A Decomposition Theory for Matroids.
I. General Results*

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A new matroid decomposition with several attractive properties leads to a new
theorem of alternatives for matroids. A strengthened version of this theorem for
binary matroids says roughly that to any binary matroid at least one of the follow-
ing statements must apply: (1) the matroid is decomposable, (2) several elements
can be removed (in any order) without destroying 3-connectivity, (3) the matroid
belongs to one of 2 well-specified classes or has 10 elements or less. The latter
theorem is easily specialized to graphic matroids. These theorems seem particularly
useful for the determination of minimal violation matroids, a subject discussed in
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1. INTRODUCTION

We propose a new decomposition for matroids. The scheme is defined
for matroids of arbitrary connectivity. In this part we investigate in detail
the cases up to and including 4-connectivity. The decomposition has
several attractive properties: (1) If a $k$-connected matroid ($k \geq 3$) is decom-
posed, then the "pieces" are proper 3-connected minors; (2) the decom-
position can be dualized, i.e., if $M$ can be decomposed into $M_1$ and $M_2$,
then $M^*$ can be decomposed into $M_1^*$ and $M_2^*$, where the asterisk indicates
the dual; (3) if $M$ is representable over a given field, or if the decom-
position is based on 1- or 2-separability of $M$, then one easily composes $M_1$
and $M_2$ to obtain $M$ again.

With the aid of the decomposition we derive a new theorem of alter-
 natives for matroids which makes precise the intuitive notion that one can
either decompose a matroid or remove a number of specified elements in
any order without destroying 3-connectivity. The actual situation is not
quite as simple, but the exceptions can be well characterized. The theorem

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can be considerably strengthened when the matroid is binary. In that case roughly one of 3 situations must occur: (1) the matroid is decomposable, (2) a number of specified elements can be removed without destroying 3-connectivity, or (3) the matroid belongs to one of 2 well-described classes or has at most 10 elements. The latter theorem is sufficiently complex that a version restricted to graphs is still of interest.

The theorems seem particularly useful for, and indeed were motivated by, investigations of minimal violation matroids of inherited properties. Let \( P \) be such a property and \( M \) be a minimal violation matroid, i.e., all proper minors of \( M \) have \( P \) but \( M \) does not. The theorems given here then may be useful to establish certain structural properties of \( M \). In part II, this notion will be made precise for binary matroids. There we define \( P \) to have the composition property if any decomposable matroid of (1) above has \( P \) provided the components have \( P \). We say that \( P \) has the extension property if any matroid of (2) above has \( P \) provided all proper minors have \( P \). We then obviously have the theorem that any binary minimal violation matroid \( M \) of any inherited property \( P \) is an instance of the cases of (3) if \( P \) has the composition and extension properties. Graphicness, planarity, and regularity have all or almost all of the desired composition and extension properties, and thus by routine arguments one produces the well-known minimal violation matroids.

Composition/decomposition results for matroids abound in the literature (see [1, 3, 4, 6] and the references cited therein), but the schemes we are aware of address low connectivity (typically 2-separability, e.g., in [1, 4, 6]), or they apply to composition but not decomposition (e.g., in [3, 4]), or they do not allow dualizing (e.g., in [8]). The reader should not be misled by the preceding observation, which is not meant to be critical of the excellent work of the cited references. It only says that these schemes were not suitable for our purposes. We should emphasize that in this paper we treat composition under some restrictive assumptions, and that more work is needed to fully understand the general composition case. We further note that for 2-separable matroids our approach is effectively the same as those described in the above references. For highly connected matroids, however, our decomposition/composition is materially different from any method we know.

The remainder of this section introduces relevant definitions. The subsequent Section 2 lists the decomposition theorems. Their proofs have been broken down into several steps given in Sections 3–6. First we show how a \( k \)-separation of a certain kind leads to a \( k \)-sum decomposition (Sect. 3). In Section 4 we treat 3-connected matroids with triangles and triads, then analyze 4-connected matroids in Section 5. Finally, in Section 6 we combine these results to obtain the desired proofs.

Let \( X \) be a base of a matroid \( M \) on a set \( S \), and \( Y = S - X \). Construct a
{0, 1}-matrix $\tilde{B} = [I \mid B]$ as follows. $S$ is to be the set of column indices of $\tilde{B}$; in particular $X$ is to index the columns of the identity $I$, say, in the order $x_1, x_2, \ldots, x_m$. Then we index the rows by $x_1, x_2, \ldots, x_m$ as well. Let $y \in Y$, and suppose $\overline{X}$ is the subset of $X$ that forms a circuit with $y$. Then in the column of $B$ with index $y$, we set element $B_{xy}$ equal to 1 if $x \in \overline{X}$, and equal to 0 otherwise. Any $\tilde{B}$ that may be so constructed from $M$ is a partial representation of $M$, and it is nothing but a matrix representation of the fundamental circuit set of Whitney [15]. Note that this construction can always be carried out unless $M$ consists only of loops. In the latter case we may formally take $\tilde{B}$ to be a matrix without rows (we call a matrix without rows or columns empty).

We have utilized partial representations in prior work [9–11], and the arguments to follow rely on the matrix theory for such representations developed in [11]. The definitions are motivated by the well-known relationships between the bases of a representable matroid and a related standard representation matrix $\tilde{A} = [I \mid A]$. That is, the bases of the matroid are in one-to-one correspondence to the nonsingular submatrices of $A$ save for the base corresponding to the submatrix $I$ of $\tilde{A}$. Thus we define for any square submatrix $B$ of $\tilde{B}$, say specified by $\overline{X} \subseteq X$ and $\overline{Y} \subseteq Y$, a determinant, $\det B$, which is declared to be 1 if $(X - \overline{X}) \cup \overline{Y}$ is a base of $M$, and to be 0 otherwise. This definition is extended to square submatrices $\tilde{B}$ of $\tilde{B}$, say specified by $\overline{X} \subseteq X$ and $\overline{Z} \subseteq X \cup Y$, by defining $\det \tilde{B}$ to be equal to 1 if the set $(X - \overline{X}) \cup \overline{Z}$ is a base of $M$, and to be equal to 0 otherwise. There may be another submatrix of $\tilde{B}$ that (possibly after row and/or column permutations) is numerically identical to $\tilde{B}$. The related row and column index sets are not both identical to $\overline{X}$ and $\overline{Z}$, respectively, and for this reason we will consider such a matrix to be different from $\tilde{B}$. Thus for mathematical exactness we could specify $\tilde{B}$ of $\tilde{B}$ by the triple $(\overline{X}, \overline{Z}, X)$.

We avoid this cumbersome notation since confusion of $\tilde{B}$ with some other matrix seems unlikely. We also use expressions like "$B$ is singular (nonsingular)" with the obvious interpretation. If $\tilde{B}$ is a not necessarily square submatrix of $B$, we define $\text{rank}(\tilde{B})$ to be the order of the largest nonsingular submatrix of $B$. If $\tilde{B}$ is indexed by $\overline{X} \subseteq X$ and $\overline{Z} \subseteq X \cup Y$, then it is easily verified that $\text{rank}(\tilde{B}) = r((X - \overline{X}) \cup \overline{Z}) - |X - \overline{X}|$, where $r(\cdot)$ is the rank function of $M$. Submodularity of $r(\cdot)$ of $M$ (i.e., $r(S_1 \cap S_2) + r(S_1 \cup S_2) \leq r(S_1) + r(S_2)$) is equivalent to a useful inequality involving the function $\text{rank}(\cdot)$. Suppose $\tilde{B}$ has a submatrix

\[
\tilde{B} = \begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}
\]
Compose $\overline{B}^1$ ($\overline{B}^2$) from $B^{12}, B^{22}, B^{32}$ ($B^{21}, B^{22}, B^{23}$), and define $\overline{B}^1$ ($\overline{B}^2$) from $B^{11}, B^{12}, B^{21}, B^{22}$ ($B^{22}, B^{23}, B^{32}, B^{33}$). Using the relationship given above between rank($\cdot$) and $r(\cdot)$ as well as the submodularity of $r(\cdot)$ one then concludes that rank($\overline{B}^1$) + rank($\overline{B}^2$) $\leq$ rank($\overline{B}^1$) + rank($\overline{B}^2$). We refer to this property as the submodularity of the function rank($\cdot$). Note that rank($\overline{B}^1$) + rank($\overline{B}^2$) $\leq$ rank($B^{22}$) + rank($\overline{B}$) is a special case of the submodularity inequality, where $\overline{B}$ has effectively been subdivided into just 4 submatrices. A blocktriangular submatrix is nonsingular if and only if each block is nonsingular. In particular, any square matrix that may be partitioned as

![Matrix Diagram](image)

is singular. A pivot on $\hat{B}_{xy} = 1$ of $\hat{B}$ transforms $\hat{B}$ into the partial representation corresponding to the basis $(X - \{x\}) \cup \{y\}$. We may evaluate determinants by pivots as follows. Let a square submatrix $\overline{B}$ of $B$ of order at least 2 be indexed by $\overline{X}$ and $\overline{Z}$ as before. If we pivot on the $(x, y)$-element of $\overline{B}$, then in the resulting partial representation the submatrix indexed by $\overline{X} - \{x\}$ and $\overline{Z} - \{y\}$ has the same determinant as $\overline{B}$.

Two nonzero rows (columns) of a $\overline{B}$ are parallel if the submatrix consisting of these rows (columns) has rank equal to 1. The matrix $[B' | I]$ is a partial representation of $M^*$, the dual of $M$, and any square submatrix $\overline{B}$ of $B$ is nonsingular if and only if $(\overline{B})'$ is nonsingular in $B'$ (in the triple notation mentioned above, $(\overline{B})'$ becomes $(Z, X, Y)$). If we delete a column with index $y \in Y$ from $\overline{B}$, we obtain a partial representation of $M \setminus y$, where \ denotes deletion. (If $e$ is an element, we write $M \setminus e$ and $M/e$ instead of $M \setminus \{e\}$ and $M/\{e\}$ to unclutter the notation.) The determinants of the square submatrices of the reduced matrix are unchanged by such a column deletion. By duality a deletion of a row and column of $\overline{B}$ with index $x \in X$ produces a partial representation of $M/\{x\}$, where / denotes contraction. Again, the determinants are not affected by this operation. An addition (expansion) is the inverse of a deletion (contraction). An extension is an addition or an expansion. Note that the definition of contraction differs from that by Tutte (see, e.g., [12]), and that another definition of extension is given by Welsh [14, p. 321].

Let $A$ be a matrix. Then $\hat{A}$ denotes $[I | A]$, where $I$ is an identity of appropriate order. In the display of matrices unspecified entries are always
to be taken as 0. We typically write the index sets of the columns above a matrix and those of the rows to the left of it, and for $\tilde{A}$ the column indices of $I$ are always the same as the row indices of $\tilde{A}$. If $A$ has size $m \times n$, then the length of $A$ is 0 if $m$ or $n$ is 0, and it is equal to $m + n$ otherwise. We define $G(A)$ to be the following bipartite graph. Each row and each column of $A$ generates a node, and each nonzero $A_{ij}$ leads to an edge connecting nodes $i$ and $j$. We say that $A$ is connected if $G(A)$ is connected. Partial representations allow a simple characterization of matroid connectivity as follows.

