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On α-Symmetric Multivariate Characteristic Functions

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An n-dimensional random vector is said to have an α-symmetric distribution, $\alpha > 0$, if its characteristic function is of the form $\varphi((|u_1|^\alpha + \cdots + |u_n|^\alpha)^{1/\alpha})$. We study the classes $\Phi_n(\alpha)$ of all admissible functions φ: $[0, \infty) \to \mathbb{R}$. It is known that members of $\Phi_n(2)$ and $\Phi_n(1)$ are scale mixtures of certain primitives Ω_n and ω_n, respectively, and we show that ω_n is obtained from Ω_{2n-1} by $n-1$ successive

solution to a question of D. St. P. Richards: If $\varphi(0) = 1$, φ is continuous, $\lim_{t\to\infty} \varphi(t) = 0$, and $\varphi^{(2n-2)}(t)$ is convex, then $\varphi \in \Phi_n(1)$. The paper closes with various criteria for the unimodality of an α-symmetric distribution. © 1998 Academic Press

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1. INTRODUCTION

An n-dimensional random vector $X = (X_1, ..., X_n)'$ is said to have an α-*symmetric distribution*, $\alpha > 0$, if its characteristic function is of the form

$$E \exp(i(u_1 X_1 + \cdots + u_n X_n)) = \varphi((|u_1|^\alpha + \cdots + |u_n|^\alpha)^{1/\alpha}) \tag{1}$$

for all $u = (u_1, ..., u_n)' \in \mathbb{R}^n$. Whenever (1) holds, we write $X \sim S_n(\alpha, \varphi)$, and we denote by

$$\Phi_n(\alpha)$$

the class of all admissible functions φ: $[0, \infty) \to \mathbb{R}$. Our definition of $\Phi_n(\alpha)$ and $S_n(\alpha, \varphi)$ is consistent with the more recent literature but deviates from the fundamental work of Cambanis, Keener, and Simons [8], who considered

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functions of the argument $|u_1|^\alpha + \cdots + |u_n|^\alpha$. The main advantage of working with functions of the l_α-(quasi-)norm

$$\|u\|_\alpha = (|u_1|^\alpha + \cdots + |u_n|^\alpha)^{1/\alpha}, \qquad u = (u_1, ..., u_n)' \in \mathbb{R}^n,$$

is the natural incorporation of the maximum norm $\|u\|_\infty = \max(|u_1|, ..., |u_n|)$ for $\alpha = \infty$.

Before we introduce our own contributions, a brief literature review will be useful. An α-symmetric distribution in one dimension is simply a symmetric distribution. Marginals of α-symmetric distributions are α-symmetric, and for any given α the classes $\Phi_n(\alpha)$ are nonincreasing in n, such that

$$\Phi_n(\alpha) \downarrow \Phi_\infty(\alpha) = \bigcap_{n \geqslant 1} \Phi_n(\alpha). \tag{2}$$

2-symmetric distributions are also called *spherically symmetric* or *isotropic*. They are of particular importance in applications and have been studied extensively (see, for example, Cambanis *et al.* [7] and the text by Fang *et al.* [11]). A classical result due to Schoenberg [20] is that $\varphi \in \Phi_n(2)$ if and only if

$$\varphi(t) = \int_{[0, \infty)} \Omega_n(rt) \, dF(r), \tag{3}$$

with F a distribution function on $[0, \infty)$, admits a representation as a scale mixture of the *primitive*

$$\Omega_n(t) = \Gamma(n/2) \left(\frac{2}{t}\right)^{(n-2)/2} J_{(n-2)/2}(t). \tag{4}$$

Here, $\Omega_n((u_1^2 + \cdots + u_n^2)^{1/2})$ is the characteristic function of a random vector $U = (U_1, ..., U_n)'$ uniformly distributed on the surface of the unit sphere in $\mathbb{R}^n$, and J denotes a Bessel function. Equivalently, spherically symmetric random vectors have a stochastic representation of the form

$$X \stackrel{\mathrm{d}}{=} RU, \tag{5}$$

where R is some nonnegative random variable independent of U. The symbol $\stackrel{\mathrm{d}}{=}$ stands for equality of distributions.

The characterization of the classes $\Phi_n(1)$ has been achieved by Cambanis *et al.* [8]: When $n \geqslant 2$, the function φ belongs to the class $\Phi_n(1)$ if and only if it is of the form

$$\varphi(t) = \int_{[0, \infty)} \omega_n(rt) \, dF(r), \tag{6}$$

where F is some distribution function on $[0, \infty)$, and ω_n is given by

$$\omega_n(t) = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)} \int_1^\infty \Omega_n(v^{1/2}t)\, v^{-n/2}(v-1)^{(n-3)/2}\, dv. \tag{7}$$

In analogy to Schoenberg's representation (3) for $\alpha = 2$, the members of $\Phi_n(1)$ are scale mixtures of a primitive ω_n. Expressed equivalently in terms of random vectors, an n-dimensional random vector X has a 1-symmetric distribution if and only if

$$X \stackrel{\mathrm{d}}{=} R \left(\frac{U_1}{D_1^{1/2}}, ..., \frac{U_n}{D_n^{1/2}} \right)', \tag{8}$$

where R is a nonnegative random variable, $U = (U_1, ..., U_n)'$ is uniformly distributed on the surface of the unit sphere in $\mathbb{R}^n$, $D = (D_1, ..., D_n)'$ has Dirichlet distribution with parameters $(1/2, ..., 1/2)'$, and R, U, and D are independent.

