_X} f(\zeta)\, r(\zeta)\, d\zeta = 0 \tag{5}$$

for all $r(\zeta) \in R(X)$.

Proof. Let $f \in R(X)$. Then, for each $r(\zeta) \in R(X)$, $f \cdot r \in R(X)$. Therefore,

$$\int_{B_X} f(\zeta)\, r(\zeta)\, d\zeta = 0$$

($d\zeta\,|_{B_X}$ is orthogonal to $R(X)$). Conversely, assume that (5) holds. Then,

$$\int_{B_X} f(\zeta) \cdot \frac{1}{\zeta - z}\, d\zeta = 0$$

for all $z \in \mathbb{C} \backslash X$. Therefore, in accordance with (2) we obtain

$$\iint_X \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z}\, dm_2 \equiv 0 \tag{6}$$

for all $z \in \mathbb{C} \backslash X$. But $\partial f / \partial \bar{\zeta} \in L^\infty$, $1/\zeta \in L^1_{\mathrm{loc}}$. So, the convolution $(\partial f / \partial \bar{\zeta}) * (1/\zeta)$ is a continuous function in $\mathbb{C}$. Since X is nowhere dense, from (6) we obtain that

$$\iint_X \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z}\, dm_2 \equiv 0$$

for all $z \in \mathbb{C}$. Thus $(\partial f / \partial \bar{\zeta})\,|_X \equiv 0$ a.e. Applying Theorem 3.1, we complete the proof. The following corollary has been first observed by Luecking (see [15]).

COROLLARY 3.1 (F. & M. Riesz theorem for Swiss cheeses). *Let K be a Swiss cheese. Let $f \in \mathrm{Lip}(1, \mathbb{C})$. Then $f \in R(K)$ if and only if*

$$\int_{\partial \Delta_0} f(\zeta)\, r(\zeta)\, d\zeta - \sum_{j=1}^{\infty} \int_{\partial \Delta_j} f(\zeta)\, r(\zeta)\, d\zeta = 0$$

for all $r(\zeta) \in R(K)$.

4. Algebra H^∞ on the Sets of Finite Perimeter

Let us recall the definition of the algebra H^∞ on an arbitrary compact set X (see [11]).

Definition. Let $Q \subset X$ be the set of all nonpeak points of $R(X)$. $H^\infty(X)$ is defined as the weak $(*)$ closure of $R(X)$ in $L^\infty(Q, m_2)$.

Remark. If X has a finite perimeter, then according to Theorem 4 in [14] $m_2(Q) = m_2(X)$ and, therefore, Q has full area density at each of its points. So, in that case $H^\infty(\bar{X})$ is the weak $(*)$ closure of $R(\bar{X})$ in $L^\infty(\bar{X})$. We shall call $\bar{X}$ an *essential set* for $R(\bar{X})$ if $\bar{X}$ is the closure of Q (cf. [1, Chap. III]).

Theorem 4.1. *Let X be an essential compact set with a finite perimeter. Let f be an arbitrary function in $H^\infty(X)$. Then, there exists a unique function $\tilde{f} \in L^\infty(B_X, d\zeta)$ such that*

$$\frac{1}{2\pi i}\int_{B_X} \frac{\tilde{f}(\zeta)}{\zeta - z}\, d\zeta = \begin{cases} f(z) & \text{for almost all } z \in X, \\ 0, & z \in \mathbb{C}\backslash X. \end{cases} \tag{7}$$

Moreover, $\|\tilde{f}\|_{L^\infty(B_X, d\zeta)} = \|f\|_{L^\infty(X)}$.

Proof. Since $f \in H^\infty(X)$, there exists a sequence $\{f_n\}_1^\infty$, $f_n \in R(X)$ such that $f_n \to f$ weak $(*)$ in $L^\infty(X, m_2)$. Then, obviously, $f_n \to f$ in the weak topology of $L^1(X, m_2)$. Moreover, since $f_n \to f$ weak $(*)$ in $L^\infty(X)$ and X is an essential set, $\|f_n\|_{L^\infty(X)} = \|f_n\|_{C(X)} \leqslant \text{const.} < +\infty$. As $f_n \to f$ weakly in L^1, there exists a sequence of their convex combinations

$$g_n = \sum_{i=1}^{j_n} \alpha_i^{(n)} f_i, \qquad \alpha_i^{(n)} \geqslant 0, \qquad \sum_{i=1}^{j_n} \alpha_i^{(n)} = 1,$$

such that $g_n \to f$ in the normed topology of L^1 on X (see [22, Chap. III, Sect. 3.13]). Furthermore, $g_n \in R(X)$ and $\|g_n\|_{C(X)} \leqslant \sum_{i=1}^{j_n} \alpha_i^{(n)} \|f_i\|_{C(X)} \leqslant$ const. for all n. Taking a subsequence, we can assume that