**Lemma 1.1** (Cunningham [4] and Krogdahl [7]). Let $\tilde{B}$ be a partial representation of a matroid $M$. Then $M$ is connected if and only if $\tilde{B}$ is connected.

Let $M$ be a matroid with rank function $r(\cdot)$ on a set $S$. If 2 elements of $S$ form a circuit in $M (M^*)$, they are said to be parallel (series) elements. Any circuit of cardinality equal to 3 in $M (M^*)$ is a triangle (triad). $M$ is $k$-separable [12] if $S$ can be partitioned into $S_1$ and $S_2$ such that $|S_1|, |S_2| \geq k$ and $r(S_1) + r(S_2) \leq r(S) + k - 1$. The pair $(S_1, S_2)$ is then a $k$-separation of $M$, which manifests itself in the previously defined $\tilde{B}$ as follows. Let $X_i = X \cap S_i$ and $Y_i = Y \cap S_i, i = 1, 2$. If we partition $B$ as

\[
\begin{array}{c|c|c}
\hline
& Y_1 & Y_2 \\
\hline
X_1 & B^{11} & B^{12} \\
\hline
X_2 & B^{21} & B^{22} \\
\hline
\end{array}
\]

then by the previously mentioned relationship between rank($\cdot$) and $r(\cdot)$ we have $\text{rank}(B^{12}) = r(X_2 \cup Y_2) - |X_2|$ and $\text{rank}(B^{21}) = r(X_1 \cup Y_1) - |X_1|$, and therefore $r(S_1) + r(S_2) \leq r(S) + k - 1$ if and only if $\text{rank}(B^{12}) + \text{rank}(B^{21}) \leq k - 1$. $M$ is $k$-connected if it has no $l$-separation, $l \leq k - 1$; in the case of $k = 2$, $M$ is also said to be connected. For a given $k \geq 2$, $M$ is $(k+)$-separable if

1. $M$ is $(\lceil k/2 \rceil + 1)$-connected,
2. both the rank and corank of $M$ are at least $k$, and
3. $M$ has a $k$-separation where the sets $S_1$ and $S_2$ satisfy $|S_1|, |S_2| \geq k + 1$.

Here $\lceil n \rceil$ denotes the smallest integer greater than or equal to $n$. The pair $(S_1, S_2)$ is then a $(k+)$-separation of $M$. A $k$-separation or $(k+)$-separation $(S_1, S_2)$ is exact if $r(S_1) + r(S_2) = r(S) + k - 1$. Throughout parts I and II we are not interested in any differences between isomorphic
matroids, and hence consider any 2 such matroids to be equal. However, in part III differences between isomorphic matroids are important, and we use \( \cong \) to denote "is isomorphic to." \( M \) is a 1-sum if it is the disjoint union of 2 matroids \( M_1 \) and \( M_2 \). This situation is denoted by \( M = M_1 \oplus M_2 \). \( M \) is a \( k \)-sum, \( k \geq 2 \), if \( M \) has a partial representation \( \tilde{B} \) where

\[
B = \begin{array}{c|c}
A^1 & \bar{O} \\
\hline
C^1 & \bar{D} \\
\hline
D^1 \bar{C}^2 & A^2 \\
\end{array}
\]

(\( \varnothing.1 \))

observes the following conditions:

(a) \( C_1 (C^2) \) is a connected nonempty proper submatrix of \( A_1 (A^2) \), and it has no nested rows (columns).

(b) \( \bar{D} \) is a nonsingular matrix and \( \text{rank} (\bar{D}) = \text{rank} (D) = k - 1 \), where

\[
D = \begin{array}{c|c}
\bar{n}^1 & \bar{n} \\
\hline
\bar{D}^{12} & \bar{D}^2 \\
\end{array}
\]

(\( \varnothing.2 \))

We recall that 2 \( \{0, 1\} \)-vectors \( c \) and \( d \) are nested if \( c_j = 1 \) implies \( d_j = 1 \), \( \forall j \), or \( d_j = 1 \) implies \( c_j = 1 \), \( \forall j \). By the previous observations the matrices \( \tilde{B}^1 \) and \( \tilde{B}^2 \) defined by

\[
\begin{align*}
\tilde{B}^1 &= \begin{array}{c|c}
A^1 & \bar{O} \\
\hline
C^1 & \bar{D} \\
\hline
D^1 \bar{C}^2 & A^2 \\
\end{array} \\
\tilde{B}^2 &= \begin{array}{c|c}
\bar{C}^1 & \bar{O} \\
\hline
\bar{D} & \bar{C}^2 \\
\hline
\bar{D}^2 & A^2 \\
\end{array}
\end{align*}
\]

(\( \varnothing.3 \))

are partial representations of \( M_1 = M/(X_2 - \bar{X}_2) \backslash (Y_2 - \bar{Y}_2) \) and \( M_2 = \)
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M(J(X1 - X1) \setminus (Y1 - Y1)) respectively, and the determinants of their submatrices agree with those of B. We call M1 and M2 the components of the k-sum and write M = M1 ⊕ k M2.

It seems worthwhile that we motivate this decomposition with a specific example. Suppose M is the polygon matroid of a 3-connected graph G, and that we can create G by identifying 4 nodes of a 3-connected graph H1 with 4 nodes of another 3-connected graph H2. We then may claim that G is decomposable into H1 and H2.

Suppose now we are given just H1 and H2, and not G. To create the latter graph, we must know the 4 nodes of H1 and of H2, and also the way in which these nodes are to be identified. For graphs this information can be provided rather compactly, but this is no longer so when we only have the polygon matroids for H1 and H2. The difficulty comes from the fact that a polygon matroid does not explicitly display the nodes of any graph creating it.

One may try to overcome this difficulty by attaching to the 4 nodes of Hi, i = 1, 2, some other graph Gi (typically Gi is proposed to be a complete graph). We then create G by first identifying the edges of Gi of H1 ∪ Gi with the edges of Gi of H2 ∪ Gi, then deleting the identified edges. This idea can be translated to matroids (see, e.g., the 3-sum definition of [S]), but it has 2 serious drawbacks: (1) the composition/decomposition generally is not dualizable, and (2) no Gi may exist such that (2a) both H1 ∪ Gi and H2 ∪ Gi are minors of G, and (2b) identification of the edges of Gi in the polygon matroid of H1 ∪ Gi with the edges of Gi in the polygon matroid of H2 ∪ Gi produces the polygon matroid of G.

The decomposition given by (9.1)-(9.3) shows a way out of this dilemma. The main idea looks rather awkward in graph terminology, but becomes pleasantly simple when described via partial representations for matroids. Suppose G contains a minor Gi such that some edges of Gi occur in H1, and others in H2. Indeed, we want G to have 4 nodes such that removal of these nodes from Gi creates 2 or more connected components each of which contains only edges of H1 or only edges of H2. Let B of (9.1) be a partial representation of the polygon matroid of G. Then Xi ∪ Yi is the edge set of Hi, i = 1, 2, and X1 ∪ X2 ∪ Y1 ∪ Y2 is the edge set of Gi. Thus we can derive Gi from G by contracting (X1 - X1) ∪ (X2 - X2) and by deleting (Y1 - Y1) ∪ (Y2 - Y2). Equivalently, Gi, the polygon matroid of Gi, is the minor defined by the submatrix of B containing C1, C2, and D. Instead of the Hi ∪ Gi above, we now derive graphs G1, i = 1, 2, from G by carrying out some but not all of the above contractions and deletions. Specifically, we contract (X2 - X2) and delete (Y2 - Y2) to create G1. For G2 the respective sets are (X1 - X1) and (Y1 - Y1). The partial representations of the related polygon matroids are given in (9.3), where B' defines the polygon matroid of G1, i = 1, 2, We assemble G from G1 and G2 by
identifying the minor $\tilde{G}$ of $G_1$ with the minor $\tilde{G}$ of $G_2$, a rather awkward process when worked out on an example. Theorem 2.3, part (d.4), of the next section shows a typical situation. The construction becomes simple and appealing when translated to the partial representations of the polygon matroids. We just overlay $B^1$ and $B^2$ such that $C^1$, $C^2$, and $D$ of $B^1$ exactly cover $C^1$, $C^2$, $D$ respectively, then fill in $D^{12}$ and define appropriate determinants for all submatrices intersecting $D^{12}$, to get the matrix $B$ of (2.1) for the polygon matroid of $G$.

In our example $\tilde{M} = M/(X_1 - \tilde{X}_1) \cup (X_2 - \tilde{X}_2) \cup (Y_1 - \tilde{V}_1) \cup (Y_2 - \tilde{V}_2)$ is the polygon matroid of $\tilde{G}$, and for this reason we call $\tilde{M}$ the connecting matroid of the decomposition. The conditions (a), (b) of (2.2) imply some attractive properties for $\tilde{M}$. Ruling out nestedness in (a) guarantees that $\tilde{M}$ is not unnecessarily large, while (b) makes the fill-in process for $D^{12}$ mentioned previously straightforward if both $B^1$ and $B^2$ are actually matrices over a given field; details are given in the next paragraph. Together (a) and (b) assure $\tilde{M}$ to be 3-connected if $k \geq 3$, a fact proved in Section 3.

We now describe some properties and general results of the decomposition. It is trivial to verify that $\oplus_k$ is not commutative, for all $k \geq 2$, and that $M = M_1 \oplus_k M_2$ if and only if $M^* = M_2^* \oplus_k M_1^*$. To date we have just begun to explore conditions for composition of $M_1$ and $M_2$, and much more work is needed to fully understand the situation. From our example it is clear that the fill-in of $D^{12}$ is the difficult step when one intends to compose $B^1$ and $B^2$ of (2.3) to $B$ of (2.1). Here we discuss just 2 situations where the latter step is very easily carried out. In the first case we have $k = 2$, i.e., $D = [1]$. We define $D^{12}$ to be the real matrix product $D^2D^1$ and declare $\text{rank}(D) = 1$. With these definitions the determinant of any submatrix of $[D_B]$ or $[D_A^2]$ is obviously established. Simple checking reveals that the determinant of any other submatrix of $B$ can now be uniquely deduced, using the previously cited determinant rule for block triangular matrices plus at most one pivot. Indeed, the list of determinants so generated is that of a matroid, and by the arguments just made only one matroid $\tilde{M}$ (which we denote by $M_1 \oplus_2 M_2$) can be created in this way. Of course, the above procedure is nothing but a translation of a well-known composition result into our notation. In the second situation we deal with any $k \geq 2$, but suppose that $M_1$ and $M_2$ are representable over a given field, and that they have standard representation matrices $B^1$ and $B^2$ (not necessarily $\{0, 1\}$), where $B^1$ and $B^2$ are given by (2.3). If these matrices satisfy (2.2) with $D^{12} = D^2DD^1$, where $D$ is the inverse of $D$, then we can compose $M_1$ and $M_2$ to a matroid $M$ (which we denote by $M_1 \oplus_k M_2$) by defining $M$ to be the matroid represented by $\tilde{B}$ with $B$ of (2.1) over that field, where the submatrices of $B$ are those of (2.3) and $D^{12}$ is the matrix just specified. If both $M_1$ and $M_2$ are representable over 2 or more fields, this composition procedure may generate several matroids depending on
the field over which $B^1$ and $B^2$ are expressed. For example, if we choose $B^1$ and $B^2$ as

$$B^1 = \begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}$$

and

$$B^2 = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}$$

then for any field both $M_1$ and $M_2$ are equal to the regular matroid $R_{10}$ of $[2,8]$. $M_1$ and $M_2$ can be assembled to a 4-sum, and $D^{12}$ (a scalar) is equal to 0 or 2 depending on whether we view $B^1$ and $B^2$, for example, as matrices over $GF(2)$ or the reals. Incidentally, this example also demonstrates that a 4-sum composed from two regular matroids need not be regular since regularity is equivalent to representability over every field. Though the notation "$M_1 \oplus_k M_2$" may be ambiguous when it is used for composition and $k \geq 3$, this will not cause any difficulty since the underlying field will always be clear from the context.