It is not known whether stochastic decompositions in analogy to (5) and (8) generally exist for α-symmetric random vectors, and only partial results are available. Bretagnolle, Dacunha Castelle, and Krivine [6] studied the classes $\Phi_\infty(\alpha)$ defined in (2). When $\alpha > 2$, $\Phi_\infty(\alpha) = \{1\}$, that is, $\Phi_\infty(\alpha)$ does not contain nondegenerate members. When $\alpha \leqslant 2$, φ belongs to $\Phi_\infty(\alpha)$ if and only if it admits a representation of the form

$$\varphi(t) = \int_{[0,\infty)} \exp(-rt^\alpha)\, dF(r) \tag{9}$$

for F some distribution function on $[0, \infty)$. Kuritsyn and Shestakov [15] showed that $\Phi_2(\infty) = \Phi_2(1)$ and gave examples of nondegenerate members of $\Phi_2(\alpha)$ for all $\alpha > 0$. Misiewicz [17] demonstrated that $\Phi_n(\infty) = \{1\}$ if $n \geqslant 3$. Both her result and that of Bretagnolle *et al.* for $\alpha > 2$ are now covered by Zastavnyi's [23] ingenious proof that $\Phi_n(\alpha) = \{1\}$ if $n \geqslant 3$ and $\alpha \in (2, \infty]$.

The paper is organized as follows: In Section 2, we establish some curious relations between 1-symmetric distributions in $\mathbb{R}^n$ and spherically symmetric distributions in Euclidean spaces of odd dimension. In particular, the primitive ω_n of $\Phi_n(1)$ is obtained from the primitive Ω_{2n-1} of $\Phi_{2n-1}(2)$ by $n-1$ successive integrations, which leads to closed form expressions for ω_n. Theorems 2.2 and 2.3 point at close links between the Fourier transforms of 1- and 2-symmetric functions.

Section 3 concerns the problem of easily applicable tests of whether a given function φ belongs to some class $\Phi_n(\alpha)$. A beautiful result of this type is due to R. Askey [1]: If φ is continuous, $\varphi(0) = 1$, $\lim_{t \to \infty} \varphi(t) = 0$, and $\varphi^{(k)}(t)$ is convex for $k = [n/2]$, the greatest integer less than or equal to

$n/2$, then $\varphi \in \Phi_n(2)$. D. St. P. Richards [19] asked for "α-analogues" of Askey's theorem, and we provide these for $\alpha = 1$ and $\alpha = \infty$. Richards' question naturally leads to the *Richards–Askey Problem* of studying the function

$$\Delta(n, \alpha) = \min\{\delta \geqslant 1 \mid (1-t)_+^\delta \in \Phi_n(\alpha)\}, \qquad (n, \alpha) \in \mathbb{Z}^+ \times \mathbb{R}^+.$$

Theorem 3.6 summarizes what is presently known about $\Delta(n, \alpha)$. Many questions remain open and make challenging problems.

With an α-symmetric distribution $S_n(\alpha, \varphi)$ specified by its characteristic function, we rely on φ for a discussion of its statistical properties. A primary issue here is unimodality, and in Section 4 we reflect on various notions of multivariate unimodality. The paper closes with a number of simple criteria for the unimodality of an α-symmetric distribution.

2. THE CLASSES $\Phi_n(1)$ AND $\Phi_{2n-1}(2)$

This section is devoted to some surprising relations between 1- and 2-symmetric distributions. We start with Theorem 2.1 which provides a representation of the primitive (7) of $\Phi_n(1)$ in terms of the primitive (4) of $\Phi_{2n-1}(2)$ and the operator

$$I\beta(t) = \lim_{x \to \infty} \int_t^x \beta(v)\, dv, \qquad t \geqslant 0, \tag{10}$$

defined for any function β: $[0, \infty) \to \mathbb{R}$ for which $I\beta(0)$ exists and is finite.

THEOREM 2.1. *For the primitive ω_n of $\Phi_n(1)$ and the primitive Ω_{2n-1} of $\Phi_{2n-1}(2)$,*

$$\omega_n(t) = \frac{\Gamma^2(n/2)}{\sqrt{\pi}\, \Gamma((2n-1)/2)}\, I^{n-1} \Omega_{2n-1}(t), \qquad t \geqslant 0. \tag{11}$$

The proof of Theorem 2.1 will be completed in the course of two further theorems, each of interest for its own sake. Beforehand, some comments are due. Clearly, the operator I can be inverted by differentiation, and a statement equivalent to (11) is that

$$\Omega_{2n-1}(t) = (-1)^{n-1} \frac{\sqrt{\pi}\, \Gamma((2n-1)/2)}{\Gamma^2(n/2)}\, \omega_n^{(n-1)}(t), \qquad t \geqslant 0. \tag{12}$$

Here and in the remainder of this section, derivatives taken at the origin are understood as right-hand limits. Equation (11) easily leads to closed form expressions for $\omega_n(t)$ in terms of powers of t, trigonometric functions, and the sine integral function

$$\mathrm{si}(t) = \lim_{x \to \infty} \int_t^x \frac{\sin v}{v}\, dv, \qquad t \geqslant 0.$$

Indeed, by formula (4) and the results in Section 8.46 of Gradshteyn and Ryzhik [13], $\Omega_{2n-1}(t)$ can be expressed in terms of trigonometric functions and inverse powers of t, and $n-1$ successive integrations yield closed form expressions for $\omega_n(t)$. We present these in Table 1 for $n = 2$, 3, and 4, along with elementary expressions for the associated primitive Ω_{2n-1} of $\Phi_{2n-1}(2)$. (The expression for ω_3 in Table 1 differs from that given by Cambanis *et al.* [8, p. 224], which has a sign error. Another typographical error is in the expression for the Bessel function $J_{5/2}$ in Gradshteyn and Ryzhik [13, 8.464.5].)

