Inequalities for sums and direct sums of Hilbert space operators

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Abstract

We prove several singular value inequalities and norm inequalities involving sums and direct sums of Hilbert space operators. It is shown, among other inequalities, that if $X$ and $Y$ are compact operators, then the singular values of $\frac{X+Y}{2}$ are dominated by those of $X \oplus Y$. Applications of these inequalities are also given.

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1. Introduction

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on a separable Hilbert space $H$. For a compact operator $X \in B(H)$, let $s_1(X) \geq s_2(X) \geq \cdots$ denote the singular values of $X$, i.e., the eigenvalues of $|X| = (X^*X)^{1/2}$ (the absolute value of $X$), arranged in decreasing order and repeated according to multiplicity. The usual operator (or the spectral) norm of an operator $X \in B(H)$ is designated by $\|X\|$. Thus, if $X$ is compact, then $\|X\| = s_1(X)$.

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In addition to the usual operator norm, which is defined on all of $B(H)$, we consider unitarily invariant (or symmetric) norms $||| \cdot |||$. Each of these norms is defined on a norm ideal contained in the ideal of compact operators, and, for the sake of brevity, we will make no explicit mention of this ideal. Thus, when we consider $|||X|||$, we are assuming that the operator $X$ belongs to the norm ideal associated with $||| \cdot |||$. Moreover, each unitarily invariant norm $||| \cdot |||$ is a symmetric gauge function of the singular values, and is characterized by the equality $|||X||| = |||UXV|||$ for all operators $X$ and for all unitary operators $U$ and $V$ in $B(H)$. For the general theory of unitarily invariant norms, we refer to [2] or [9].

It has been shown by Bhatia and Kittaneh [5] that if $X$ and $Y$ are positive operators in $B(H)$, then

\[ |||X - Y||| \leq |||X \oplus Y|||. \]  

Here we use the direct sum notation $\bigoplus_{i=1}^{n} X_i$ for the block-diagonal operator defined on $\bigoplus_{i=1}^{n} H$ (the direct sum of $n$ copies of $H$) with operators $X_i$ as its diagonal entries.

In view of the fact that the unitarily invariant norms are increasing with respect to the singular values (see, e.g., [2, p. 52] or [9, p. 71]), the inequality (1) has been improved by Zhan [15] (see, also, [16, p. 33] and [17]), in the finite-dimensional setting, so that

\[ s_j(X - Y) \leq s_j(X \oplus Y) \] for $j = 1, 2, \ldots$

In this paper we present several singular value inequalities and norm inequalities involving sums and direct sums of operators. In Section 2 we establish a general Cauchy–Schwarz type inequality for singular values. This inequality has several consequences and applications. In particular, it will be shown that if $X$ and $Y$ are compact operators in $B(H)$, then

\[ s_j \left( \frac{X + Y}{2} \right) \leq s_j(X \oplus Y) \] for $j = 1, 2, \ldots$, which supplements the inequality (2).

A very useful related norm inequality (see, e.g., [2, p. 97], [3,4]) asserts that if $X$ and $Y$ are operators in $B(H)$, then

\[ \left| \left| \left( \frac{X + Y}{2} \right) \oplus \left( \frac{X + Y}{2} \right) \right| \right| \leq |||X \oplus Y||| \] for every unitarily invariant norm. In Section 3 we give a weighted mean norm inequality, which is a considerable generalization of the inequality (4). Related norm inequalities are also obtained.

2. Singular value inequalities

The aim of this section is to establish singular value inequalities relating sums and direct sums of compact operators. The following lemma is essential in our analysis (see e.g., [2, p. 75] or [9, p. 27]).