A $k$-sum is proper if both submatrices $A^1$ and $A^2$ of $B$ of (9.1) are connected; it is semi-proper if one of the following 2 situations prevails: either $A^1$ is connected and $A^2$ is equal to $C^2$ with one additional zero row adjoined to the bottom of $C^2$, or $A^2$ is connected and $A^1$ is equal to $C^1$ with one additional zero column adjoined to the left-hand side of $C^1$. Some results of part II motivated these definitions as follows. Suppose we know that $M_1$ and $M_2$ of (2.3) are graphic and 3-connected, and we want to claim that $M$ of (9.1) is graphic if it is binary. In part II submatrix $C^1$ of $B^1$ is just a vector of 1s, so $\bar{Y}_1 \cup \bar{Y}_1$ is a cutset of the graph $G^2$ producing $M^2$. It turns out that we want that cutset to be a star of $G^2$ to claim graphicness for $M$. Obviously the cutset is a star if and only if removal of the cutset edges from $G_2$ produces a 2-connected graph. The latter requirement is equivalent to the demand that $A^2$ be connected. These considerations plus a few others too long to be discussed here, led us to believe that connectedness of $A^1$ and/or $A^2$ is an attractive property, and motivated the above definitions.

To reduce confusion we derive the $k$-connectivity definitions ($k \geq 2$) for graphs from those for matroids, so a graph is $k$-connected, $k \geq 2$, if that is true for the associated graphic matroid. However, when a graph is claimed to be connected we mean that every pair of nodes is joined by a path. This is not the same as saying that the (graphic) matroid is connected; the latter statement implies that the matroid (and hence the graph) is 2-connected. Note that this definition of $k$-connectivity for graphs, $k \geq 2$, implies the customary one (defined via removal of nodes), and hence the existence of $k$ internally node-disjoint paths between any two disjoint node sets. The exact relationships are nicely proved in [5].
Let $G$ be a 2-connected graph. An edge $e$ is parallel to an edge $f$ if $e$ has the same endpoints as $f$. If $e$ has a degree 2 endpoint in common with $f$, or if $e$ is connected to $f$ by a path all of whose nodes have degree 2 in $G$, then $e$ and $f$ are in series. Let $M$ be the (connected) polygon matroid defined by $G$. Then 2 edges of $G$ are parallel if and only if the related elements of $M$ are parallel, and 2 series edges of $G$ give rise to 2 series elements in $M$. However, the converse of the latter statement is not true. Thus special attention is needed when statements about series elements of $M$ are translated to claims about $G$.

$K_n (K_{n,m})$ refers to the complete (complete bipartite) graph on $n$ ($m$ and $n$) nodes. $W_m$, $m \geq 3$, is the wheel with $m$ spokes. From $W_m$ one derives a nongraphic matroid, the whirl $\mathcal{W}_m$, by declaring the rim of the wheel to be independent. It is convenient to consider $\mathcal{W}_3$, the rank 2 uniform matroid on 4 elements, to be a whirl and to designate it by $\mathcal{W}$.

Now and then we use the term "efficient algorithm." By this we mean an appropriate Turing machine which relies on an independence black box to decide dependence/independence of the sets of the given matroid, and whose total effort for producing the desired answer is bounded by a polynomial in the size of the groundset of the matroid. Any other matroid terminology used later may be found in the book by Welsh [14].

2. Decomposition Theorems

In this section we state the main theorems of this part. Their proofs rely on the lemmas and theorems of Sections 3–5, and are given in Section 6. The notation used below in connection with $k$-sums is that of (2.1)–(2.3).

Theorem 2.1 (General matroids). Every matroid $M$ observes at least one of the following conditions, where (d.1)–(d.4) imply decomposability or $(k^+)$-separability, (r.1)–(r.3) state that some elements can be removed while maintaining certain connectivity conditions, and (s.1), (s.2) define 2 special classes:

(d.1) $M$ is a 1-sum.

(d.2) $M$ is a connected proper 2-sum, and has no series or parallel elements.

(d.3) $M$ is a 3-connected 3-sum.

(d.4) $M$ has a $(4^+)$-separation.

(r.1) $M$ has series or parallel elements.

(r.2) $M$ is 3-connected and has at least one triangle or triad. For every triangle $\{e, f, g\}$: $M \setminus e$, $M \setminus f$, $M \setminus g$, $M/e \setminus g$, $M/f \setminus e$, $M/g \setminus f$, $M/\{e, f, g\}$ are
all 3-connected, and $M \setminus \{e, f, g\}$ is connected. For every triad $\{e, f, g\}: M / e$, $M / f$, $M / g$, $M \setminus e / g$, $M \setminus f / e$, $M \setminus g / f$, $M \setminus \{e, f, g\}$ are all 3-connected, and $M \setminus \{e, f, g\}$ is connected.

(r.3) $M$ is 4-connected. For all pairs $\{e, f\}$ of distinct elements, $M \setminus \{e, f\}$ is 3-connected or has series elements, $M / \{e, f\}$ is 3-connected or has parallel elements, and $M \setminus e / f$ is 3-connected. There exist 2 distinct elements $e$ and $f$ such that $M \setminus \{e, f\}$ or $M / \{e, f\}$ is 3-connected; for any 2 such elements there exists a third one, say $g \neq e, f$, such that at least one of the minors $M \setminus \{e, f, g\}$, $M \setminus \{e, f\}/g$, $M / \{e, f, g\}$, $M / \{e, f\}\setminus g$ as well as all extensions of that minor in $M$ are 3-connected.

(s.1) $M$ is 4-connected, and the rank and corank of $M$ are at most 6. For all pairs $\{e, f\}$ of distinct elements $M \setminus \{e, f\}$ has series elements, $M / \{e, f\}$ has parallel elements, and $M \setminus e / f$ is 3-connected.

(s.2) $M$ has at most 9 elements, or the rank or corank of $M$ is 3 or less.

On the surface (d.3) and (d.4) permit a wide range of possibilities. However, the "shifting" algorithm of Section 3 reduces that range to just a few well-described cases (see Theorem 3.3).

Much more can be said when we assume the matroid $M$ to be binary. Note, however, that (r.2) and (r.2*) below are weaker than (r.2) of Theorem 2.1. In return we gain the interesting case (s.2) of Theorem 2.2.

THEOREM 2.2 (Binary matroids). Every binary matroid $M$ observes at least one of the following conditions, where (d.1)–(d.4) describe proper and semi-proper decomposition cases, (r.1)–(r.3) concern removal of certain elements, and (s.1)–(s.3) define 3 special classes:

(d.1) $M$ is a 1-sum.

(d.2) $M$ is a connected proper 2-sum, and has no series or parallel elements.

(d.3) $M$ is 3-connected and one of (a), (b) below applies.

(a) $M$ is a proper 3-sum with 3-connected components $M_1$ and $M_2$.

(b) $M$ is a semi-proper 3-sum, and the component $M_i$ with connected $A^i$ is 3-connected.

(d.4) $M$ is a 3-connected proper 4-sum with 3-connected components $M_1$ and $M_2$.

(r.1) $M$ has series or parallel elements.

(r.2) $M$ is 3-connected and has a triangle $\{e, f, g\}$ such that $M \setminus e$, $M \setminus f$, $M \setminus g$, $M/e \setminus g$, $M/f \setminus e$, $M/g \setminus f$, $M / \{e, f, g\}$ are all 3-connected, and $M \setminus \{e, f, g\}$ is connected.

(s.1) $M$ is 4-connected, and the rank and corank of $M$ are at most 6. For all pairs $\{e, f\}$ of distinct elements $M \setminus \{e, f\}$ has series elements, $M / \{e, f\}$ has parallel elements, and $M \setminus e / f$ is 3-connected.

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On the surface (d.3) and (d.4) permit a wide range of possibilities. However, the "shifting" algorithm of Section 3 reduces that range to just a few well-described cases (see Theorem 3.3).

Much more can be said when we assume the matroid $M$ to be binary. Note, however, that (r.2) and (r.2*) below are weaker than (r.2) of Theorem 2.1. In return we gain the interesting case (s.2) of Theorem 2.2.

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(a) $M$ is a proper 3-sum with 3-connected components $M_1$ and $M_2$.

(b) $M$ is a semi-proper 3-sum, and the component $M_i$ with connected $A^i$ is 3-connected.

(d.4) $M$ is a 3-connected proper 4-sum with 3-connected components $M_1$ and $M_2$.

(r.1) $M$ has series or parallel elements.

(r.2) $M$ is 3-connected and has a triangle $\{e, f, g\}$ such that $M \setminus e$, $M \setminus f$, $M \setminus g$, $M/e \setminus g$, $M/f \setminus e$, $M/g \setminus f$, $M / \{e, f, g\}$ are all 3-connected, and $M \setminus \{e, f, g\}$ is connected.
(r.2*) M is 3-connected and has a triad \( \{e, f, g\} \) such that \( M/e, M/f, M/g, M\setminus e/g, M\setminus f/e, M\setminus g/f, M\setminus \{e, f, g\} \) are all 3-connected, and \( M/\{e, f, g\} \) is connected.

(r.3) (As in Theorem 2.1).

(s.1) \( M = R_{10} \).

(s.2) There exists a representation matrix \( B \) of \( M \), where \( B \) or \( B' \) is one of the matrices below.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

There are at least 3 blocks of type \([11]\).

Each \( E' \) is nonsingular, and \( m \geq 1 \). If \( m = 1 \) \((m \geq 2)\), one \((one or both)\) of the columns labelled \( y, z \) need not be present. If \( M \) is regular, then each \( E' \) is the matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

and \( y \) must be absent.

(s.3) \( M \) has at most 9 elements.

Below we include a graph version of Theorem 2.2, where the cases are numbered as in Theorem 2.2 to simplify comparisons.

To display any graph, say \( G \), we utilize points for the nodes and line segments for the edges as usual. The drawing of \( G \) is frequently simplified by omission of most nodes and edges. Instead we create one or more connected closed areas using heavy lines. All nodes of \( G \) not explicitly shown are understood to be in the interior of these areas, and any edge not explicitly shown connects 2 nodes (explicitly shown or not) of the same closed area. If a closed area is not labelled, then its interior does not con-
tain any nodes. Paths are indicated by dashed lines connecting 2 nodes. Any 2 such paths have disjoint internal nodes.

**Theorem 2.3 (Graphs).** Every graph \( G \) observes at least one of the following conditions, where (d.1)-(d.4) involve proper or semi-proper decomposition, (r.1)-(r.3) concern removal of elements, and (s.2), (s.3) define 2 special classes:

(d.1) \( G \) is not 2-connected.

(d.2) \( G \) is a 2-connected proper 2-sum, has no series or parallel edges, and may be decomposed as follows:

\[
\begin{align*}
&G \\
&G_1 \\
&G_2
\end{align*}
\]

\( G_1, G_2, G_1/\{e, f\} \) and \( G_2/\{g, h\} \) are 2-connected. \( P_1, P_2 \) may be null paths.