Let us now turn to the proof of Theorem 2.1. We start by recalling some basic facts on divided differences and Fourier transforms. For a function β: $[0, \infty) \to \mathbb{R}$ and distinct numbers $y_1, ..., y_n \geqslant 0$ we let

$$[y_1, ..., y_n; \beta(\cdot)] = \sum_{k=1}^{n} \frac{\beta(y_k)}{\prod_{j=1,\, j \neq k}^{n} (y_k - y_j)}, \tag{13}$$

the $(n-1)$th *divided difference* of β. If β has $n-1$ bounded and continuous derivatives,

$$[y_1, ..., y_n; \beta(\cdot)] = \frac{\beta^{(n-1)}(\xi)}{(n-1)!}, \qquad \min(y_1, ..., y_n) \leqslant \xi \leqslant \max(y_1, ..., y_n), \tag{14}$$

and we can extend definition (13) by continuity to arbitrary arguments $y_1, ..., y_n \geqslant 0$. See Section 6.1 of Isaacson and Keller [14] for the proof of (14) and many other interesting facts on divided differences.

If η is an integrable function from $\mathbb{R}^n$ to $\mathbb{R}$, we call the function

$$h(x_1, ..., x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-i(u_1 x_1 + \cdots + u_n x_n)} \eta(u_1, ..., u_n)\, du_1 \cdots du_n \tag{15}$$

its *Fourier transform*. Whenever $\eta(u_1, ..., u_n) = \beta((u_1^2 + \cdots + u_n^2)^{1/2})$ has spherical symmetry, its Fourier transform is a spherically symmetric function too. Then

$$h(x_1, ..., x_n) = h_n(r) = (2\pi)^{-n/2}\, r^{1-n/2} \int_0^{\infty} \beta(v)\, v^{n/2} J_{(n-2)/2}(rv)\, dv, \tag{16}$$

TABLE I

Closed form expressions for the primitives of $\Phi_n(1)$ and $\Phi_{2n-1}(2)$

n	$\omega_n(t)$	$\Omega_{2n-1}(t)$
2	$-\frac{2}{\pi}\,\mathrm{si}(t)$	$\frac{\sin t}{t}$
3	$\frac{1}{2}\left(\frac{\sin t}{t}+\cos t+t\,\mathrm{si}(t)\right)$	$3\,\frac{\sin t-t\cos t}{t^3}$
4	$-\frac{1}{2\pi}\left(\left(1-\frac{2}{t^2}\right)\sin t\right.$ $\left.+\left(t+\frac{2}{t}\right)\cos t+(t^2+4)\,\mathrm{si}(t)\right)$	$15\,\frac{(3-t^2)\sin t-3t\cos t}{t^5}$
$\vdots$	$\vdots$	$\vdots$
∞	$\exp(-t)$	$\exp(-t^2)$

where $r=(x_1^2+\cdots+x_n^2)^{1/2}$ and J denotes a Bessel function (Bochner and Chandrasekharan [5, Section II.7]). We may therefore write the Fourier transform of a spherically symmetric function as a radial function

$$h_n(r), \qquad r\geqslant 0,$$

of the nonnegative argument r. If $n=1$, h_1 corresponds to the cosine transform of the function $\beta(|u|)$. In the case $n=2m-1$ is odd, we recall from Theorem 40 of Bochner and Chandrasekharan [5] that h_1 and h_{2m-1} are related through

$$h_{2m-1}(r)=\frac{(-1)^{m-1}}{\pi^{m-1}}\frac{d^{m-1}}{dx^{m-1}}h_1(\sqrt{x})\bigg|_{x=r^2}. \tag{17}$$

We are now ready for Theorem 2.2 which relates the Fourier transforms of 1- and 2-symmetric functions in a curious way.

THEOREM 2.2. *Let n be a positive integer. Assume $\beta\colon[0,\infty)\to\mathbb{R}$ is a continuous function with $n-1$ bounded and continuous derivatives satisfying*

$$\int_0^\infty t^{n+k-1}\,|\beta^{(k)}(t)|\,dt<\infty, \qquad k=0,1,...,n-1. \tag{18}$$

Denote by g, h_1, and h_{2n-1} the Fourier transforms (15) *and* (16) *of the functions*

$$\beta(|u_1|+\cdots+|u_n|), \qquad \beta^{(n-1)}(|u|), \qquad \beta^{(n-1)}((u_1^2+\cdots+u_{2n-1}^2)^{1/2})$$

defined on $\mathbb{R}^n$, $\mathbb{R}$, *and* $\mathbb{R}^{2n-1}$. *Then* g, h_1, *and* h_{2n-1} *are bounded and continuous on their domains and related as follows*:

$$g(x_1, ..., x_n) = \frac{1}{\pi^{n-1}} [x_1^2, ..., x_n^2; h_1(\sqrt{\cdot})] = \frac{(-1)^{n-1}}{(n-1)!} h_{2n-1}(\xi),$$
$$x_1, ..., x_n \in \mathbb{R}, \tag{19}$$

where ξ *is some nonnegative number satisfying* $\min(|x_1|, ..., |x_n|) \leqslant \xi \leqslant \max(|x_1|, ..., |x_n|)$. *In particular*,

$$h_{2n-1}(r) = \frac{1}{(2\pi)^{n-1}} \left(-\frac{d}{r\,dr}\right)^{n-1} h_1(r)$$
$$= (-1)^{n-1} (n-1)!\, g(r, ..., r), \qquad r \geqslant 0. \tag{20}$$