Lemma 2.1. Let $A, B, X \in B(H)$ such that $X$ is compact. Then $s_j(AXB) \leq \|A\|\|B\|s_j(X)$ for $j = 1, 2, \ldots$

Based on this lemma, we have the following Cauchy–Schwarz type inequality for singular values.
Theorem 2.1. Let $A_i, B_i, X_i \in B(H)$ such that $X_i$ is compact, $i = 1, \ldots, n$. Then
\[
s_j \left( \sum_{i=1}^{n} A_i X_i B_i \right) \leq \left\| \sum_{i=1}^{n} |A_i^*|^2 \right\|^{1/2} \left\| \sum_{i=1}^{n} |B_i|^2 \right\|^{1/2} s_j \left( \bigoplus_{i=1}^{n} X_i \right)
\]
for $j = 1, 2, \ldots$

Proof. On $\bigoplus_{i=1}^{n} H$, define the operators
\[
A = \begin{bmatrix} A_1 & \cdots & A_n \\
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 \\
\vdots & \ddots \\
0 & \cdots & X_n \\
\end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots \\
B_n & 0 & \cdots & 0 \\
\end{bmatrix}.
\]
Then $AXB = \left( \sum_{i=1}^{n} A_i X_i B_i \right) \oplus \bigoplus_{j=1}^{n-1} 0, \|A\| = \left\| \sum_{i=1}^{n} |A_i^*|^2 \right\|^{1/2}, \|B\| = \left\| \sum_{i=1}^{n} |B_i|^2 \right\|^{1/2}$, and $s_j(X) = s_j \left( \bigoplus_{i=1}^{n} X_i \right)$ for $j = 1, 2, \ldots$. Now, the result follows by applying Lemma 2.1 to the operators $A$, $B$, and $X$. □

Corollary 2.1. Let $A, B, X, Y \in B(H)$ such that $X$ and $Y$ are compact, and $A$ and $B$ are contractions. Then
\[
s_j \left( AXB + \left( I - |A^*|^2 \right)^{1/2} Y \left( I - |B|^2 \right)^{1/2} \right) \leq s_j (X \oplus Y)
\]
for $j = 1, 2, \ldots$

Corollary 2.2 admits the following important special case.

Corollary 2.2. Let $X, Y \in B(H)$ be compact, and $0 < \alpha < 1$. Then
\[
s_j (\alpha X + (1 - \alpha) Y) \leq s_j (X \oplus Y)
\]
for $j = 1, 2, \ldots$ In particular (letting $\alpha = 1/2$), we have
\[
s_j \left( \frac{X + Y}{2} \right) \leq s_j (X \oplus Y)
\]
for $j = 1, 2, \ldots$

The following noncommutative Bohr inequality has been proved in [10].

Lemma 2.2. Let $A, B \in B(H)$, and $0 < \alpha < 1$. Then
\[
|\alpha A + (1 - \alpha) B|^2 \leq \alpha |A|^2 + (1 - \alpha) |B|^2.
\]
Using this, together with Corollary 2.2, we have the following relevant inequality.

Theorem 2.2. Let $X, Y \in B(H)$ be compact, and $0 < \alpha < 1$. Then
\[
s_j (\alpha X + (1 - \alpha) Y) \leq s_j \left( \sqrt{2\alpha} X \oplus \sqrt{2(1 - \alpha)} Y \right)
\]
for $j = 1, 2, \ldots$
Proof. It follows, from Lemma 2.2, Corollary 2.2, and the min–max principle (see, e.g., [2, p. 58] or [9, p. 25]), that
\[ s_j^2(\alpha X + (1 - \alpha)Y) = s_j(|\alpha X + (1 - \alpha)Y|^2) \leq s_j(\alpha |X|^2 + (1 - \alpha)|Y|^2) \leq 2s_j(\alpha |X|^2 \oplus (1 - \alpha)|Y|^2) = s_j^2\left(\sqrt{2\alpha X \oplus \sqrt{2(1-\alpha)Y}}\right) \]
for \( j = 1, 2, \ldots \) Consequently,
\[ s_j(\alpha X + (1 - \alpha)Y) \leq s_j\left(\sqrt{2\alpha X \oplus \sqrt{2(1-\alpha)Y}}\right) \]
for \( j = 1, 2, \ldots \)

Letting \( Y = X^* \) in (3), we obtain the following singular value inequality comparing the singular values of \( X \) to those of its real part \( \text{Re} X = \frac{X + X^*}{2} \).