(d.3) \( G \) is 3-connected and a proper or semi-proper 3-sum, and may be decomposed in one of the 3 ways shown below:

(a) Proper 3-sum:

\[
\begin{align*}
&G \\
&P_1 \\
&P_2
\end{align*}
\]

\( G_1 \) and \( G_2 \) are 3-connected, and \( G_1/\{e, f, g\} \) and \( G_2/\{h, i, j\} \) are 2-connected. \( P_1, P_2, \) and \( P_3 \) may be null paths.

(b) Semi-proper 3-sum, case 1:

\[
\begin{align*}
&G \\
&G_1
\end{align*}
\]

\( G_1 \) is 3-connected and \( G_1/\{e, f, g\} \) is 2-connected.
(c) \textit{Semi-proper 3-sum, case 2:}

\[
\begin{array}{c}
\text{G} \\
\text{G}_1 \\
\text{G}_2
\end{array}
\]

\(G_2\) is 3-connected and \(G_2 \backslash \{h, i, j\}\) is 2-connected. \(P_1, P_2, P_3\) may be null paths.

(d.4) \(G\) is a 3-connected proper 4-sum, and may be decomposed as follows:

\[
\begin{array}{c}
\text{G}_1 \\
\text{G}_2
\end{array}
\]

\(G_1\) and \(G_2\) are 3-connected, and \(G_1 \backslash \{e, f, g, h\}\) and \(G_2 \backslash \{i, j, k, l\}\) are 2-connected. \(P_0, P_1, ..., P_4\) may be null paths.

(r.1) \(G\) has series or parallel edges.

(r.2) \(G\) is 3-connected and has a triangle \(\{e, f, g\}\) such that \(G \backslash e, G \backslash f, G \backslash g, G \backslash e \backslash g, G \backslash f \backslash e, G \backslash g \backslash f, G \backslash \{e, f, g\}\) are all 3-connected, and \(G \backslash \{e, f, g\}\) is 2-connected.

(r.2*) \(G\) is 3-connected and has a star \(\{e, f, g\}\) such that \(G \backslash e, G \backslash f, G \backslash g, G \backslash e \backslash g, G \backslash f \backslash e, G \backslash g \backslash f, G \backslash \{e, f, g\}\) are all 3-connected, and \(G \backslash \{e, f, g\}\) is 2-connected.

(r.3) \(G\) is 4-connected. For all pairs \(\{e, f\}\) of distinct edges, \(G \backslash \{e, f\}\) is 3-connected or has series elements, \(G \backslash \{e, f\}\) is 3-connected or has parallel elements, and \(G \backslash e \backslash f\) is 3-connected. There exist 2 distinct edges \(e\) and \(f\) such that \(G \backslash \{e, f\}\) or \(G \backslash \{e, f\}\) is 3-connected; for any 2 such edges there exists a third one, say \(g \neq e, f\), such that at least one of the minors \(G \backslash \{e, f, g\}\),
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If $m \geq 2$, the edge labelled $z$ need not be present.

(s.3) $G$ is one of the following graphs: $K_n, 1 \leq n \leq 4$; the wheel with four spokes; $K_5$ with one edge deleted; the (planar) dual of the latter graph; $K_{3,3}$.

The development of the above theorems led to several variations, which we will not include here since they are easily produced by the methods of the subsequent sections. In Section 6 we sketch an efficient algorithm that selects one of the alternatives of the preceding theorems for a given matroid.

3. SEPARATIONS AND SUMS

In this section we establish the structure of $(k+)$-separations, and thus obtain easily checked sufficient conditions for the existence of $k$-sums, $k \leq 4$. Throughout this section $K$ is defined to be equal to $\lceil k/2 \rceil$.

**Lemma 3.1.** If a matroid $M$ has a $(k+)$-separation, $k \geq 2$, then $M$ has a partial representation $\mathcal{B}$ with

\[
\mathcal{B} = \begin{array}{c|c|c}
X_1 & Y_1 & A^1 \\
X_2 & Y_2 & A^2 \\
\end{array}
\]

such that the $X_i$ and $Y_i$ are nonempty and $|X_i \cup Y_i| \geq k + 1$, $i = 1, 2$, and $\bar{K} \leq \text{rank}(D) \leq k - 1$. 
Conversely, a matroid \( M \) is \((k+\lambda)-\)separable, \( k \geq 2 \), if it is \((k+1)-\)connected and has a partial representation \( \hat{B} \) with \( B \) of (3.2) such that the \( X_i \) and \( Y_i \) are nonempty and \(|X_i \cup Y_i| \geq k + 1\), \( i = 1, 2 \), and \( \text{rank}(D) = k - 1 \). In particular, every \((k+2)-\)connected \( k \)-sum, \( k \geq 2 \), is \((k+\lambda)-\)separable.

Proof. Let \((S_1, S_2)\) be a \((k+\lambda)-\)separation of \( M \) with, say, groundset \( S \), and let \( r(\cdot) \) and \( r^*(\cdot) \) be the rank functions of \( M \) and \( M^* \), respectively. Thus \( r(S_1) + r(S_2) = r(S) + l - 1 \), or equivalently \( r(S_i) + r^*(S_i) = |S_i| + l - 1 \), \( i = 1, 2 \), where \( \bar{k} + 1 \leq l \leq k \). Further \( |S_i| \geq k + 1 \), \( i = 1, 2 \) and \( r(S), r^*(S) \geq k \). If \( r(S_1) = r(S), \) then \( r(S_1) + r(S_2) = r(S) + l - 1 \) and \( l - 1 < k \leq r(S) \), \( |S_i| \) imply that \( r(S_1) < |S_1| < r(S) \). If \( r(S_2) = |S_2| \), then \( r(S_2) + r^*(S_2) = |S_2| + l - 1 \) implies \( r^*(S_2) < r^*(S), |S_2| \). Thus by the symmetry and duality we may suppose without loss of generality that \( r(S_2) < r(S), |S_2| \). Choose a base \( X_2 \) of \( S_2 \), and let \( Y_2 = S_2 - X_2 \). Select \( X_1 \subseteq S_1 \) so that \( X_1 \cup X_2 \) is a base of \( M \), and define \( Y_1 = S_1 - X_1 \). The related partial representation \( \hat{B} \) has \( B \) of (3.2). The sets \( X_i, Y_i, i = 1, 2 \) must all be nonempty since \( M \) is connected and \( r(S_2) < r(S), |S_2| \), and clearly \(|X_i \cup Y_i| \geq k + 1, i = 1, 2, \) as well as \( k \leq \text{rank}(D) \leq k - 1 \). The proof of the 2 converse statements involves trivial checking of the definitions.

**Theorem 3.3** (Structure of \((k+\lambda)-\)separations). Let a matroid \( M \) be \((k+\lambda)-\)separable for some \( k \geq 2 \). Then \( M \) has a partial representation \( \hat{B} \), where \( B \) is the matrix of (3.2) such that (1) the sets \( X_i, Y_i \) of (3.2) are nonempty, \( i = 1, 2 \), (2) the sets \( X_i \cup Y_i, i = 1, 2 \) define an \((l+\lambda)-\)separation of \( M \), where \( \bar{k} + 1 \leq l \leq k \), (3) \( A^1 \) of (3.2) is connected or it is one of the matrices of (3.4) below;

\[
\begin{cases}
(i) & \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}; \\
& \text{size } 1 \times l; \bar{k} \leq p < l, \text{ where } p \text{ is the number of } 1s; \\
(ii) & \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\
& 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix}; \\
& \bar{k} \leq p_i < l, \text{ where } p_i \text{ is the number of } 1s \text{ of row } i, i = 1, 2;
\end{cases}
\]

and (4) \( A^2 \) of (3.2) is connected or is the transpose of one of the matrices of (3.4).

Proof. By Lemma 3.1 we can assume a partial representation \( \hat{B} \) with \( B \) of (3.2). Let \( A^{11}, A^{12}, \ldots, A^{1m} \) (\( A^{21}, A^{22}, \ldots, A^{2n} \)) be the maximal connected submatrices of \( A^1 \) (\( A^2 \)). We will also call them blocks, as is customary. Define \( D^{1j} \) (\( D^{2j} \)) to be the column (row) submatrix of \( D \) with the same column (row) indices as \( A^{1j} \) (\( A^{2j} \)), and denote by \( D^y \) the submatrix of \( D \) specified by the column indices of \( D^{1j} \) and the row indices of \( D^{2j} \). Collect in a matrix \( E^1 \) (\( E^2 \)) the columns (rows) of \( D \) that are not included in any \( D^{1i} \)
It is claimed that \( \text{rank}(D^{1i}) \geq \kappa \), so each block of \( A^1 \) and \( A^2 \) has length of at least \( \kappa + 1 \). By duality we need only consider the case where \( \text{rank}(D^{1i}) < \kappa \), for some \( i \). We then define \( S_1 \) to be the set of indices of the rows and columns of \( A^{1i} \), and let \( S_2 = S - S_1 \), where \( S \) is the groundset of \( M \). With \( r(\cdot) \) denoting the rank function of \( M \), we see that 
\[
 r(S_1) + r(S_2) = r(S) + \text{rank}(D^{1i}) \leq r(S) + \kappa - 1,
\]
so \( M \) is not \((k+)\)-separable, a contradiction. Next we show that each \( D_{ij} \) is nonzero. Submodularity of \( \text{rank}(\cdot) \) implies that 
\[
 \text{rank}(D^{1i}) + \text{rank}(D^{2j}) \leq \text{rank}(D^{1i}) + \text{rank}(D^{2j}) \leq \kappa - 1.
\]
Thus \( 2\kappa \leq \text{rank}(D^{1i}) + \kappa - 1 \) must hold, and \( D_{ij} \) is necessarily nonzero. In passing we note that the \((\kappa + 1)\)-connectedness of \( M \) is essential for the desired conclusion about \( D_{ij} \), and that this fact motivated condition (1) of the definition of \((k+)\)-separability in Section 1.

In the discussion below we repeatedly repartition \( B \) of (3.2), and a simplified terminology is appropriate. "Shifting a column of \( E^1 \)" means adjoining that column of \( E^1 \) to \( A^2 \), and "Shifting a row of \( E^2 \)" means adjoining that row of \( E^2 \) to \( A^1 \). "Shifting except for specified blocks," all of which reside either in \( A^1 \) or in \( A^2 \), is interpreted as follows. If the specified blocks are in \( A^1 \) (\( A^2 \)), then we adjoin all rows and columns of \( A^1 \) (\( A^2 \)) that do not intersect those blocks, to \( A^2 \) (\( A^1 \)). We now give an algorithm that efficiently produces a partition of \( B \) that satisfies (1)-(4) of the theorem.

1. If \( A^1 \) properly contains a block \( A^{1i} \) with length of at least \( \text{rank}(D^{1i}) + 2 \), or if \( A^2 \) properly contains a block \( A^{2j} \) with length of at least \( \text{rank}(D^{2j}) + 2 \), then select one such block arbitrarily, shift except for that block, and return to the beginning of this step.

2. If \( A^1 \) or \( A^2 \) properly contains 2 or more blocks, then select 2 such blocks arbitrarily, shift except for those 2 blocks, and go to step 1.