$$g_n \to f \qquad \text{a.e. on } X. \tag{8}$$

Put $\tilde{g}_n = g_n|_{B_X}$. Then, $\|\tilde{g}_n\|_{L^\infty(B_X, d\zeta)} \leqslant \|g_n\|_{C(X)} \leqslant \text{const.}$ for all n. Then, $\{\tilde{g}_n\}_1^\infty$ contains a subsequence which converges in the weak $(*)$ topology of $L^\infty(B_X, d\zeta)$ to a certain function $\tilde{f}$. We also denote this subsequence by $\tilde{g}_n$. We have

$$\|\tilde{f}\|_{L^\infty(B_X, d\zeta)} \leqslant \varliminf_{n\to\infty} \|\tilde{g}_n\| \leqslant \text{const.}$$

Fix $z_0 \in \mathbb{C}\setminus X$. Since $g_n \cdot 1/(\zeta - z_0) \in R(X)$ for all n, and $1/(\zeta - z_0)$ is continuous near X, we have

$$\frac{1}{2\pi i}\int_{B_X} \frac{\tilde{f}(\zeta)}{\zeta - z_0}\, d\zeta = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{B_X} \frac{\tilde{g}_n(\zeta)}{\zeta - z_0}\, d\zeta = 0$$

($d\zeta\,|_{B_X}$ is orthogonal to $R(X)$). We have proved the second equality in (7). To prove the remaining part, find $z_0 \in X$ such that the following conditions hold

(a) $(1/2\pi i)\int_{B_X} d\zeta/(\zeta - z_0) = 1$,

(b) $\int_{B_X} d\,|\zeta|/(|\zeta - z_0|) < \infty$,

(c) $\lim_{n\to\infty} g_n(z_0) = f(z_0)$.

As we saw in the proof of Theorem 3.1, the measure $(1/2\pi i)(1/(\zeta - z_0))\, d\zeta\,|_{B_X}$ is a representing measure for $R(X)$ at z_0. Then, according to (4) we have

$$g_n(z_0) = \frac{1}{2\pi i}\int_{B_X} \frac{\tilde{g}_n(\zeta)}{\zeta - z_0}\, d\zeta$$

for all n. In view of (b) $1/(\zeta - z_0) \in L^1(B_X, d\zeta)$. So, by (c) we obtain

$$f(z_0) = \lim_{n\to\infty} g_n(z_0) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{B_X} \frac{\tilde{g}_n(\zeta)}{\zeta - z_0}\, d\zeta = \frac{1}{2\pi i}\int_{B_X} \frac{\tilde{f}(\zeta)}{\zeta - z_0}\, d\zeta.$$

According to (8)(a), (b), (c) hold for a.a. $z \in X$. Thus, (7) holds a.e. on X. To establish the uniqueness of $\tilde{f}$, suppose there exists another function $\tilde{f}_1 \in L^\infty(B_X, d\zeta)$ satisfying (7). Then

$$\int_{B_X} \frac{\tilde{f}(\zeta) - \tilde{f}_1(\zeta)}{\zeta - z}\, d\zeta \equiv 0 \qquad \text{a.e. in } \mathbb{C}.$$

Hence, $\tilde{f} = \tilde{f}_1$ a.e. with respect to $d\zeta$ on B_X. Since H^∞ is an algebra, (7) holds for all powers f^n. So, for almost all $z \in X$ (a), (b), (c) hold, and (7) is valid for all f^n, $n = 1, \ldots$. Fix such $z_0 \in X$. As $\|\tilde{f}^n\| \leqslant \|\tilde{f}\|^n$, we have

$$|f(z_0)|^n = |f^n(z_0)| \leqslant \frac{1}{2\pi}\int_{B_X} \frac{\|\tilde{f}\|^n}{|\zeta - z_0|}\, d\,|\zeta|.$$

Hence,

$$\begin{aligned} |f(z_0)| &\leqslant \|\tilde{f}\|_{L^\infty(B_X, d\zeta)} \cdot \sqrt[n]{(1/2\pi)\int_{B_X} \frac{d\,|\zeta|}{|\zeta - z_0|}} \\ &\leqslant \|\tilde{f}\|_{L^\infty(B_X, d\zeta)} \cdot \sqrt[n]{\text{const.}} \end{aligned}$$

As $n \to \infty$, we obtain

$$|f(z_0)| \leqslant \|\tilde{f}\|_{L^\infty(B_X, d\zeta)}.$$

Therefore, $|f(z)| \leqslant \|\tilde{f}\|$ a.e. on X. So, $\|f\|_{L^\infty(X)} \leqslant \|\tilde{f}\|_{L^\infty(B_X, d\zeta)}$. To prove the inverse inequality, i.e., $\|\tilde{f}\|_{L^\infty(B_X, d\zeta)} \leqslant \|f\|_{L^\infty(X)}$ we quote Davie's result according to which there exists a sequence $f'_n \in R(X)$ such that $\|f'_n\|_{C(X)} \leqslant \|f\|_{L^\infty(X)}$ and $f'_n \to f$ weak $(*)$ in L^∞ (see [2, 12]). Then, starting the construction of $\tilde{f}$ with the sequence $\{f'_n\}$, we immediately obtain our inequality.