**Corollary 2.3.** Let \( X \in B(H) \) be compact. Then
\[ s_j(\text{Re} X) \leq s_j(X \oplus X) \]
for \( j = 1, 2, \ldots \)

It should be mentioned here that the inequality \( s_j(\text{Re} X) \leq s_j(X) \) is false for \( j > 1 \). To see this, consider the two-dimensional example \( X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). Then \( s_2(\text{Re} X) = \frac{1}{2} > 0 = s_2(X) \). However, if \( \lambda_1(\text{Re} X) \geq \lambda_2(\text{Re} X) \geq \cdots \) are the eigenvalues of \( \text{Re} X \), arranged in decreasing order and repeated according to multiplicity, then a result of Fan and Hoffman [8] asserts that \( \lambda_j(\text{Re} X) \leq s_j(X) \) for \( j = 1, 2, \ldots \)

It has been shown in [5] that if \( X \) and \( Y \) are compact operators in \( B(H) \), then
\[ 2s_j(XY^*) \leq s_j(|X|^2 + |Y|^2) \] (7)
for \( j = 1, 2, \ldots \). This noncommutative arithmetic-geometric mean inequality is equivalent to the inequality (2) (see [16, p. 36]).

Replacing \( X \), in Corollary 2.3, by \( XY^* \) and using the inequality (7), we have the following related inequality.

**Corollary 2.4.** Let \( X, Y \in B(H) \) be compact. Then
\[ s_j(XY^* + YX^*) \leq s_j((|X|^2 + |Y|^2) \oplus (|X|^2 + |Y|^2)) \]
for \( j = 1, 2, \ldots \)

Based on the inequality (7), Bhatia and Kittaneh [5] proved that if \( X, Y \in B(H) \), then
\[ |||XY^* + YX^*||| \leq |||X||| \cdot |||Y||| \] (8)
for every unitarily invariant norm. In view of the inequality (8) and Corollary 2.4, one may ask if the inequality \( s_j(XY^* + YX^*) \leq s_j(|X|^2 + |Y|^2) \) for \( j = 1, 2, \ldots \) is true. In fact, this inequality is false for \( j > 1 \). To see this, consider the two-dimensional example \( X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \). Then \( s_2(XY^* + YX^*) = 1 > 0 = s_2(|X|^2 + |Y|^2) \).
Theorem 2.3. Let $A, B, X, Y \in B(H)$ such that $X$ and $Y$ are compact, $AA^* + BB^* \leq A^*A + B^*B$, and $A^*A + B^*B$ is invertible. Then

$$s_j((AA^* + BB^*)^{-1/2}(AXB + BYA)(AA^* + BB^*)^{-1/2}) \leq s_j(X \oplus Y)$$
for $j = 1, 2, \ldots$

Proof. Let $C = A^*A + B^*B$, $D = AA^* + BB^*$, $A_1 = C^{-1/2}A$, $A_2 = C^{-1/2}B$, $B_1 = BC^{-1/2}$, and $B_2 = AC^{-1/2}$. Then

$$|A_1|^2 + |A_2|^2 = C^{-1/2}DC^{-1/2} \leq I \quad \text{and} \quad |B_1|^2 + |B_2|^2 = I. \quad (9)$$
Now,

$$s_j((AA^* + BB^*)^{-1/2}(AXB + BYA)(AA^* + BB^*)^{-1/2})$$
$$= s_j(A_1XB_1 + A_2YB_2)$$
$$\leq ||A_1|^2 + |A_2|^2||B_1|^2 + |B_2|^2||s_j(X \oplus Y) \quad \text{by Theorem 2.1}$$
$$\leq s_j(X \oplus Y) \quad \text{by (9)}$$
for $j = 1, 2, \ldots \Box$

Corollary 2.5. Let $A, B, X, Y \in B(H)$ such that $X$ and $Y$ are compact, $A$ and $B$ are positive, and $A + B$ is invertible. Then

$$s_j((A + B)^{-1/2}(A^{1/2}XB^{1/2} + B^{1/2}YA^{1/2})(A + B)^{-1/2}) \leq s_j(X \oplus Y)$$
for $j = 1, 2, \ldots$