Proof. It is clear from assumption (18) that the Fourier transforms $g(x_1, ..., x_n)$, $h_1(r)$, and $h_{2n-1}(r)$ are bounded and continuous functions on their domains. The case $n = 1$ is trivial, and we proceed to prove (19) and (20) for $n \geqslant 2$. By symmetry,

$$g(x_1, ..., x_n) = \frac{1}{\pi^n} \int_0^\infty \cdots \int_0^\infty \cos(u_1 x_1) \cdots \cos(u_n x_n)$$
$$\times \beta(u_1 + \cdots + u_n)\, du_1 \cdots du_n. \tag{21}$$

Literally the same arguments as in the proof of Theorem 3.1 of Cambanis *et al.* [8, pp. 225–226] show that

$$g(x_1, ..., x_n) = \frac{1}{\pi^n} [x_1^2, ..., x_n^2; B_{n,\beta}(\cdot)], \qquad x_1, ..., x_n \in \mathbb{R}, \tag{22}$$

where

$$B_{n,\beta}(\cdot) = \begin{cases} (-1)^{(n-2)/2} (\cdot)^{(n-1)/2} \int_0^\infty \sin(v\sqrt{\cdot})\, \beta(v)\, dv & n \text{ even} \\ (-1)^{(n-1)/2} (\cdot)^{(n-1)/2} \int_0^\infty \cos(v\sqrt{\cdot})\, \beta(v)\, dv & n \text{ odd} \end{cases}. \tag{23}$$

While the arguments leading from (21) to (22) are word-for-word identical to those of Cambanis *et al.*, we note two differences. Contrary to Cambanis *et al.*, we do not assume that β is a member of $\Phi_n(1)$, such that g may attain negative values. Also, our initial remarks on divided differences justify that we do not restrict to pairwise distinct arguments in formula (22).

In view of the assumptions (18) on β, integration by parts gives

$$B_{n,\beta}(\cdot) = \begin{cases} B_{n-1,\beta'}(\cdot) + (-1)^{(n-2)/2}\beta(0)(\cdot)^{(n-2)/2} & n \text{ even} \\ B_{n-1,\beta'}(\cdot) & n \text{ odd} \end{cases}. \tag{24}$$

Let us continue with $B_{n-1,\beta'}(\cdot), ..., B_{2,\beta^{(n-2)}}(\cdot)$ in the same fashion, and insert the resulting expressions into (22). By formula (14), the polynomial terms do not contribute to the divided differences, and we find that

$$\begin{aligned} g(x_1, ..., x_n) &= \frac{1}{\pi^n}[x_1^2, ..., x_n^2; B_{n,\beta}(\cdot)] = \frac{1}{\pi^n}[x_1^2, ..., x_n^2; B_{n-1,\beta'}(\cdot)] \\ &= \cdots = \frac{1}{\pi^n}[x_1^2, ..., x_n^2; B_{1,\beta^{(n-1)}}(\cdot)]. \end{aligned} \tag{25}$$

From (23), $B_{1,\beta^{(n-1)}}(\cdot) = \pi h_1(\sqrt{\cdot})$, and the integrability conditions (18) imply that h_1 has $n-1$ derivatives. Thus (25), (14), and (17) give

$$\begin{aligned} g(x_1, ..., x_n) &= \frac{1}{\pi^{n-1}}[x_1^2, ..., x_n^2; h_1(\sqrt{\cdot})] \\ &= \frac{1}{\pi^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}h_1(\sqrt{x})\bigg|_{x=\xi^2} \\ &= \frac{(-1)^{n-1}}{(n-1)!}h_{2n-1}(\xi) \end{aligned} \tag{26}$$

where ξ is some number satisfying $\min(|x_1|, ..., |x_n|) \leqslant \xi \leqslant \max(|x_1|, ..., |x_n|)$. We have proved (19). Finally, (20) is immediate from the passage to equal arguments $x_1 = \cdots = x_n = r \geqslant 0$ and a change of variables in (26). ∎

As an illustration of Theorem 2.2, the reader may want to check formulas (19) and (20) for the simple example $\beta(t) = \exp(-t)$.

Theorem 2.3. (a) *Let φ be a member of $\Phi_n(1)$ with $n-1$ bounded and continuous derivatives and $\lim_{t\to\infty}\varphi(t) = 0$. Then $(-1)^{n-1}\varphi^{(n-1)}(0) > 0$, and the function*

$$\psi(t) = \frac{\varphi^{(n-1)}(t)}{\varphi^{(n-1)}(0)}, \qquad t \geqslant 0, \tag{27}$$

belongs to the class $\Phi_{2n-1}(2)$. *Moreover, if* F *and* G *are the distribution functions in the canonical representations*

$$\varphi(t)=\int_{[0,\infty)} \omega_n(rt)\,dF(r), \qquad \psi(t)=\int_{[0,\infty)} \Omega_{2n-1}(rt)\,dG(r), \qquad t\geqslant 0, \tag{28}$$

then $F(0+)=G(0+)=0$, *and*

$$(-1)^{n-1}\,\varphi^{(n-1)}(0)\,dG(r)=\frac{\Gamma^2(n/2)}{\sqrt{\pi}\,\Gamma((2n-1)/2)}\,r^{n-1}\,dF(r), \qquad r>0. \tag{29}$$