Remark. The trick applied to prove the inequality $\|f\| \leqslant \|\tilde{f}\|$ goes back to Landau.

COROLLARY 4.1. *Let X be a set of finite perimeter. Let $f_1, f_2 \in H^\infty(X)$ and $\tilde{f}_1 = \tilde{f}_2$ a.e. on B_X. Then, $f_1 \equiv f_2$ (a.e.).*

Note. This corollary shows that although, in general, a function f in $H^\infty(X)$ is defined almost everywhere with respect to m_2, there exists a universal set B_X of m_2-measure zero (even more, $m_1(B_X) < \infty$) and a function $\tilde{f}$ on B_X such that $\tilde{f}$ defines f uniquely. It seems natural to call $\tilde{f}$ the *boundary values* of f on X.

COROLLARY 4.2. *Let K be a Swiss cheese. Then,* (7) *and Corollary* 4.1 *hold for $H^\infty(K)$, with*

$$B_K = \bigcup_{j=0}^{\infty} (\partial \Delta_j).$$

ACKNOWLEDGMENTS

This work is a part of the author's Ph.D. thesis [13] submitted to Brown University. The author is indebted to his research supervisor, Professor John Wermer, for his constant help, support and encouragement. Professors H. Federer and W. Fleming were very helpful discussing their results in Geometric Measure Theory with the author. Also, the author wants to thank Professor Andrew Browder for many important remarks and suggestions, and for his constant interest and encouragement.

REFERENCES

1. A. BROWDER, "Introduction to Function Algebras," Benjamin, New York, 1969.
2. A. M. DAVIE, Bounded limits of analytic functions, *Proc. Amer. Math. Soc.* **32** (1) (1972), 127–133.
3. E. DE GIORGI, Su una teoria generale della misura $(r-1)$ dimensionale in uno spazio ad r dimensioni, *Ann. Mat. (4)* **3b** (1954), 191–213.

4. E. De Giorgi, Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni, *Ricerche Mat.* **4** (1955), 95–113.
5. P. Duren, "Theory of H^p Spaces," Academic Press, New York, 1970.
6. H. Federer, The Gauss–Green theorem, *Trans. Amer. Math. Soc.* **58** (1945), 44–76.
7. H. Federer, "Geometric Measure Theory," Springer-Verlag, Heidelberg/New York, 1969.
8. H. Federer, Colloquium lectures on geometric measure theory, *Bull. Amer. Math. Soc.* **84** (3) (1978), 291–338.
9. W. H. Fleming, Functions whose partial derivations are measures, *Illinois J. Math.* **4** (3) (1960), 452–478.
10. J. Garnett, "Analytic Capacity and Measure," Lecture Notes in Mathematics, Vol. 291, Springer-Verlag, Berlin, 1972.
11. I. Glicksberg, "Recent Results on Function Algebras," Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, Vol. 11, Amer. Math. Soc., Providence, R.I., 1972.
12. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice–Hall, Englewood Cliffs, N.J., 1962.
13. D. Khavinson, "On a Geometric Approach to Problems Concerning Cauchy Integrals and Rational Approximation," Ph.D. thesis, Brown Univ., Providence, R.I., 1983.
14. D. Khavinson, Sets of finite perimeter, Cauchy integrals and rational approximation, *in* "Approximation Theory IV," pp. 567–573, Academic Press, New York, 1983.
15. D. Luecking, "Research Problems in Function Theory" (A. Giroux and Q. I. Rahman, Eds.), Ann. Sci. Math. du Quebec, Vol. 6, p. 76, No. 1, 1982.
16. A. O'Farell, Estimates for capacities and approximation in Lipschitz norms, *J. Reine Angew. Math.* **311** (1979), 101–116.
17. A. O'Farell, Lip 1 rational approximation, *J. London Math. Soc.* **11** (1975), 159–164.
18. A. O'Farell, Hausdorff content and rational approximation in fractional Lipschitz norms, *Trans. Amer. Math. Soc.* **228** (1977), 187–206.
19. A. O'Farell, written communication.
20. K. Pietz, Cauchy transforms and characteristic functions, *Pacific J. Math.* **58** (2) (1975), 563–568.
21. K. Pietz, A geometric property of certain plane sets, *Proc. Amer. Math. Soc.* **54** (1976), 197–200.
22. W. Rudin, "Functional Analysis," McGraw–Hill, New York, 1973.

Printed in Belgium