Corollary 2.6. Let $A, B, X, Y \in B(H)$ such that $X$ and $Y$ are compact, $A$ and $B$ are self-adjoint, $\text{Re} (AB)$ is positive, and $A + B$ is invertible. Then

$$s_j((A + B)^{-1}(A - B)(A + B)^{-1} + (A - B)(A + B)^{-1}Y(A + B)^{-1}(A - B))$$
$$\leq 2s_j(X \oplus Y)$$
for $j = 1, 2, \ldots$

Proof. Let $C = (A + B)^{-1}(A - B)$. Since $\text{Re} (AB) \geq 0$, it follows that $(A - B)^2 \leq (A + B)^2$, and so $(A + B)^{-1}(A - B)^2(A + B)^{-1} \leq I$. Thus,

$$||C|| = ||(A - B)(A + B)^{-1}|| \leq 1. \quad (10)$$
Now,

$$s_j ((A + B)^{-1}(A - B)(A + B)^{-1} + (A - B)(A + B)^{-1}Y(A + B)^{-1}(A - B))$$
$$= s_j(CXC^* + CYC)$$
$$\leq ||C|^2 + |C|^2||C^*|^2 + |C|^2||^2s_j(X \oplus Y) \quad \text{by Theorem 2.1}$$
$$= ||C|^2 + |C|^2||s_j(X \oplus Y)$$
$$\leq 2||C||s_j(X \oplus Y) \quad \text{by the triangle inequality}$$
$$\leq 2s_j(X \oplus Y) \quad \text{by (10)}$$
for $j = 1, 2, \ldots \Box$
Another application of Theorem 2.1 can be seen as follows.

**Theorem 2.4.** Let $A, B, X, Y \in B(H)$ such that $A$ and $B$ are normal contractions, and $n$ is a natural number with $n > 1$. Then

$$ s_j \left( (I - |A|^2)^{1/2} (X - A^{2n-1} Y B^{2n-1}) (I - |B|^2)^{1/2} \right) $$

$$ \leq s_j \left( \bigoplus_{i=1}^{n} (X - A Y B) \oplus \bigoplus_{i=1}^{n-1} (Y - A X B) \right) $$

for $j = 1, 2, \ldots$

**Proof.** Let $A_i = (I - |A|^2)^{1/2} A^{i-1}$ and $B_i = B^{i-1} (I - |B|^2)^{1/2}$ for $i = 1, \ldots, 2n - 1$. Then

$$ \sum_{i=1}^{2n-1} |A_i^*|^2 = \sum_{i=1}^{2n-1} |A_i|^2 \quad (\text{since } A_i \text{ is normal for } i = 1, \ldots, 2n - 1) $$

$$ = \sum_{i=1}^{2n-1} (A^{i-1})^* (I - |A|^2) A^{i-1} $$

$$ = I - |A|^{2n} $$

$$ \leq I. \quad (11) $$

Similarly,

$$ \sum_{i=1}^{2n-1} |B_i|^2 \leq I. \quad (12) $$

Now,

$$ s_j ((I - |A|^2)^{1/2} (X - A^{2n-1} Y B^{2n-1}) (I - |B|^2)^{1/2}) $$

$$ = s_j \left( \sum_{i=1}^{n} A_{2i-1} (X - A Y B) B_{2i-1} + \sum_{i=1}^{n-1} A_{2i} (Y - A X B) B_{2i} \right) $$

$$ \leq \left\| \sum_{i=1}^{2n-1} |A_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^{2n-1} |B_i|^2 \right\|^{1/2} \quad s_j \left( \bigoplus_{i=1}^{n} (X - A Y B) \oplus \bigoplus_{i=1}^{n-1} (Y - A X B) \right) $$

$$ \leq s_j \left( \bigoplus_{i=1}^{n} (X - A Y B) \oplus \bigoplus_{i=1}^{n-1} (Y - A X B) \right). \quad (13) $$

Here (13) follows from Theorem 2.1, and (14) follows from (11) and (12). \hfill \square

By an argument similar to that used in the proof of Theorem 2.4, one can establish the following related inequalities.