(b) *Let* ψ *be a member of* $\Phi_{2n-1}(2)$ *such that* $I^{n-1}\psi$ *exists, with the operator* I *defined by* (10). *Then* $I^{n-1}\psi(0)>0$, *and the function*

$$\varphi(t)=\frac{I^{n-1}\psi(t)}{I^{n-1}\psi(0)}, \qquad t\geqslant 0, \tag{30}$$

belongs to $\Phi_n(1)$. *Moreover, if* F *and* G *are the distribution functions in* (28), *then* $F(0+)=G(0+)=0$, *and*

$$dG(r)=I^{n-1}\psi(0)\,\frac{\Gamma^2(n/2)}{\sqrt{\pi}\,\Gamma((2n-1)/2)}\,r^{n-1}\,dF(r), \qquad r>0. \tag{31}$$

Proof. We restrict ourselves to proving part (a); the proof of part (b) is analogous. The assumption that $\lim_{t\to\infty}\varphi(t)$ is zero ensures that $F(0+)=0$ for the distribution function F in the representation (28). To prove (27) and (29), let us assume for the while that $\varphi\in\Phi_n(1)$ satisfies the assumptions of Theorem 2.2. Then formula (20) applies, and the Fourier transform of the function $(-1)^{n-1}\,\varphi^{(n-1)}((u_1^2+\cdots+u_{2n-1}^2)^{1/2})$ exists and takes only nonnegative values. Thus $(-1)^{n-1}\,\varphi^{(n-1)}(0)>0$, and ψ belongs to $\Phi_{2n-1}(2)$. A result on p. 226 of Cambanis *et al.* [8] implies that the distribution F in (28) is absolutely continuous with density

$$f(r)=\frac{2}{\Gamma^2(n/2)}\,r^{n-1}B_{n,\varphi}^{(n-1)}(r^2),$$

where $B_{n,\varphi}$ is defined in (23). By (24) and the fact that higher order derivatives of polynomials vanish,

$$B_{n,\varphi}^{(n-1)}(r^2)=B_{n-1,\varphi'}^{(n-1)}(r^2)=\cdots=B_{1,\varphi^{(n-1)}}^{(n-1)}(r^2).$$

Let h_1 and h_{2n-1} denote the Fourier transforms (16) of $\psi(|u|)$ and $\psi((u_1^2+\cdots+u_{2n-1}^2)^{1/2})$. From (23), the definition of ψ in (27), and (17),

$$B^{(n-1)}_{1,\,\varphi^{(n-1)}}(r^2) = \pi\varphi^{(n-1)}(0)\,\frac{d^{n-1}}{dx^{n-1}}\,h_1(\sqrt{x})\bigg|_{x=r^2}$$
$$= (-1)^{n-1}\,\pi^n\varphi^{(n-1)}(0)\,h_{2n-1}(r),$$

and we obtain

$$f(r) = (-1)^{n-1}\,\frac{2\pi^n}{\Gamma^2(n/2)}\,\varphi^{(n-1)}(0)\,r^{n-1}h_{2n-1}(r), \qquad r>0.$$

By Theorem 2.9 of Fang *et al.* [11], the distribution G in (28) is absolutely continuous with density

$$g(r) = \frac{2\pi^{(2n-1)/2}}{\Gamma((2n-1)/2)}\,r^{2n-2}h_{2n-1}(r), \qquad r>0.$$

Comparing the preceding two equalities, we see that (29) holds. This proves part (a) of the theorem whenever φ satisfies (18).

Now let φ be any member of $\Phi_n(1)$ that has $n-1$ bounded and continuous derivatives. For $k>0$, consider $\varphi_k(t) = \varphi(t)\exp(-t/k)$, $t \geqslant 0$. Clearly, φ_k belongs to $\Phi_n(1)$ and satisfies the assumptions of Theorem 2.2. Thus

$$\psi_k(t) = \frac{\varphi_k^{(n-1)}(t)}{\varphi_k^{(n-1)}(0)}, \qquad t \geqslant 0,$$

is a member of $\Phi_{2n-1}(2)$. Since $\psi_k(t)$ tends to $\psi(t)$ as k tends to infinity, the continuity theorem shows that ψ is a member of $\Phi_{2n-1}(2)$ too. Also, the relation (29) carries over from the distributions F_k and G_k corresponding to φ_k and ψ_k, respectively, to their weak limits F and G corresponding to φ and ψ, respectively, and it is easily seen that $G(0+) = 0$. The proof is complete. ▮

Proof of Theorem 2.1. The idea is to apply part (b) of Theorem 2.3 to $\psi = \Omega_{2n-1} \in \Phi_{2n-1}(2)$. We start by showing that $I^{n-1}\Omega_{2n-1}$ exists. When $n=2$, this is evident. When $n \geqslant 3$, note from (4) and Eq. 8.461.1 of Gradshteyn and Ryzhik [13] that $\Omega_{2n-1}(t)$ is a linear combination of trigonometric functions multiplied by rational functions of order $\mathcal{O}(1/t^{n-1})$. By Formula 8.472.4 of the same reference and partial integration,

$$I\Omega_{2n-1}(t) = \lim_{x\to\infty}\int_t^x \Omega_{2n-1}(v)\,dv = -(2n-3)\lim_{x\to\infty}\int_t^x \frac{1}{v}\,\Omega'_{2n-3}(v)\,dv$$
$$= (2n-3)\left(\frac{1}{t}\,\Omega_{2n-3}(t) - \lim_{x\to\infty}\int_t^x \frac{1}{v^2}\,\Omega_{2n-3}(v)\,dv\right).$$

This shows that $I\Omega_{2n-1}(t)$ is of order at most $\mathcal{O}(1/t^{n-1})$ as t tends to infinity. Thus it allows for $n-2$ further applications of the operator I. In particular, $\psi(t)=\Omega_{2n-1}(t)$ satisfies the assumptions of Theorem 2.3(b) with the distribution G in the representation (28) concentrated at $r=1$. By formula (31), the function $\varphi(t)=I^{n-1}\Omega_{2n-1}(t)/I^{n-1}\Omega_{2n-1}(0)$ must coincide with the primitive $\omega_n(t)$ of $\Phi_n(1)$. Finally, the constant in (11) follows upon equating the factor in (31) with 1. ∎

3. α-ANALOGUES OF ASKEY'S THEOREM

In contrast to the progress that has been made in the characterization problem for the classes $\Phi_n(\alpha)$, little is known about easily applicable sufficient conditions for membership in $\Phi_n(\alpha)$. The only widely known result of this type is the beautiful Askey theorem [1]. It reduces to the celebrated criterion of Pólya [12, p. 509] when $n=1$.