3. If \( E^1 \) is nonempty, then shift columns of \( E^1 \) one-by-one until the shifting of any remaining column of \( E^1 \) would produce a new \( D \) with independent columns. If any shifting has been done, go to step 1.

4. If \( E^2 \) is nonempty, then shift rows of \( E^2 \) one-by-one until the shifting of any remaining row of \( E^2 \) would produce a \( D \) with independent rows. If any shifting has occurred, go to step 1. Otherwise, stop.

Since \( D_{ij} \neq 0 \), for all \( i \) and \( j \), one rather easily verifies that the sum of the number of blocks of \( A^1 \) and \( A^2 \), of the number of columns of \( E^1 \), and of the number of rows of \( E^2 \), is reduced by each shifting. Thus we must stop in step 4 at some time. Examine the immediately preceding pass through steps 1-4. If \( A^1 \) has a block \( A^{1i} \) with length at least \( \text{rank}(D^{1i}) + 2 \), then \( A^1 = A^{1i} \) by step 1, and case (1) of the theorem applies. If \( A^1 \) has 2 or more blocks, then by step 2 \( A^1 \) has exactly 2 blocks, and every row and column of \( A^1 \) intersects those blocks. Moreover, neither block satisfies the rank condition of step 1, and we must have case (3.4)(ii). Finally suppose \( A^1 \) has just one
block, and that the rank condition of step 1 does not hold for that block. By step 3 we then must have case (3.4)(i). By the symmetry analogous conclusions can be drawn for $A^2$.

We employed a rather simple device to bound the number of passes through step 1. Careful examination of the algorithm reveals that at most 5 such passes may take place before the algorithm stops.

We need 2 lemmas before we can put Theorem 3.3 and the shifting algorithm of its proof to use.

**Lemma 3.5.** Let $B$ with $B$ of (3.2) be a partial representation of a $(k+)$-separable matroid $M$, $k = 2, 3,$ or $4$. If both $A^1$ and $A^2$ of $B$ are connected, then $M$ is a proper $k$-sum, and $B$ of (3.2) may actually be chosen as in (3.1) with $C^1 (C^2)$ as a row (column) vector of $k - 1$ Is. For $k = 3$ only: If exactly one of $A^1$, $A^2$ is connected and if the other $A^1$ has length equal to 4, then $M$ is a semi-proper 3-sum, and in (3.1) the $C^i$ are as just specified.

**Proof.** First assume $A^1$ and $A^2$ to be connected. The case $k = 2$ is trivial. For $k = 3$ consider all paths in $G(A^1)$ between all pairs of distinct nodes $y$ and $z$ of $Y_i$ for which the 2 columns of $D$ with indices $y$ and $z$ are independent. In a shortest such path all intermediate nodes of $Y_i$ correspond to zero columns of $D$, so if necessary that path can be reduced by pivots in $A^1$ to one having exactly 2 arcs, which yields $C^1$. Similarly we obtain $C^2$. By submodularity the column indices of $C^1$ and the row indices of $C^2$ define a nonsingular $2 \times 2$ submatrix $D$ of $D$. Since the current $A^1$ and $A^2$ are still connected, $M$ is a proper 3-sum with the desired $C^1$ and $C^2$. For $k = 4$, we may suppose by the preceding arguments that $A^1$ has a [11] submatrix, say with column index set $Z$, such that the column submatrix of $D$ specified by $Z$, say $E$, has rank 2. Let $U$ be the set of column indices of $D$ such that each column $d$ of $D$ indexed by $u \in U$ forms a rank 3 matrix with $E$. In $G(A^1)$ select a shortest path from $U$ to $Z$. That path produces the desired $C^1$ matrix, or $B$ has one of the following matrices as submatrix, where in both cases $u \in U$.

\[
\begin{array}{ccc}
  & & 1 \\
1 & 1 & 1 \\
\end{array}
\]

By the dual statement of Theorem 2.1(b) of [11], pivots on the circled
entries produce in either case a new partial representation $\tilde{B}$ whose $B$ has a submatrix

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{B}
\end{bmatrix}
$$

where the vector of 1s belongs to the new $A^1$, and where $\tilde{B}$ is a rank 3 column submatrix of the new $D$. Thus the desired $C^1$ is easily selected. The remaining arguments for $k = 4$ are the same as for $k = 3$ above. For proof of the final claim of the lemma suppose without loss of generality that $A^2$ is connected, while $A^1$ is not and has length equal to 4. First we produce $C^2$ as given above. Then $A^1$ can be permuted to $[011]$ since otherwise $M$ is not 3-connected. The two 1s of $A^1$ give $C^1$. The column indices of $C_1$ and the row indices of $C^2$ again define a nonsingular $\tilde{D}$ of $D$, so $M$ is a semi-proper 3-sum.

The next lemma, though easily checked, is of fundamental importance for the proof of the decomposition Theorem 2.2 for binary matroids.

**Lemma 3.6.** Let $M$ be a connected binary matroid on a set $S$ without series or parallel elements. If $S_1$, $S_2 \subseteq S$ are a 2-separation of $M$, then $|S_i| \geq 5, i = 1, 2$.

The preceding results lead to the following sufficient conditions for certain $k$-sums.

**Theorem 3.7.** (Sufficient condition for proper/semi-proper $k$-sums, $2 \leq k \leq 4$). Let $M$ be a matroid:

(a) Suppose $M$ is connected and 2-separable, but has no series or parallel elements. Then $M$ is a proper 2-sum. If in addition $M$ is binary, then the sets $X_i$, $Y_i$, $i = 1, 2$, of (3.1) satisfy $|X_i \cup Y_i| \geq 5, i = 1, 2$.

(b) Let $M$ be $(3+\ast)$-separable. If $A^1$ or $A^2$ of (3.2) contains a block of length at least 4, and if the other $A^1$ has length equal to 4 or has a block of length at least 4, then $M$ is a proper or semi-proper 3-sum.

(c) Assume $M$ is $(4+\ast)$-separable. If both $A^1$ and $A^2$ of (3.2) contain a block of length at least 5, then $M$ is a proper 3- or 4-sum.

In all of the above cases $C^1$ and $C^2$ of the claimed proper 2-, 3-, 4-sums or of the semi-proper 3-sums may be taken to be vectors of 1s of appropriate dimension. Finally each claimed $k$-sum is $(k+\ast)$-separable for $k = 2, 3, 4$.

**Proof.** (a) By the assumptions $M$ is $(2+\ast)$-separable. The conclusion then follows from Theorem 3.3 and Lemma 3.6.
(b) By duality we may suppose that \( A^1 \) has a block of length at least 4. If \( A^2 \) also has such a block, then with the shifting algorithm and Lemma 3.5 we get a proper 3-sum. Otherwise \( A^2 \) has length equal to 4 and is a column vector with two 1s and one 0. If \( A^1 \) is indeed connected, then \( M \) is a semi-proper 3-sum. If \( A^1 \) is not connected, then the shifting algorithm again establishes a proper 3-sum.

(c) Follows from the shifting algorithm and Lemma 3.5.

The final claim follows from Lemmas 3.1 and 3.5.

Next we give sufficient conditions that assure decomposition of 1- or 2-element extensions of \( k \)-sums, \( k = 2 \) or 3.

**Theorem 3.8.** (Sufficient conditions for decomposition of 1- and 2-element extensions of \( k \)-sums, \( k = 2 \) or 3):

(a) Suppose that \( M \) is binary, connected, and 2-separable, and that \( M \) has no series or parallel elements or has a 2-separation \((S_1, S_2)\) for which \( M \setminus S_1 \) and \( M/S_2 \) are connected and \(|S_i| \geq 5, i = 1, 2\). Denote by \( M^1 \) a 3-connected (not necessarily binary) \( j \)-element extension of \( M \), \( i \leq j \leq 3 \); in the case of \( j = 3 \) the additional elements are to form a triangle or triad of \( M \). Then \( M^1 \) is a proper 3-sum for \( j = 1 \), and a proper 3- or 4-sum for \( j = 2, 3 \).

(b) Let \( M \) be a 3-connected proper 3-sum such that the sets \( X_i, Y_i, i = 1, 2 \), of (2.1) observe \(|X_i \cup Y_i| \geq 5, i = 1, 2\). Then any 3-connected 1-element extension of \( M \) is a proper 3- or 4-sum.

The statements in the last paragraph of Theorem 3.7 (concerning \( C^1, C^2 \), and \((k + )\)-separability) apply here as well.

**Proof.** (a) First suppose that \( M \) has no series or parallel elements. By Theorem 3.3 and Lemma 3.6 we may choose a \( B \) of (3.2) for \( M \) such that \( A^1 \) and \( A^2 \) are connected and \(|X_i \cup Y_i| \geq 5, i = 1, 2\). We can draw the same conclusion if \( M \) has a 2-separation \((S_1, S_2)\) for which \( M \setminus S_1 \) and \( M/S_2 \) are connected and \(|S_i| \geq 5, i = 1, 2 \) (define \( B \) by any base \( X_1 \cup X_2 \) such that \( X_2 \) is a base of \( S_2 \) and \( X_1 \subseteq S_1 \)). Since \( M \) can be derived from \( M^1 \) by \( j \) deletions/contractions, there exists a partial representation matrix \( B_1 \) of \( M \) such that (1) \( B^1 \) contains \( B \), and (2) \( B^1 \) has a total of \( j \) rows and columns beyond those of \( B \). We adjoin the rows (columns) at the bottom (to the left) of \( B \) and extend the partition of \( B \) in the obvious way. An application of the shifting algorithm and of Lemma 3.5 then produces the desired conclusions.

(b) Similar to (a).

Lemmas 3.1 and 3.5 prove the final claim as before.

We would like to find sufficient conditions that assure that the com-
ponents of a $k$-sum have high connectivity if $k$ is large. Here we prove that the components of any 3-connected proper $k$-sum must be 3-connected, for any $k \geq 3$. First we show that the connecting matroid $\mathcal{M}$ of any 3-connected $k$-sum, $k \geq 3$, is 3-connected.

**Lemma 3.9.** Let $\mathcal{B}$ be a partial representation of a matroid $\mathcal{M}$ with

$$
\mathcal{B} = \begin{array}{c|c|c}
\bar{X}_1 & \bar{X}_2 & \mathcal{C}^1 \\
\hline
\bar{Y}_1 & \bar{C}^2 & \mathcal{D} \\
\bar{Y}_2 & & \\
\hline
\end{array}
$$

where $\mathcal{D}$ is a nonsingular matrix of order at least 2, and $\mathcal{C}^1$ ($\mathcal{C}^2$) is connected and has no nested rows (columns). Then $\mathcal{M}$ is 3-connected.