**Theorem 2.5.** Let $A, B, X, Y \in B(H)$ such that $A$ and $B$ are normal contractions, and $n$ is a natural number with $n > 1$. Then
\[ s_j \left( (I - |A|^2)^{1/2} (X - A^{2n} Y B^{2n}) (I - |B|^2)^{1/2} \right) \]

\[ \leq s_j \left( \bigoplus_{i=1}^{n} (X - A Y B) \oplus \bigoplus_{i=1}^{n} (Y - A X B) \right) \]

and

\[ s_j \left( (I - |A|^2)^{1/2} (X - A^n Y B^n) (I - |B|^2)^{1/2} \right) \leq s_j \left( \bigoplus_{i=1}^{n-1} (X - A X B) \oplus (X - A Y B) \right) \]

for \( j = 1, 2, \ldots \)

For commuting sequences of normal operators, we have the following improvement of Theorem 2.1.

**Theorem 2.6.** Let \( \{A_i\}_{i=1}^{n} \) and \( \{B_i\}_{i=1}^{n} \) be two sequences of commuting normal operators in \( B(H) \), and \( X_i \in B(H) \) is compact, \( i = 1, \ldots, n \). Then

\[ s_j \left( \sum_{i=1}^{n} A_i X_i B_i \right) \leq s_j \left( \bigoplus_{k=1}^{n} \left( \sum_{i=1}^{k} |A_i|^2 \right)^{1/2} X_k \left( \sum_{i=1}^{k} |B_i|^2 \right)^{1/2} \right) \]

for \( j = 1, 2, \ldots \)

**Proof.** Let \( A = (\sum_{i=1}^{n} |A_i|^2)^{1/2} \) and \( B = (\sum_{i=1}^{n} |B_i|^2)^{1/2} \). Since \( \{A_i\}_{i=1}^{n} \) is a sequence of commuting normal operators, it follows, from [7, p. 55], that there is a measure space \( (Z, \Omega, \mu) \), functions \( \phi_1, \phi_2, \ldots, \phi_n \) in \( L^\infty(\mu) \), and an isomorphism \( U : H \rightarrow L^2(\mu) \) such that \( U^{-1} M_{\phi_i} U = A_i \) for \( i = 1, \ldots, n \), where \( M_{\phi_i} \) is the multiplication operator defined by \( M_{\phi_i} f = \phi_i f \) for all \( f \in L^2(\mu) \). Let \( \phi = \sum_{i=1}^{n} |\phi_i|^2 \) and let \( E = \{ z \in Z : \phi(z) \neq 0 \} \). Define

\[ \psi_i(z) = \begin{cases} \frac{\phi_i(z)}{\phi(z)} & \text{if } z \in E, \\ 0 & \text{if } z \notin E. \end{cases} \]

If \( C_i = U^{-1} M_{\psi_i} U \) for \( i = 1, \ldots, n \), then \( \{C_i\}_{i=1}^{n} \) is a sequence of commuting normal operators such that \( A_i = AC_i = C_i A \) for \( i = 1, \ldots, n \), and \( \sum_{i=1}^{n} |C_i|^2 \) is the orthogonal projection on \( \text{ran } A \).

By a similar argument, one can prove that there is a sequence \( \{D_i\}_{i=1}^{n} \) of commuting normal operators such that \( B_i = BD_i = D_i B \) for \( i = 1, \ldots, n \), and \( \sum_{i=1}^{n} |D_i|^2 \) is the orthogonal projection on \( \text{ran } B \).

Now, it follows, from Theorem 2.1, that

\[ s_j \left( \sum_{i=1}^{n} A_i X_i B_i \right) = s_j \left( \sum_{i=1}^{n} C_i A X_i B D_i \right) \]

\[ \leq \left\| \sum_{i=1}^{n} |C_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^{n} |D_i|^2 \right\|^{1/2} s_j \left( \bigoplus_{k=1}^{n} A X_k B \right) \]

\[ = \left\| \sum_{i=1}^{n} |C_i|^2 \right\|^{1/2} \left\| \sum_{i=1}^{n} |D_i|^2 \right\|^{1/2} s_j \left( \bigoplus_{k=1}^{n} A X_k B \right) \]
\[ s_j \left( \bigoplus_{k=1}^{n} AX_k B \right) \]
\[ = s_j \left( \bigoplus_{k=1}^{n} \left( \sum_{i=1}^{n} |A_i|^2 \right)^{1/2} X_k \left( \sum_{i=1}^{n} |B_i|^2 \right)^{1/2} \right) \]

for \( j = 1, 2, \ldots \) \( \Box \)

The following lemma follows from the proof of Theorem 8 in [13].