THEOREM 3.1 (Askey). *If $\varphi\colon [0,\infty)\to\mathbb{R}$ is such that $\varphi(0)=1$, φ is continuous, $\lim_{t\to\infty}\varphi(t)=0$, and $(-1)^k\,\varphi^{(k)}(t)$ is convex for $k=[n/2]$, the greatest integer less than or equal to $n/2$, then $\varphi\in\Phi_n(2)$.*

Note that Askey's criterion considers φ a function of the argument $\|u\|_2=(u_1^2+\cdots+u_n^2)^{1/2}$. The formulation with φ a function of $\|u\|_2^2$ in a number of references results from an oversight in notation.

D. St. P. Richards [19] asked for "α-analogues" of Askey's theorem. We present these for $\alpha=1$ and $\alpha=\infty$.

THEOREM 3.2. *If $\varphi\colon [0,\infty)\to\mathbb{R}$ is such that $\varphi(0)=1$, φ is continuous, $\lim_{t\to\infty}\varphi(t)=0$, and $\varphi^{(2n-2)}(t)$ is convex, then $\varphi\in\Phi_n(1)$.*

COROLLARY 3.3. *If $\varphi\colon [0,\infty)\to\mathbb{R}$ is such that $\varphi(0)=1$, φ is continuous, $\lim_{t\to\infty}\varphi(t)=0$, and $\varphi''(t)$ is convex, then $\varphi\in\Phi_2(\infty)$.*

Generally, any proof of an α-analogue of Askey's theorem will fall into two parts: First, find the smallest positive integer δ for which the function

$$(1-t)_+^\delta=\begin{cases}(1-t)^\delta & 0\leqslant t\leqslant 1\\ 0 & t>1\end{cases} \tag{32}$$

belongs to $\Phi_n(\alpha)$. For $\alpha=2$ and δ a positive integer, it is well-known that $(1-t)_+^\delta\in\Phi_n(2)$ if and only if $\delta\geqslant[n/2]+1$ (Chanysheva [9], Zastavnyi [22]). The analogue for $\alpha=1$ is an immediate consequence of Theorem 2.3 and the result for $\alpha=2$. We omit the straightforward proof.

Proposition 3.4. *Let n and δ be positive integers. Then $(1-t)_+^\delta \in \Phi_n(1)$ if and only if $\delta \geqslant 2n-1$.*

Once shown that $(1-t)_+^\delta \in \Phi_n(\alpha)$, the key observation is that scale mixtures of this function belong to $\Phi_n(\alpha)$, too (see, for example, Misiewicz [18, pp. 33–34]). The following characterization of the scale mixtures of $(1-t)_+^\delta$ adapted from Lévy [16, Théorème 4] then provides a powerful tool to obtain α-analogues of Askey's theorem. Williamson [21] has almost identical results.

Proposition 3.5 (Williamson, Lévy). *Let δ be a positive integer. The function $\varphi: [0, \infty) \to \mathbb{R}$ has an integral representation of the form*

$$\varphi(t) = \int_{(0,\infty)} (1-rt)_+^\delta \, dF(r) \tag{33}$$

for F some distribution function on $(0, \infty)$ if and only if $\varphi(0)=1$, φ is continuous, $\lim_{t\to\infty} \varphi(t)=0$, and $(-1)^{\delta-1} \varphi^{(\delta-1)}(t)$ is convex.

In view of the scale mixture argument, combining Propositions 3.4 and 3.5 gives Theorem 3.2. Corollary 3.3 is then immediate from the equality of the classes $\Phi_2(1)$ and $\Phi_2(\infty)$ (Kuritsyn and Shestakov [15]). The result for $n=2$ in the corollary is the only case of interest if $\alpha=\infty$, because $\Phi_n(\infty)=\{1\}$ if $n\geqslant 3$ (Misiewicz [17]). For α different from 1, 2, or ∞, α-analogues of Askey's theorem remain open.

Let us slightly generalize our considerations and introduce a challenging problem:

The Richards–Askey Problem. *Study the Richards–Askey function*

$$\Delta(n,\alpha) = \min\{\delta \geqslant 1 \mid (1-t)_+^\delta \in \Phi_n(\alpha)\}, \qquad (n,\alpha) \in \mathbb{Z}^+ \times \mathbb{R}^+. \tag{34}$$

Ultimately, find its values on $\mathbb{Z}^+ \times \mathbb{R}^+$.

Note that δ is no longer assumed to be an integer in (34). The restriction $\delta \geqslant 1$ is justified below: Lemma 3.7 shows that $(1-|u|)_+^\delta$ is not a univariate characteristic function if $\delta<1$. The existence of the minimum in (34) follows from the continuity theorem, and the usual convention of setting the minimum of the empty set to ∞ will be employed. The connection between the Richards–Askey function and α-analogues of Askey's theorem is obvious: Given some explicit finite value of $\Delta(n,\alpha)$, Proposition 3.5 and the scale mixture argument lead to an α-analogue of Askey's theorem.