**Proof.** $\mathcal{B}$ is connected, and does not contain a unit vector column/row or 2 parallel columns/rows. Hence $\mathcal{M}$ is connected, has no series or parallel elements, and any 2-separation of $\mathcal{M}$ produces a partition of $\mathcal{B}$ of the form

$$
\begin{array}{c|c|c|c|c}
\mathcal{O} & \mathcal{C}^{11} & \mathcal{C}^{12} & \mathcal{C}^{13} & \mathcal{C}^{14} \\
\hline
\mathcal{C}^{21} & \mathcal{D}^1 & \mathcal{D}^2 & \mathcal{C}^{22} & \mathcal{C}^{23} \\
\hline
\mathcal{C}^{23} & \mathcal{D}^3 & \mathcal{D}^4 & \mathcal{C}^{24} & \mathcal{O} \\
\hline
\mathcal{O} & \mathcal{C}^{13} & \mathcal{C}^{14} & \mathcal{O} & \\
\hline
\end{array}
$$

where the $\mathcal{C}^{ij}$, $j = 1, 2, 3, 4$, come from $\mathcal{C}^i$, and similarly the $\mathcal{C}^{2j}$ and $\mathcal{D}^j$ from $\mathcal{C}^2$ and $\mathcal{D}$. Let $E^j$, $j = 1, 2, 3, 4$ be the submatrix of $\mathcal{B}$ composed of $\mathcal{C}^{ij}$, $\mathcal{C}^{2j}$, $\mathcal{D}^j$, and the 0 matrix adjacent to $\mathcal{C}^{ij}$ and $\mathcal{C}^{2j}$. If one of the $E^j$ is empty, then we have partitioned $\mathcal{B}$ by either one horizontal or one vertical line. But then one of the 2 submatrices of $\mathcal{B}$ so created has rank 1 and consists of 2 or more rows or columns of $\mathcal{B}$, which implies that $\mathcal{M}$ has series or parallel elements, a contradiction. Thus all $E^j$ are nonempty, and by the symmetry we may assume that $E^2 = 0$ and rank($E^3$) = 1. Almost all arguments to follow rely on the connectedness of $\mathcal{C}^1$ and $\mathcal{C}^2$, so we will omit repeated references to that fact. If $\mathcal{C}^{13}$ ($\mathcal{C}^{23}$) is nonempty, it must be nonzero, since $\mathcal{C}^{12}$ ($\mathcal{C}^{22}$) is empty or 0. Rank($E^3$) $\geq 2$ if both $\mathcal{C}^{13}$ and $\mathcal{C}^{23}$ are nonzero, so by the symmetry we may assume that $\mathcal{C}^{13}$ is empty. Then $\mathcal{C}^{12}$ is also empty, and $\mathcal{C}^{11}$ or $\mathcal{C}^{14}$ is empty. If $\mathcal{C}^{11}$ is empty, $\mathcal{D}$ contains a zero row, a contradiction. So suppose that $\mathcal{C}^{14}$ is empty. If $E^3$ consists of just one row vector, then $\mathcal{C}^2$ is not connected or contains 2 nested columns,
a contradiction. Otherwise $E^3$ contains at least 2 rows of $\bar{D}$ and has rank $(E^3) \geq 2$, also a contradiction.

**Theorem 3.11.** Let $M$ be a $k$-sum, $k \geq 3$, with components $M_1$ and $M_2$, and $B$ be the matrix of (2.1). If $M$ is 3-connected and $A_1$ ($A_2$) has no zero column (row), then $M_1$ ($M_2$) is 3-connected. Conversely, if $M_1$ and $M_2$ are 3-connected, then $M$ is 3-connected.

**Proof.** First assume that $M$ is 3-connected, and that $A_1$ has no zero column. Since any column submatrix of $D_1$ of $B$ spans the related column submatrix of $D^{12}$ (see Theorem 2.1(g) of [11]), one easily verifies that $B_1$ of (2.3) is connected and does not have unit vector columns/rows or 2 parallel columns/rows. Thus $M_1$ is connected and has no series or parallel elements. Any 2-separation of $M_1$ thus corresponds to a partition of $B_1$ into 4 nonempty submatrices. By Lemma 3.9 the induced partition of the submatrix $\bar{B}$ defined by (3.10) may divide $\bar{B}$ into at most 2 submatrices, one of which is just a row or column of $\bar{B}$. If $\bar{B}$ is not partitioned at all, one very easily extends the partition of $B_1$ to one for $B$ of $M$ and establishes a 2-separation of $M$. Thus 4 cases remain, depending on whether a column of $C_2$ or $[C_1^C]$, or a row of $C_1$ or $[\bar{D}|C_2^C]$, is separated from the remainder of $\bar{B}$. Here we discuss just one case in detail since all others are just as easily handled. That is, we assume that a row of $[\bar{D}|C_2^C]$ is separated from the rest of $\bar{B}$, so $B_1$ is partitioned as

\[
\begin{array}{c|cccc}
 & d_1 & d_2 & \bar{d} & C^{22} \\
\hline
A^{11} & A^{12} & 0 & \\
A^{13} & A^{14} & 0 & C^1 \\
D^{11} & D^{12} & \bar{D}^1 & C^{21} \\
\end{array}
\]

Here $A^1$ has been partitioned into the $A^1$, $D^{11}$ into $D^{11}$, $D^{12}$, $d_1$, and $d_2$; $\bar{D}$ into $\bar{D}^1$ and $\bar{D}$; and $C^1$ into $C^{21}$ and $C^{22}$. Since both $C_1$ and $C_2$ are connected, $C^1$, $C^{21}$, and $C^{22}$ are nonzero, and the rank of $\bar{B}$ after deletion of row $x$ is at least 2. For a 2-separation $A^{13}$ and $D^{11}$ (which are nonempty) must be zero. Now $A^1$ has no zero column, so $A^{11}$ must be nonzero and hence non-empty, and we may move row $x$ of $B_1$ below the double line to get another 2-separation of $M_1$ where $\bar{B}$ is not partitioned at all, a case already treated. The 3-connectivity of $M_2$ follows by duality. Now let $M_1$ and $M_2$ be 3-connected. First we verify that $M$ is connected and that it has no series or parallel elements. Let $T_i$ be the groundset of $M_i$, $i = 1, 2$, $\bar{T}$ be that of $\bar{M}$,
the matroid of Lemma 3.9, and suppose the sets $S_1, S_2$ of cardinality at least 3 establish a 2-separation of $M$. By the assumptions and Lemma 3.9 we have without loss of generality $|\bar{T} \cap S_1|, |T_2 \cap S_1| \leq 1, \text{ so } |T_2 \cap S_1| \leq 1$ as well since $\bar{T} \subseteq T_2$ and $|\bar{T}| \geq 6$. But then $|S_1| \leq 2$, a contradiction.  

**Corollary 3.12.** (a) Let $M$ be a $k$-sum, $k \geq 3$, with components $M_1$ and $M_2$, and $B$ be the matrix of (2.1). If $M$ is 3-connected, then $M_1$ ($M_2$) can be turned into a 3-connected matroid by deletion of elements in $Y_1 - \bar{Y}_1$ that are parallel to any element in $X_2 \cup \bar{Y}_2$ (by contraction of elements in $X_1 - \bar{X}_1$ that are in series with any element in $X_1 \cup \bar{Y}_1$).

(b) A proper $k$-sum, $k \geq 3$, is 3-connected if and only if its components are 3-connected.

**Proof.** By Theorem 3.11 and duality only part (a) concerning the $M_1$-case deserves discussion. Let $\tilde{M}_1$ be the matroid derived from $M_1$ as specified in (a), and define $\bar{M}_1$ to be the matroid derived from $M_1$ by deleting all elements corresponding to zero columns of $A^1$. By Theorem 3.11, $\tilde{M}_1$ is 3-connected, and its groundset is contained in the one of $\bar{M}_1$. Suppose we can add $l \geq 0$ elements of $\bar{M}_1$ to $\tilde{M}_1$ and get a 3-connected matroid, but addition of a set with $l+1$ elements is not possible. Let $y$ be an arbitrary element among such a set, say $\bar{Y}$, with $l+1$ elements. Then $y \in Y_1 - \bar{Y}_1$, and $y$ must be parallel to an element of $X_2 \cup \bar{Y}_2$ or to another element, say $z$, of $\bar{Y}$. The former case is impossible by definition of $\tilde{M}_1$. In the latter case columns $y$ and $z$ of $D^1$, and hence of $D$, are parallel, and $M$ is not 3-connected, a contradiction.  

4. **Triangles and Triads**

Any triangle or triad of a 3-connected $M$ on 6 or more elements induces a 3-separation. Thus we may be faced with the situation where the connectivity of $M$ is not high, and yet $M$ may not be decomposable. This dilemma can be resolved in several ways. In the approach described below we introduce an alternative that assures that (roughly speaking) the triangle or triad can be removed from the matroid while 3-connectivity is maintained.

**Theorem 4.1.** (Structure of matroids with triangles/triads). If $M$ is a 3-connected matroid containing at least one triangle or triad, then at least one of the cases (d.3), (d.4), (r.2), (s.2) of Theorem 2.1 applies. If in addition $M$ is binary, then at least one of (d.3), (d.4), (r.2), (r.2*), (s.2), (s.3) of Theorem 2.2 holds.

**Proof.** We will only present detailed arguments for the binary case.
since the ones for the general case follow from (in fact, are easier than) these. The proof proceeds roughly as follows. We pick a triangle \( \{ e, f, g \} \), check connectivity of \( M/e\setminus g \), and find 2 situations, given by (4.2) and (4.3) below, that cannot be readily classified into one of the cases of Theorem 4.1. Similarly we examine \( M/\{ e, f, g \} \) and \( M\setminus \{ e, f, g \} \), and arrive at (4.4) below as an exception. However, a few additional arguments then completely dispose of (4.4) via (4.5). Thus we are faced with (4.2) and (4.3) only. Case (4.2) is easily reduced to (4.4), which has already been settled. The remaining case (4.3) requires a bit more effort, but eventually it, too, is classified as claimed in the theorem. With this overview in hand we are now ready for the detailed arguments.

By (s.3) of Theorem 2.2 we may suppose that \( M \) has at least 10 elements. Then the rank and corank of \( M \) must be at least 4 since otherwise \( M \) has series or parallel elements. Throughout \( B \) is a matrix of a binary representation matrix \( \tilde{B} \) of \( M \).

Let \( \{ e, f, g \} \) be a triangle of \( M \), and first suppose \( M/e\setminus g \) to be 2-separable. It is clearly connected, so if it has series elements, \( B \) may be selected as

\[
\begin{array}{ccc}
A' & 0 \\
\hline \\
e & b & 1 \\
\hline \\
f & a & 1 \\
\hline \\
o & 0 \\
\end{array}
\]  

(4.2)

where \( A' \) is nonempty and \( a, b \) are independent.

In the case of parallel elements we have

\[
\begin{array}{ccc}
A' & 0 \\
\hline \\
e & 1 & 1 \\
\hline \\
g & 1 & 0 \\
\hline \\
e & 1 & 1 \\
\end{array}
\]  

(4.3)

where \( A' \) is nonempty and \( \text{rank}(D) = 2 \) or 3.

In the remaining case \( M/e\setminus g \) is a proper 2-sum, and by Theorem 3.8(a) \( M \) is a proper 3- or 4-sum. We will deal with (4.2) and (4.3) later, so for the
time being suppose $M/e\backslash g$, $M/f\backslash e$, and $M/g\backslash f$ are 3-connected. It is easy to see that $M\backslash e$, $M\backslash f$, and $M\backslash g$ are then 3-connected as well, and that $M/\{e, f, g\}$ cannot have parallel elements. Thus $M/\{e, f, g\}$ must be a proper 2-sum if it is 2-separable, and $M$ is then a proper 3- or 4-sum by Theorem 3.8(a).