**Lemma 2.3.** Let \( A, B, C, D, X \in B(H) \) such that \( X \) is compact, \( 0 \leq A \leq C \), and \( 0 \leq B \leq D \). Then

\[ s_j(A^{1/2}XB^{1/2}) \leq s_j(C^{1/2}XD^{1/2}) \]

for \( j = 1, 2, \ldots \)

The following lemma can be found in [2, p. 123].

**Lemma 2.4.** For \( r \in (0, 1] \), the function \( f(t) = t^r \) is operator concave on \([0, \infty)\).

Using these two lemmas, we obtain the following application of Theorem 2.6.

**Theorem 2.7.** Let \( A, B, X, Y \in B(H) \) such that \( X \) and \( Y \) are compact, \( A \) and \( B \) are normal, and \( r \) is a real number with \( r \geq 2 \). Then

\[ s_j(AX + YB) \leq 2^{1-1/r} s_j \left( (|A|^r + I)^{1/r} X (|B|^r + I)^{1/r} \oplus (|A|^r + I)^{1/r} Y (|B|^r + I)^{1/r} \right) \]

for \( j = 1, 2, \ldots \)

**Proof.** It follows, from Theorem 2.6, that

\[ s_j \left( \frac{AX + YB}{2} \right) \]
\[ = s_j \left( \frac{AXI + IYB}{2} \right) \]
\[ \leq s_j \left( \left( \frac{|A|^2 + I}{2} \right)^{1/2} X \left( \frac{|B|^2 + I}{2} \right)^{1/2} \oplus \left( \frac{|A|^2 + I}{2} \right)^{1/2} X \left( \frac{|B|^2 + I}{2} \right)^{1/2} \right) \]  

(15)

for \( j = 1, 2, \ldots \) By Lemma 2.4, we have \( \frac{|A|^2 + I}{2} \leq \left( \frac{|A|^r + I}{2} \right)^{2/r} \) and \( \frac{|B|^2 + I}{2} \leq \left( \frac{|B|^r + I}{2} \right)^{2/r} \). Now, using these together with Lemma 2.3 and the inequality (15), we have the required result. \( \Box \)

Our final result in this section can be stated as follows. Its proof is similar to that of Theorem 2.7.

**Theorem 2.8.** Let \( A, B, X, Y \in B(H) \) such that \( X \) and \( Y \) are compact, \( A \) and \( B \) are normal, and \( r \) is a real number with \( r \geq 2 \). Then
$$s_j(AXB + Y) \leq 2^{1-1/r}s_j\left((|A|^r + I)^{1/r} X(|B|^r + I)^{1/r} \oplus (|A|^r + I)^{1/r} Y(|B|^r + I)^{1/r}\right)$$

for $j = 1, 2, \ldots$

### 3. Norm inequalities

In this section we give some norm inequalities relating sums and direct sums of operators. A basic result in this direction is the inequality (4). For an extension to $n$-tuples of positive operators and applications of this inequality, we refer to [3,4,12].

In view of the inequalities (3), (4), and (6), one might conjecture that if $X, Y \in B(H)$, and $0 < \alpha < 1$, then

$$|||(\alpha X + (1 - \alpha)Y) \oplus (\alpha X + (1 - \alpha)Y)||| \leq \sqrt{2\alpha X \oplus 2(1 - \alpha)Y}$$

for every unitarily invariant norm on $B(H \oplus H)$. However, this inequality is refuted by the two-dimensional example $X = I$ and $Y = 0$, with the trace norm and $\frac{1}{2} < \alpha < 1$.

Next, we give a generalization of the inequality (4) along the lines of the inequality (5). To do this, we need the following four lemmas. The first lemma is a special case of Theorem 2 in [14]. The second and the fourth lemmas follow from Theorems 1 and 3 in [11], respectively. The third lemma is an extension to $n$-tuples of positive operators of a special case of Corollary 1 in [1] (see, also [6]).