Theorem 3.6 summarizes what is presently known on the Richards–Askey problem.

THEOREM 3.6. *The Richards–Askey function* $\Delta(n, \alpha) = \min\{\delta \geqslant 1 \mid (1-t)_+^\delta \in \Phi_n(\alpha)\}$ *has the following properties*:

(a) *Let* δ *be a positive number. The function* $(1-t)_+^\delta$ *belongs to the class* $\Phi_n(\alpha)$ *if and only if* $\delta \geqslant \Delta(n, \alpha)$.

(b) $\Delta(n, 2) = (n+1)/2$ *and* $\Delta(n, 1) = 2n-1$ *for* $n \geqslant 1$; $\Delta(2, \infty) = 3$.

(c) *For every fixed* $\alpha > 0$, $\Delta(n, \alpha)$ *is a nondecreasing function of* n.

If $\alpha < 1$, *then* $\Delta(n, \alpha) \geqslant 1 + \log_2 n$. *Moreover, there exists some finite constant* $n_\alpha \geqslant 2$ *such that* $\Delta(n, \alpha) = \infty$ *for* $n \geqslant n_\alpha$.

If $\alpha \in [1, 2]$, *then* $\Delta(n, \alpha) \geqslant -\ln 2n / \ln(1 - 2^{-1/\alpha})$.

If $\alpha \in (2, \infty]$ *and* $n \geqslant 3$, *then* $\Delta(n, \alpha) = \infty$.

(d) *If* $n \geqslant 2$, *then* $\lim_{\alpha \to 0} \Delta(n, \alpha) = \infty$.

The proof of the inequalities in part (c) requires the subsequent lemma. In the univariate case, $n = 1$, it has been given before (Boas and Kac [4], Theorem 1), although with less transparent arguments.

LEMMA 3.7. *Suppose* $\varphi \in \Phi_n(\alpha)$ *is such that* $\varphi(t) = 0$ *for* $t \geqslant 1$. *Then*

$$|\varphi(t)| \leqslant \frac{1}{2n}, \qquad t \geqslant t_\alpha = \max\left(\frac{1}{2}, \frac{1}{2^{1/\alpha}}\right). \tag{35}$$

Proof. Let $t \geqslant t_\alpha$ and consider a regular grid in $\mathbb{R}^n$ with k^n nodes of form

$$u = (u_1 t, ..., u_n t)', \qquad (u_1, ..., u_n)' \in \{1, ..., k\}^n.$$

Denote the grid by G_k, and assign its nodes alternating constants $\xi_u = (-1)^{u_1 + \cdots + u_n}$. The positive definiteness of the function $\varphi(\|\cdot\|_\alpha)$ implies that

$$\sum_{u,\, v \in G_k} \xi_u \xi_v \varphi(\|u - v\|_\alpha) = \sum_{u,\, v \in G_k} \xi_u \xi_v \varphi\left(t\left(\sum_{i=1}^n |u_i - v_i|^\alpha\right)^{1/\alpha}\right) \geqslant 0.$$

As $\varphi(t) = 0$ for $t \geqslant 1$, the terms on the left-hand side of the inequality vanish except for $u = v$ and possibly for "nearest neighbors" u, v satisfying $\sum_{i=1}^n |u_i - v_i| = 1$. This leads to the inequality

$$k^n \varphi(0) - 2n(k-1)\, k^{n-1} \varphi(t) \geqslant 0,$$

or equivalently,

$$\varphi(t) \leqslant \frac{k}{2n(k-1)}.$$

The passage to the limit as k tends to infinity shows that $\varphi(t) \leqslant 1/(2n)$. The same reasoning with identical constants $\xi_u = 1$ gives $\varphi(t) \geqslant -1/(2n)$. ▮

Proof of Theorem 3.6.

(a) If $(1-t)_+^\delta \in \Phi_n(\alpha)$, we must have $\delta \geqslant \Delta(n, \alpha)$. Conversely, if $\delta \geqslant \delta_0 = \Delta(n, \alpha)$, Theorem 8 of Williamson [21] shows that $(1-t)_+^\delta$ has a representation as a scale mixture of $(1-t)_+^{\delta_0}$. Thus it is a member of $\Phi_n(\alpha)$.

(b) The assertion for $\alpha = 2$ is due to Zastavnyi [22]. The statements for $\alpha = 1$ and $\alpha = \infty$ are then immediate from Theorem 2.3 and the fact that $\Phi_2(\infty) = \Phi_2(1)$ [15]. The result for $\alpha = 1$ and $n = 2$ is also found in [22].

(c) For $\alpha > 0$, the classes $\Phi_n(\alpha)$ are nonincreasing in n. Thus $\Delta(n, \alpha)$ is a nondecreasing function of n. The inequality for $\alpha < 1$ follows from Lemma 3.7 applied to the function $(1-t)_+^\delta$. The second statement for $\alpha < 1$ is proved by contradiction: Let $\alpha < 1$ and suppose $\Delta(n, \alpha)$ is finite for all n. As $\exp(-t)$ is a scale mixture of $(1-t)_+^\delta$ for any finite $\delta \geqslant 1$, we would conclude that $\exp(-t) \in \bigcap_{n \geqslant 1} \Phi_n(\alpha) = \Phi_\infty(\alpha)$, a contradiction to (9). The inequality for $\alpha \in [1, 2]$ again follows from Lemma 3.7. If $n \geqslant 3$ and $\alpha \in (2, \infty]$, then $\Phi_n(\alpha) = \{1\}$ (Zastavnyi [23]), such that $\Delta(n, \alpha) = \infty$.