Hence we now assume in addition that $M/\{e, f, g\}$ is 3-connected. If $M\backslash \{e, f, g\}$ is connected, we have (r.2) of Theorem 2.2, so suppose this is not so. Thus we have

\[
B = \begin{pmatrix} \{e, f, g\} \\ B^1 & B^2 \end{pmatrix}
\]

(4.4)

where rank$(B^2) = 2$ and $B^1$ is nonempty but not connected. If $B^1$ has a block of length at least 4, then by Theorem 3.7(b), $M$ is a proper or semi-proper 3-sum. Hence assume all blocks have length 3, so we have

\[
B = \begin{pmatrix} \{e, f, g\} \\ B^1 & B^2 \end{pmatrix}
\]

(4.5)

$E$ must be empty since otherwise each of its rows has exactly two 1s, which in turn implies that $M\backslash e$, $M\backslash f$, $M\backslash g$ are not all 3-connected. Due to the 3-connectedness of $M$ each row of $D$ has exactly two 1s. To achieve 10 or more elements we must have at least 3 blocks in $B^1$, and pivots in those blocks plus row exchanges (if necessary) convert $B$ to the transpose of (s.2)(i) of Theorem 2.2.

By the above discussion all situations lead to the desired conclusion except for the case of $B$ of (4.2) or (4.3). A pivot on element $B_{eg}$ of $B$ of (4.2) produces a new $B$ whose last 3 rows are dependent. If we index these 3 rows by $e$, $f$, $g$, we have the transpose of an instance of (4.4). By the above discussion $M$ is then a proper or semi-proper 3-sum, or $B^1$ is of type (4.5). If $E$ of (4.5) is empty, the above arguments for (4.5) apply. Otherwise
$M$ is a semi-proper 3-sum. Thus we may presume that every triangle $\{e,f,g\}$ of $M$ leads to a $B$ of (4.3). Equivalently we may say: for every triangle $\{e,f,g\}$, at least one of $M/e\setminus g$, $M/f\setminus e$, $M/g\setminus f$ has parallel elements. The corresponding dual statement must hold for every triad $\{e,f,g\}$. If $A^1$ of (4.3) has a block of length at least 5, then by Theorem 3.7(b, c), $M$ is a proper 3- or 4-sum. Otherwise suppose $A^1$ has a block of length 3. By the above conclusion about triads the deletion of the columns of that block from $B$ reduces the submatrix $[D|E]$ to a matrix with two parallel rows (these must be the ones with index $g, f$) or with a unit vector (this must be the last row). This implies that $A^1$ has no zero column, and that it has at least two blocks. If there is also a block of length 4, the submatrix of $D$ specified by the columns of the latter block has rank 2, and $M$ is a proper 3-sum by Theorem 3.7(b). Thus we may assume that there is a second block of length 3. The previous arguments may be applied to this latter block, and we see that block 1 must induce parallel rows and block 2 must induce a row unit vector in the appropriate column submatrix of $[D|E]$, or vice versa. But then $M$ is a semi-proper 3-sum. This concludes the case where $A^1$ has a length 3 block, so we finally address the situation where all blocks have length 4. But then we have (s.2)(ii) of Theorem 2.2 or $M$ is a proper 3-sum. Finally we establish the claim about the $E^i$, $i \geq 1$, of Theorem 2.2(s.2)(ii) when $M$ is regular. Trivial enumeration reveals that a binary matroid represented by $B$ with

\[
\begin{array}{|c|c|c|c|}
\hline
& y_1 & & \\
\hline
x & 1 & 1 & 0 \\
\hline
x_1 & E^1 & 1 & 1 \\
\hline
\end{array}
\]

\det E^1 \neq 0$, is regular (so has no $F_7$ or $F^*$ minor) if and only if at most one pivot in row $x$ and exchanges among $Y_1$ columns and among $X_1$ rows convert $E^1$ to

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]

(4.6)

Thus we may assume that $E^1$ of (s.2)(ii) of Theorem 2.2 is of this form. Enumeration of the possibilities for $E^i$, $i \geq 2$, and for the last 3 columns of (s.2)(ii) then show that due to pivots and column exchanges each $E^i$ may also be taken to be the matrix of (4.6), and that column $y$ must be absent. Any matroid so defined by (s.2)(ii) is indeed regular since it is graphic (see Sect. 6).
5. 4-Connectivity

Intuitively one would expect that a highly connected matroid $M$ on a sufficiently large set $S$ has a minor $\overline{M}$ whose groundset is much smaller than $S$ such that $\overline{M}$ and all extensions of $\overline{M}$ in $M$ are at least 3-connected. In this section we make this notion precise for 4-connected matroids.

**Theorem 5.1.** (Structure of 4-connected matroids). If $M$ is a 4-connected matroid, then at least one of the cases (d.4), (r.3), (s.1), (s.2) of Theorem 2.1 applies. If $M$ is also binary, then at least one of (d.4), (r.3), (s.1), (s.3) of Theorem 2.2 is satisfied.

The main part of the proof of this theorem is handled by the following lemma.

**Lemma 5.2.** Let $M$ be a 4-connected matroid. If for all distinct elements $e$ and $f$, $M/\{e,f\}$ has parallel elements and $M\setminus\{e,f\}$ has series elements, then both the rank and corank of $M$ are at most 6.

We note that the lemma is false if we require $M$ to be only 3-connected instead of 4-connected. For example, let $B$ be a $k \times k$ matrix, $k \geq 7$, containing only 1s except for the diagonal entries, which are all 0. Simple checking confirms that the binary matroid $M$ represented by $B$ satisfies the assumptions of Lemma 5.2 except that it is only 3-connected instead of 4-connected.

**Proof of Lemma 5.2.** Due to the length and complexity of our proof we will only present a simplified version that shows that the rank and corank of $M$ are at most 11. Suppose this is not so. By duality we may assume that the corank of $M$ is not less than the rank of $M$. Thus in any partial representation $\hat{B}$ the matrix $B$ has at least as many columns as rows. We first show that there is a $B$ of the form

$$
\begin{array}{c|c|c|}
| & Y_1 & Y_2 \\
\hline
x_1 & 1 & 1 \\
\hline
x_2 & 0 & 1 \\
\hline
z & n & n \\
\end{array}
$$

(5.3)

where $\text{rank}(D) = 3$ and $|Y_1|$ is bounded by 6 (more careful arguments produce a bound of 5). $M$ must have a circuit with 4 elements, so a $B$ of
type (5.3) does exist when the condition $|Y_1| \leq 6$ is ignored. Among all such matrices choose a $B$ such that the submatrix $D$ is arranged as

$$
D = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} \\
1 & \cdots & 1 \\
1 & \cdots & 1 \\
0 & 1 & \cdots
\end{bmatrix}
$$

and $m \cdot |Y_{11}| + n \cdot |Y_{12}| + |Y_{13}|$ is minimal, where the constants $m$ and $n$ obey $m \gg n \gg 0$. It is claimed that $|Y_{11}| = 2$ and $|Y_{ij}| < 2$, $j = 2, 3$. Note that $|Y_1 \cup Y_2| \geq 11$ since otherwise the corank of $M$ is at most 11, and that deletion of any 2 columns (rows) from $B$ must produce a unit vector row (column) or 2 parallel rows (columns) since deletion (contraction) of any 2 elements of $M$ produces series (parallel) elements. Two observations will be useful.

**Observation 1.** If deletion of columns $z$ and $y_1 \in Y_1 \cup Y_2$ from $B$ produces 2 parallel non-unit vector rows $x_1, x_2 \in X_1$, then deletion of columns $z$ and $y_2 \neq y_1$ from $B$ cannot make rows $x_1$ and $x_2$ parallel.

**Proof.** If this is not so, then $x_1, x_2$, and $z$ form a triad of $M$.

**Observation 2.** If for $i = 1, 2, 3$, we delete columns $z$ and $y_i$ from $B$, getting $B^i$, where the $y_i$ are distinct and not in $Y_{11}$ (not in $Y_{11} \cup Y_{12}$) then we cannot have in each $B^i$ some row $x_i \in X_2$ parallel to the first (second) row of $B^i$.

**Proof.** If such rows $x_i$ exist, then they must be distinct, and in the first instance, where row $x_i$ is parallel to the first row of $B^i$, $B$ must be

$$
B = \begin{bmatrix}
1 & 1 & 1 & 0 \ 
1 & 0 & 1 & 0 \ 
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Here the submatrix defined $x_1, x_2, x_3$, and $Y_{11}$ has rank 1. Deletion of rows $x_1$ and $x_2$ cannot produce a unit vector, and indeed must make columns $y_1$ and $y_2$ parallel. Then $\{x_1, x_2, y_1, y_2\}$ is $S_1$ of a 3-separation.
(S_1, S_2) of M, a contradiction. The second, parenthetic case is handled analogously.

We now establish the claim about the |Y_{ij}|. Assume |Y_{11}| \geq 3. For i = 1, 2, 3, 4, delete columns z and y_i from B, getting B^i, where the y_i are distinct and in Y_1 \cup Y_2. A row unit vector in B^i must have index x \in X_1 (otherwise M has a triad), and |Y_{11}| = 2. Suppose B^i has rows x_1 and x_2 parallel. If x_1, x_2 \in X_2, then M has a triad. If x_1 \in X_1 and x_2 \in X_2, a pivot in row x_2 of B produces the unit vector situation. Thus we are done unless all B^i have parallel rows with indices in X_1. But then one pair (x_1, x_2) must be repeated, contrary to Observation 1. To show |Y_{12}| \leq 2 we delete columns z and y_i from B, i = 1, 2, 3, 4, where the y_i are distinct and in (Y_1 \cup Y_2) - Y_{11}. The arguments rely on Observations 1 and 2, and are very similar to those above.

Analogously |Y_{13}| \leq 2 is proved; this time y_j, i = 1, 2, ..., 5, of Y_{13} \cup Y_2 are involved in the deletion process. Thus |Y_1| is indeed bounded by 6. Let Y_2 = \{ y_1, y_2, ..., y_s \}, s \geq 5. Deletion of columns z and y_1 cannot produce a row unit vector (else M has a triad), so 2 parallel rows x_1 and x_2 must be present. Indeed, one of x_1 and x_2 must be in X_1 and the other one in X_2 since otherwise M has a triad. Thus B of (5.3) is actually

\[ B = \begin{array}{ccc}
& y_1 & y_2 \\
1 & D & 0 \\
1 & 1 & 0 \\
0 & 0/1
\end{array} \]  

and d is parallel to, say, the first row of D. Let x \in X_1 be the index of the second row of B, and suppose we pivot in \tilde{B} on B_{x2}. The resulting partial representation \tilde{B}^1 has a B^1 that looks almost like B, except that D has become a matrix D^1, column 1 of B^1 is labelled x, and X_1 has become (X_1 - \{x\}) \cup \{z\}. It is easy to see that d in B^1 is not parallel to any row of D^1. We now repeat the previous column deletion argument for B^1, this time using the indices x and y_1, and conclude that B^1 must have a second row with index in X_2 which has only 0s in columns x and y \in Y_2 - \{y_1\}.

The preceding procedure can be applied iteratively to the remaining y_i \in Y_2, so |X_2| \geq 2 \cdot |Y_2|. Combining this with the previous bound we have 7 + |Y_2| \geq 1 + |Y_1| + |Y_2| \geq |X_1| + |X_2| = 3 + |X_2| \geq 3 + 2 \cdot |Y_2|, so |Y_2| \leq 4 and |Y_1 \cup Y_2| \leq 10, a contradiction. When this proof is carried out much more carefully, we get |Y_1 \cup Y_2| \leq 5. 