**Lemma 3.1.** Let $A_i, B_i, X_i \in B(H), \quad i = 1, \ldots, n$, and $r > 0$. Then

$$\left\| \sum_{i=1}^{n} A_i^* X_i B_i \right\|^r \leq \left( \sum_{i=1}^{n} A_i^* |X_i| A_i \right)^r \leq \left( \sum_{i=1}^{n} B_i^* |X_i| B_i \right)^r$$

for every unitarily invariant norm.

**Lemma 3.2.** Let $A_i, X_i \in B(H), \quad i = 1, \ldots, n$, such that $\sum_{i=1}^{n} A_i^* A_i \leq I$, and $r \geq 1$. Then

$$\left\| \sum_{i=1}^{n} A_i^* |X_i| A_i \right\|^r \leq \sum_{i=1}^{n} A_i^* |X_i| A_i$$

for every unitarily invariant norm.

**Lemma 3.3.** Let $A_i \in B(H)$ be positive, $i = 1, \ldots, n$, and $0 < r < 1$. Then

$$\left\| \left( \sum_{i=1}^{n} A_i \right)^r \right\| \leq \left\| \sum_{i=1}^{n} A_i^r \right\|$$

for every unitarily invariant norm.

**Lemma 3.4.** Let $A_i, X_i \in B(H)$ such that $A_i$ is a contraction, $0 < \alpha_i < 1$, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} \alpha_i = 1$, and $r \geq 2$. Then

$$\left\| \left( \sum_{i=1}^{n} \alpha_i X_i A_i \right)^r \right\| \leq \left\| \sum_{i=1}^{n} \alpha_i A_i^* |X_i| A_i \right\|$$

for every unitarily invariant norm.
Now, we are ready to present our main results in this section, which involve operator weighted means. The first result is a considerable generalization of the inequality (4).

**Theorem 3.1.** Let \( A_i, B_i, X_i \in B(H) \), \( i = 1, \ldots, n \), such that \( \sum_{i=1}^{n} |A_i|^2 \leq 1 \), \( \sum_{i=1}^{n} |B_i|^2 \leq 1 \), and \( r \geq 1 \). Then
\[
\left\| \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} A_i^* X_i B_i \right\|^{r} \leq \left( \sum_{j=1}^{n} \left\| A_j^* X_j^r A_j \right\| \right)^{1/r} \left( \sum_{j=1}^{n} \left\| B_j^* X_j^{1/r} B_j \right\| \right)^{1/(1/r)}
\]
for every unitarily invariant norm. In particular, if \( X_i \) is normal, \( i = 1, \ldots, n \), then
\[
\left\| \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} A_i^* X_i A_i \right\|^{r} \leq n \left\| A_i^* X_i^r A_i \right\|
\]
for every unitarily invariant norm.

**Proof.** On \( \bigoplus_{i=1}^{n} H \), define the operators \( \tilde{A}_1 = \text{diag}(A_1, \ldots, A_n) \), \( \tilde{A}_2 = \text{diag}(A_2, \ldots, A_n, A_1) \), \ldots, \( \tilde{A}_n = \text{diag}(A_n, A_1, \ldots, A_{n-1}) \). Let \( \tilde{B}_i \) and \( \tilde{X}_i \) be defined in the same manner as \( \tilde{A}_i \), \( i = 1, \ldots, n \). Then
\[
\bigoplus_{j=1}^{n} \left( \sum_{i=1}^{n} A_i X_i B_i \right) = \sum_{i=1}^{n} \tilde{A}_i \tilde{X}_i \tilde{B}_i \tag{16}
\]
\[
\left\| \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \right\| = \left\| \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \right\| = \left\| \bigoplus_{i=1}^{n} A_i^* X_i^{1/r} B_i \right\|, \tag{17}
\]
and
\[
\left\| \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \right\| = \left\| \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \right\| = \left\| \bigoplus_{i=1}^{n} A_i^* X_i^{1/r} B_i \right\| \tag{18}
\]
for \( i, j = 1, \ldots, n \). Now,
\[
\left\| \bigoplus_{j=1}^{n} \sum_{i=1}^{n} A_i^* X_i B_i \right\|^{r} = \left\| \sum_{i=1}^{n} \tilde{A}_i \tilde{X}_i \tilde{B}_i \right\|^{r} \quad \text{(by (16))}
\]
\[
\leq \left\| \left( \sum_{i=1}^{n} \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{A}_i \right)^{1/r} \right\|^{1/r} \left\| \left( \sum_{i=1}^{n} \tilde{B}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \right)^{1/r} \right\|^{1/r} \quad \text{(by Lemma 3.1)}
\]
\[
\leq \left\| \sum_{i=1}^{n} \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{A}_i \right\| \sum_{i=1}^{n} \tilde{B}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \quad \text{(by Lemma 3.2)}
\]
\[
\leq \left( \sum_{i=1}^{n} \left\| \tilde{A}_i^* \tilde{X}_i^{1/r} \tilde{A}_i \right\| \right) \left( \sum_{i=1}^{n} \left\| \tilde{B}_i^* \tilde{X}_i^{1/r} \tilde{B}_i \right\| \right) \quad \text{(by the triangle inequality)}
\]
Theorem 3.3. For every unitarily invariant norm 
\[ \sum_{i=1}^{n} A_i^* |X_i^r A_i| \] 
which yields the desired result. \( \square \)