(d) The proof is done by contradiction: Assume there exist some $\delta > 0$ and a sequence $\alpha_k \downarrow 0$, such that $\Delta(n, \alpha_k) \leqslant \delta$ for all k. Then $(1-t)_+^\delta$ belongs to $\Phi_n(\alpha_k)$, and thereby to $\Phi_2(\alpha_k)$, for all k. As positive definiteness is preserved under limits, the function

$$\psi(u_1, u_2) = \lim_{k \to \infty} \left(1 - (|u_1|^{\alpha_k} + |u_2|^{\alpha_k})^{1/\alpha_k}\right)_+^\delta = \begin{cases} (1 - |u_2|)_+^\delta & \text{if} \quad u_1 = 0 \\ (1 - |u_1|)_+^\delta & \text{if} \quad u_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

would be positive definite in $\mathbb{R}^2$, which it is not. This argument is due to J. K. Misiewicz and D. St. P. Richards [18, p. 38]. ▮

4. α-SYMMETRIC DISTRIBUTIONS AND MULTIVARIATE UNIMODALITY

In this final section, we are interested in simple tests of whether an α-symmetric distribution is unimodal. Again, a theorem of this type is due to R. Askey [2].

THEOREM 4.1 (Askey). *If* $\varphi\colon [0, \infty) \to \mathbb{R}$ *is such that* $\varphi(0) = 1$, φ *is continuous,* $\lim_{t \to \infty} \varphi(t) = 0$, *and* $-\varphi'(t)$ *is convex, then* $\varphi(|u|)$ *is the characteristic function of a unimodal distribution.*

The resemblance of Theorems 4.1 and 3.1 is striking, and the conditions of Theorem 4.1 coincide with those of Theorem 3.1 when n is set to 2 or 3. Naturally, one asks for the reasons for this coincidence, and for a generalization of Askey's univariate result to multivariate, spherically symmetric distributions.

Before answering these questions, we need to make precise our notion of unimodality, which will always be understood as unimodality at the origin. Unfortunately, the generally accepted definition of a univariate, *unimodal* distribution F having its distribution function $F(x)$ convex for $x<0$ and concave for $x>0$ does not generalize to two or more dimensions. Consequently, various notions of multivariate unimodality have been developed (Dharmadhikari and Joag-Dev [10, Section 2.2]). For spherically symmetric distributions, however, all major notions of multivariate unimodality coincide with the single exception of linear unimodality, which we discuss below. All other notions can be unified by defining a spherically symmetric, *unimodal* distribution in $\mathbb{R}^n$ as a scale mixture of (possibly degenerate) uniform distributions on n-dimensional balls (Berk and Hwang [3]). Let us introduce the class

$$\Upsilon_n(2)$$

of all functions $\varphi \in \Phi_n(2)$ which are such that $S_n(2, \varphi)$ is unimodal. By a standard calculation, a random vector uniformly distributed on the unit ball in $\mathbb{R}^n$ has characteristic function $\Omega_{n+2}((u_1^2 + \cdots + u_n^2)^{1/2})$, with Ω given by formula (4). In view of Schoenberg's representation (3) for the class $\Phi_{n+2}(2)$, it is evident that

$$\Upsilon_n(2) = \Phi_{n+2}(2), \qquad n \geqslant 1 \tag{36}$$

(cf. Zolotarev [24, p. 288]). Clearly, Askey's Criterion 3.1 and the identity (36) answer the initial questions of this section. We have the following generalization of Theorem 4.1.

Theorem 4.2. *If $\varphi: [0, \infty) \to \mathbb{R}$ is such that $\varphi(0) = 1$, φ is continuous, $\lim_{t \to \infty} \varphi(t) = 0$, and $(-1)^k \varphi^{(k)}(t)$ is convex for $k = [n/2] + 1$, then $\varphi \in \Upsilon_n(2)$.*

Let us turn to the notion of linear unimodality, which we discuss in the general context of α-symmetric distributions. Recall that a multivariate distribution is said to be *linear unimodal* if each one-dimensional marginal distribution is univariate unimodal (Dharmadhikari and Joag-Dev [10, p. 42]).

Theorem 4.3. *An α-symmetric distribution $S_n(\alpha, \varphi)$ is linear unimodal if and only if $\varphi \in \Phi_3(2)$. In particular, if $n \geqslant 3$, any* 1-*symmetric or spherically symmetric distribution is linear unimodal.*

Proof. Let $X=(X_1, ..., X_n)' \sim S_n(\alpha, \varphi)$, and consider any linear combination $Y=\sum_{i=1}^{n} a_i X_i$, where $a=(a_1, ..., a_n)' \in \mathbb{R}^n$. The characteristic function $E \exp(iuY)=\varphi(\|a\|_\alpha |u|)$ of Y is, up to a scale factor, independent of a. Thus $S_n(\alpha, \varphi)$ is linear unimodal if and only if φ belongs to $\Upsilon_1(2)=\Phi_3(2)$. The latter equality also proves the claim for $\alpha=1$ and $\alpha=2$, because $\Phi_n(1) \subseteq \Phi_n(2) \subseteq \Phi_3(2)$ if $n \geqslant 3$. ∎

The assertion for spherically symmetric distributions in Theorem 4.3 has been known before (Berk and Hwang [3]). Even for 1-symmetric distributions, the restriction $n \geqslant 3$ is essential. By Proposition 2.2 of Cambanis *et al.* [8], the 1-symmetric distribution $S_2(1, \omega_2)$ has marginal densities

$$g_0(x)=\frac{1}{\pi^2 x} \ln \left|\frac{1+x}{1-x}\right|, \qquad x \neq -1, 0, 1,$$

that are not unimodal.

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