**Proof of Theorem 5.1.** We will only cover the binary case since the
general one is handled almost identically. Simple checking reveals that any 4-connected binary matroid on 8 or more elements has at least 10 elements. Suppose for some pair \( \{ e, f \} \) of distinct elements \( M\setminus e/f \) or \( M\setminus \{ e, f \} \) is 2-separable. If one of the matroids is a proper 2-sum without series or parallel elements, then \( M \) is a proper 4-sum by Theorem 3.8(a). If \( M\setminus e/f \) has series or parallel elements, then \( M \) has a triangle or triad, a contradiction. Hence we may assume that for all distinct elements \( e \) and \( f \), \( M\setminus \{ e, f \} \) is 3-connected, and that \( M\setminus \{ e, f \} \) is 3-connected as well or has series elements. A 3-connected \( M\setminus \{ e, f \} \) cannot be a wheel or whirl (else \( M \) has a triangle), so by Tutte's wheel and whirl theorem [12] (see [11] for a simple proof) there is an element \( g \neq e, f \) such that \( M\setminus \{ e, f, g \} \) or \( M\setminus \{ e, f \}/g \) is 3-connected; let \( \tilde{M} \) be that minor. Clearly all extensions of \( \tilde{M} \) in \( M \) are 3-connected as well. So far we have shown that \( M \) satisfies (d.4), (r.3), or (s.3) of Theorem 2.2, or for all pairs \( \{ e, f \} \) of distinct elements \( M\setminus \{ e, f \} \) has series elements and \( M\setminus \{ e, f \} \) has parallel elements. By Lemma 5.2, the rank and corank of \( M \) are at most 6. Some checking proves \( R_{10} \) to be the only candidate, and thus (s.1) of Theorem 2.2 holds. (One first establishes that \( M \) has at most 10 elements using a refined version of the procedure given in the proof of Lemma 5.2. It is easy to show that \( R_{10} \) is the only 4-connected binary matroid on 10 elements, that it satisfies the assumption of Lemma 5.2, and that it is not a proper 4-sum.)

6. PROOFS OF DECOMPOSITION THEOREMS

With the results of Sections 3–5 it is now easy to prove the theorems of Section 2.

*Proof of Theorems 2.1 and 2.2.* Due to (d.1), (d.2), (r.1), (s.2) of Theorem 2.1, (d.1), (d.2), (r.1), (s.3) of Theorem 2.2, and Theorem 3.7(a) we may presume that \( M \) is 3-connected and has 10 or more elements, and that the rank and corank of \( M \) is at least 4. If \( M \) has a triangle or triad, then Theorem 4.1 provides the desired conclusion. Otherwise \( M \) has a \((3+)\)-separation or is 4-connected. In the former case \( M \) is a proper 3-sum by Theorem 3.7(b). With Theorem 5.1 we dispose of the latter case.

Next we show that Theorem 2.3 follows from Theorem 2.2. Below \( \bar{k} \) denotes \( \lceil k/2 \rceil \).

**Lemma 6.1.** Let \( M \) be a graphic proper \( k \)-sum, \( k \geq 2 \), on a set \( S \), arising from a graph \( G \). Suppose \( B \) with \( B \) of (9.1) is a representation matrix of \( M \), and define \( H_i \) to be the subgraph of \( G \) induced by the index set \( X_i \cup Y_i \) of \( B \), \( i = 1, 2 \). Then \( H_1 \) and \( H_2 \) are connected and have exactly \( k \) nodes in common.

*Proof.* Let \( H_i \) have \( n_i \) nodes, \( i = 1, 2 \), and suppose \( H_1 \) and \( H_2 \) have \( m \)
nodes in common. The submatrices $A^1$ and $A^2$ of $B$, and hence $B$ itself, are connected, so by Lemma 1.1, $G, G/(X_2 \cup Y_2)$ and $H_2 = G \setminus (X_1 \cup Y_1)$ are 2-connected. This implies that $H_1$ is connected. By the well-known relationship between node set cardinality and rank, $r(X_1 \cup Y_1) + r(X_2 \cup Y_2) = r(S) + k - 1$ implies $(n_1 - 1) + (n_2 - 1) = (n_1 + n_2 - m - 1) + k - 1$, and $m = k$ as claimed. □

**Lemma 6.2.** Let $G$ be a 2-connected graph without series or parallel edges, but suppose that $M$, the related polygon matroid, has series elements. Then $M$ has a 2-separation, say $(S_1, S_2)$, such that $M \setminus S_1$ and $M/S_2$ are connected and $|S_i| \geq 5, i = 1, 2$.

**Proof.** Two series edges of $M$, say $e$ and $f$, form a dual circuit of $M$, and thus make up a cutset of $G$. Let $G_1$ and $G_2$ be the connected components of $G \setminus \{e, f\}$. $G_2$ is the graph

$$
\begin{array}{c}
\begin{array}{c}
\bullet \\
\text{G_21}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
\text{G_22}
\end{array}
\end{array}
\cdots
\begin{array}{c}
\begin{array}{c}
\bullet \\
\text{G_2f}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
v
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
u
\end{array}
\end{array}
$$

where $u$ and $v$ are the endpoints of $e$ and $f$, and each $G_2i, 1 \leq i \leq l, l \geq 1$, is 2-connected. $G$ has no series or parallel elements, so neither $G_1$ nor $G_2$ can be paths, and one $G_2i$ has at least 5 edges. Define $S_2$ to be the set of elements of $M$ corresponding to the edges of that $G_2i$, and $S_1$ to be the set of the remaining elements. Clearly $M \setminus S_1$ and $M/S_2$ are connected, and $|S_i| \geq 5, i = 1, 2$. □

**Proof of Theorem 2.3.** Let $M$ be the polygon matroid of a graph $G$. Initially assume the following. If a statement of Theorem 2.2 holds for $M$ and claims that 2 elements, say $e$ and $f$, are in series in a certain minor of $M$, then the edges $e$ and $f$ are in series in the related minor of $G$. We then can verify (d.1)-(s.3) of Theorem 2.3 from the related statements of Theorem 2.2 as follows:

(d.1), (r.1)-(r.3), (s.3). Trivial. Note that we have listed 3-connected graphs only under (s.3) of Theorem 2.3 since all other graphs with at most 9 edges can be handled by (d.2) or (r.1).

(d.2)-(d.4). We will only discuss (d.4), since the other cases are handled similarly. Let $G, B, H_1, H_2, \ldots$ be as in the preceding Lemma 6.1. By Lemma 3.5, $B$ may be chosen such that $C^1$ and $C^2$ are vectors with exactly three 1s. Simple case checking reveals that due to the regularity of $M$, $\bar{D}$ may be selected to be the matrix of (4.6). By Lemma 6.1, $H_1$ and $H_2$ have 4 nodes in common. Suppose we delete the edges of $(Y_1 - \bar{X}_1)$ from $G$ and contract those of $(X_1 - \bar{X}_1)$. The submatrix $C^1$ of $A^1$ shows that the resulting graph, say $G_2$, has a cocycle with 4 edges. Indeed, the cocycle must be a
star with 4 nonparallel edges due to the nonsingular \( \bar{D} \) and the connected \( A^2 \). This is only possible if none of the contractions merged 2 common nodes of \( H_1 \) and \( H_2 \) into one node. Hence \( G_2 \) is the graph of (d.4). By similar reasoning \( G_1 \) of (d.4) is \( G/(X_2 - \bar{X}_2)(Y_2 - \bar{Y}_2) \). Arguing via extensions from \( G_1 \) and \( G_2 \) we get the indicated structure of paths in \( G \) of (d.4).

(s.1). This cannot occur since \( R_{10} \) is not graphic or cographic.

(s.2). One easily verifies that the graphic matroid \( M \) of \( K_{2,n}, n \geq 4, \) of (s.2)(i) of Theorem 2.3 (of the graph of (s.2)(ii) of Theorem 2.3) has a partial representation \( \hat{B} \), where \( B \) is the matrix of (s.2)(i) of Theorem 2.2 (of (s.2)(ii) of Theorem 2.2). In each case \( M \) is 3-connected and not cographic, so by \([16]\) no other graph can arise from (s.2) of Theorem 2.2.

Now suppose that a statement of Theorem 2.2 holds for \( M \), and that according to the statement a minor of \( M \) has series elements that are not in series in the related minor of \( G \). There are 2 cases where this may occur, and in each instance we show that Theorem 2.3 is still valid.

(r.1) of Theorem 2.2. If (r.1) of Theorem 2.3 does not apply to \( G \), then by Lemma 6.2, \( M \) has a 2-separation \( (S_1, S_2) \) such that \( M\backslash S_i \) and \( M/S_2 \) are connected and \( |S_i| \geq 5, i = 1, 2 \). It is then easily checked that \( M \) is a proper 2-sum, so (d.2) of Theorem 2.3 holds for \( G \).

(r.3) of Theorem 2.2. Assume that for 2 distinct elements \( e \) and \( f \), \( \bar{M} = M\backslash \{e, f\} \) has series elements, but that \( \bar{G} = G\backslash \{e, f\} \) does not. \( M \) and \( \bar{G} \) cannot have parallel elements (otherwise \( M \) is not 3-connected), so by Lemma 6.2 and Theorem 3.8(a), \( M \) is a proper 3- or 4-sum. Thus (d.3) or (d.4) of Theorem 2.3 holds for \( G \).

Finally we sketch an algorithm that efficiently selects one of the alternatives of Theorem 2.2 for a given binary matroid. The proof of validity follows from the preceding sections in a straightforward manner, and is omitted. A similar scheme can be devised for general matroids and the alternatives of Theorem 2.1. For efficient detection of 2- and 3-separability one can use the matroid intersection approach of \([4]\). The cases (d.1), (d.2), (r.1), and (s.3) of Theorem 2.2 are easy to identify, so we will assume that the given binary matroid is 3-connected and has 10 or more elements.

1. If \( M \) has no triangle or triad, go to 4.

2. Do this step for all triangles \( \{e, f, g\} \) of \( N = M, M^* \).

(a) If one \( N/e \backslash g, N/f \backslash e, N/g \backslash f \) is 2-separable: if the minor has series elements, produce \( B \) of (4.2) and go to (e). If it has parallel elements, process the next triangle. Otherwise derive a proper 3- or 4-sum from the 2-separation and stop.

(b) If \( N/\{e, f, g\} \) is 2-separable: Derive a proper 3- or 4-sum from the 2-separation and stop.
(c) If $N \setminus \{e, f, g\}$ is not connected: Produce $B$ of (4.4) and go to (e).

(d) Stop, $M$ satisfies (r.2) or (r.2*).

(e) From the transpose of $B$ of (4.2) or from $B$ of (4.4) derive a proper or semi-proper 3-sum or the transpose of a matrix of (s.2)(i) and stop.

3. Produce $B$ of (4.3) from one of the triangle cases of $M$ or $M^\star$. Derive $M$ to be a semi-proper 3-sum or a proper 3- or 4-sum, or determine $B$ or $B'$ to be a matrix of (s.2)(ii).

4. If $M$ is 3-separable: From any 3-separation determine $M$ to be a proper 3-sum.

5. Check the 3-connectivity conditions of (r.3) for $M$. If they are not satisfied, produce a proper 4-sum or show that $M = R_{10}$.

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REFERENCES