It should be mentioned here that the inequality (4) follows, from Theorem 3.1, by letting \( n = 2, r = 1 \), and \( A_i = B_i = \frac{1}{\sqrt{2}} I \) for \( i = 1, 2 \).

Based on Lemmas 3.1 and 3.3, and using an argument similar to that used in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. Let \( A_i, B_i, X_i \in B(H), i = 1, \ldots, n \), such that \( \sum_{i=1}^{n} |A_i|^2 \leq I, \sum_{i=1}^{n} |B_i|^2 \leq I \), and \( 0 < r \leq 1 \). Then
\[ \frac{1}{n} \left\| \sum_{j=1}^{n} \sum_{i=1}^{n} A_i^* X_i B_i \right\|^r \leq \left\| \sum_{i=1}^{n} (A_i^* |X_i^r A_i|) \right\|^r \leq \left\| \sum_{i=1}^{n} (B_i^* |X_i| B_i) \right\|^r \]
for every unitarily invariant norm. In particular, if \( X_i \) is normal, \( i = 1, \ldots, n \), then
\[ \left\| \sum_{j=1}^{n} \sum_{i=1}^{n} A_i^* X_i A_i \right\|^r \leq \left\| \sum_{i=1}^{n} (A_i^* |X_i| A_i)^r \right\|^r \]
for every unitarily invariant norm.

Lemma 3.4 enables us to give another result related to the inequality (4).

Theorem 3.3. Let \( A_i, X_i \in B(H) \) such that \( A_i \) is a contraction, \( 0 < \alpha_i < 1 \), \( i = 1, \ldots, n \), with \( \sum_{i=1}^{n} \alpha_i = 1 \), and \( r \geq 2 \). Then
\[ \left\| \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_i X_i A_i \right\|^r \leq \left\| \sum_{i=1}^{n} A_i^* |X_i^r A_i| \right\|^r \]
for every unitarily invariant norm.

Proof. Let \( \tilde{A}_i \) and \( \tilde{X}_i, i = 1, \ldots, n \), be defined as in the proof of Theorem 3.1. Then
\[ \left\| \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_i X_i A_i \right\|^r = \left\| \sum_{i=1}^{n} \alpha_i \tilde{X}_i \tilde{A}_i \right\|^r \]
\[ \leq \left\| \sum_{i=1}^{n} \alpha_i \tilde{A}_i^* |\tilde{X}_i^r \tilde{A}_i| \right\|^r \quad \text{(by Lemma 3.4)} \]
\[ \leq \sum_{i=1}^{n} \alpha_i \left| \tilde{A}_i^* |\tilde{X}_i^r \tilde{A}_i| \right| \quad \text{(by the triangle inequality)} \]
\[ = \left| \tilde{A}_i^* |\tilde{X}_1^r \tilde{A}_1| \right| \quad \text{(by (17))} \]
\[ = \left\| \sum_{i=1}^{n} A_i^* |X_i^r A_i| \right\|^r \quad \text{(by (17))}, \]
as required. \( \square \)
References